Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
   (a) Let $(X, \mathcal{T})$ be a topological space. Explain why the condition that X is compact is stronger than the assumption that X has a finite open cover.
   (b) Show that every topological space has a finite open cover.
   Hint: What is the first axiom of a topology?

2. Let X be a topological space.
   (a) Verify that, if X is T_1, then any subspace of X is T_1.
   (b) Verify that, if X is Hausdorff, then any subspace of X is Hausdorff.

3. (a) Prove that any set X with the indiscrete topology $\mathcal{T}$ is connected.
   (b) Let X be a set with at least two elements, endowed with the discrete topology. Prove that X is disconnected.

4. Determine the connected components of $\mathbb{R}$ with the following topologies.
   (a) the topology induced by the Euclidean metric
   (b) the discrete topology
   (c) the indiscrete topology
   (d) the cofinite topology

5. Give an example of a subsets $A \subseteq B$ of $\mathbb{R}$ such that ...
   (a) A is compact, and B is noncompact
   (b) B is compact, and A is noncompact

6. Suppose that A and B are compact. Show that $A \cup B$ is compact.

7. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Prove that the two following definitions of compactness are equivalent.
   - The subset A is *compact* if it is a compact topological space with respect to the subspace topology $\mathcal{T}_A$.
   - The subset A is *compact* if it satisfies the following property: for any collection of open subsets $\{U_i\}_{i \in I}$ of X such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite subscollection $U_1, U_2, \ldots, U_n$ such that $A \subseteq \bigcup_{i=1}^n U_i$.

Worksheet problems

(Hand these questions in!)

- Worksheet #17 Problems 1, 2(a).

Assignment questions

(Hand these questions in!)

1. Let X be an infinite Hausdorff topological space. Prove that X has an infinite discrete subspace A, i.e., an infinite subset A whose subspace topology is the discrete topology on A.

2. Let X be a topological space with the following property: for every $x \in X$, there exists a continuous function $f_x : X \to (\mathbb{R}, \text{Euclidean})$ such that $f_x^{-1}(\{0\}) = \{x\}$. Prove that X is Hausdorff.

3. (a) Prove the following result. *Hint*: See Worksheet #17 Theorems 1.6 and 1.10.

   Theorem (Generalized Intermediate Value Theorem). Let $(X, \mathcal{T}_X)$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.

   (b) Prove that any continuous function $f : [0,1] \to [0,1]$ has a *fixed point*, that is, an element $x \in [0,1]$ so that $f(x) = x$. *Hint:* Consider the function

$$\begin{aligned} g : [0,1] &\to \mathbb{R} \\ g(x) &= f(x) - x. \end{aligned}$$

4. Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are path-connected topological spaces. Show that the product $X \times Y$ with the product topology $\mathcal{T}_{X \times Y}$ is path-connected.

5. Suppose that $\{A_i\}_{i \in I}$ is a collection of path-connected subsets of a topological space $(X, \mathcal{T})$. Show that, if the intersection $\bigcap_{i \in I} A_i$ is nonempty, then the union $\bigcup_{i \in I} A_i$ is path-connected.

6. For each of the topological spaces X and subsets $A \subseteq X$ given below, state the following. State the interior $\text{Int}(A)$, closure $\overline{A}$, boundary ∂A, and set of accumulation points A' of A. State whether A is T_1, Hausdorff, connected, path-connected, and/or compact. **No justification needed.**

   (a) Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$, $A = \{a\}$.

   (b) X and $\mathcal{T}$ as above, $A = \{b, d\}$.

   (c) $X = \mathbb{R}$ with the Euclidean topology, $A = (0, 1) \cup \{2\}$.

   (d) $X = \mathbb{R}$ with the cofinite topology, $A = (0, 1) \cup \{2\}$.

   (e) $X = \mathbb{R}$ with the topology $\{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$, $A = \{-1, 1\}$.

   (f) X as above, $A = (-\infty, 0)$.

   (g) X as above, $A = (0, \infty)$.

   (h) $X = \mathbb{R}$ with the topology $\{U \mid 0 \in U\} \cup \{\varnothing\}$, $A = \mathbb{N}$.