1 Simplicial Group Actions

These notes follow Bredon "Compact transformation groups" Chapter 3.1.

Definition I. Let X be an (abstract) simplicial complex and G a group. An action of G on X is called a *simplicial group action* if each element $g \in G$ acts by a simplicial map $g : X \to X$.

Our goal is to make sense of the quotient of a simplicial group action.

Exercise 1. (Bonus) Let X be an abstract simplicial complex with a simplicial action by a group G. Show that the action of G induces an action of G on the barycentric subdivision of $\mathrm{sd}(X)$ of X, and that these actions coincide on the geometric realizations under the natural homeomorphism $|X| \cong |\mathrm{sd}(X)|$.

Definition II. Let X be an abstract simplicial complex with vertex set $V(X)$ and simplices $S(X)$. Suppose X admits a simplicial action by a group G. Define the quotient to be the abstract simplicial complex X/G defined as follows. A vertex $v^* \in V(X/G)$ is a G-orbit of a vertices in $V(X)/G$. A collection of vertices $\{v_0^*, \dots, v_p^*\}$ span a simplex in $S(X/G)$ if at least one choice of representatives $\{v_0, \dots, v_p\}$ of the orbits span a simplex in $S(X)$. The simplex $\{v_0, \dots, v_p\}$ of X is called a simplex *over* the simplex $\{v_0^*, \dots, v_p^*\}$ of X/G.

Exercise 2. (Bonus) Let X be an abstract simplicial complex with a simplicial action by a group G. Verify that the abstract simplicial structure defined on X/G is in fact a valid abstract simplicial structure.

There is a natural simplicial map

$$\begin{aligned} X &\longrightarrow X/G \\ v &\longmapsto v^* \end{aligned}$$

that gives rise to a simplicial map $|X| \to |X/G|$. This map is G-equivariant with respect to the trivial G-action on $|X/G|$, therefore the map factors through the orbit space $|X|/G$ via continuous maps

$$\begin{tikzcd} {|X|} \arrow[d] \arrow[dr] & \\ {|X|/G} \arrow[r, dashed] & {|X/G|} \end{tikzcd}$$

For a simplex $\{v_0, \dots, v_p\}$ of X, and a point $(\sum_{i=0}^p t_i v_i)$ in the simplex, consider the G-orbit of this point in $|X|/G$. The map $|X|/G \to |X/G|$ is defined as follows.

$$\begin{aligned} |X|/G &\longrightarrow |X/G| \\ G \cdot \left(\sum_{i=0}^p t_i v_i \right) &\longmapsto \sum_{i=0}^p t_i \left(G \cdot v_i \right) = \sum_{i=0}^p t_i v_i^* \end{aligned}$$

Exercise 3. (Bonus)

(a) Verify that the map $X \to X/G$ is simplicial, and G-equivariant with respect to the trivial action of G on X/G.

(b) Verify that the induced map $|X| \to |X/G|$ factors through $|X|/G$, and the factorization has the formula claimed.

Unfortunately, in general, this map $|X|/G \to |X/G|$ need not be a homeomorphism, or a homotopy equivalence, as we see in the following exercise.

Exercise 4. Consider the simplicial set R with vertices $V(R) = \{n\}_{n \in \mathbb{Z}}$, and simplices $S(R)$ consisting of the vertices and all edges of the form $\{n, n+1\}$. We can identify its geometric realization with the simplicial structure on the real line $\mathbb{R}$ that has a vertex at each integer point.

(a) Consider the group $G = \mathbb{Z}$; for notational clarity we denote its elements g_m for $m \in \mathbb{Z}$. Consider the action of $\mathbb{Z}$ on R by

$$\begin{aligned} g_m : V(R) &\longrightarrow V(R) \\ g_m \cdot n &\longmapsto m + n \end{aligned}$$

Verify that these are well-defined simplicial maps, and describe the action on the geometric realization $|R|$.

(b) Describe the simplicial complex R/G. Describe the spaces $|R/G|$ and $|R|/G$, and the map $|R|/G \to |R/G|$.

(c) Now consider the barycentric subdivision of $\mathrm{sd}(R)$ of R, with the induced action of G. Is $\mathrm{sd}(R)/G$ isomorphic to R/G? Again compare the spaces $|\mathrm{sd}(R)/G|$ and $|\mathrm{sd}(R)|/G$.

(d) Barycentrically subdivide a second time. Compare $|\mathrm{sd}^2(R)/G|$ and $|\mathrm{sd}^2(R)|/G$.

Definition III. Let X be an abstract simplicial complex with a simplicial action of a group G. The action is called *regular* if the group G and all of its subgroups $H \subseteq G$ satisfy the following Condition $(\mathcal{R})$:

$(\mathcal{R})$ Suppose $h_0, h_1, \dots, h_p$ are elements of H. Suppose that $(v_0, v_1, \dots, v_p)$ and $(h_0 \cdot v_0, h_1 \cdot v_1, \dots, h_p \cdot v_p)$ are $(p+1)$-tuples of (not necessarily distinct) vertices of X such that $\{v_0, v_1, \dots, v_p\}$ and $\{h_0 \cdot v_0, h_1 \cdot v_1, \dots, h_p \cdot v_p\}$ are simplices of X. Then there exists some $h \in H$ such that $h \cdot v_i = h_i \cdot v_i$ for all i.

In this case, we call X a *regular G-complex*.

Theorem IV. *Let X is a G-complex and $H \subseteq G$ a subgroup, and X/H the quotient complex. Then $|X|/H$ has a natural simplicial structure, such that the map $|X|/H \to |X/H|$ is a simplicial isomorphism.*

Exercise 5. This exercise will establish Theorem IV.

(a) Suppose that X is a regular G-complex, and let $\sigma = \{v_0^*, \dots, v_p^*\}$ be a simplex of X/G. Show that the set of simplices of X over σ form a G-orbit of p-simplices.

(b) Verify that the map $|X|/G \to |X/G|$ is one-to-one and onto.

(c) **(Bonus)** Verify that the topology on $|X|/G$ agrees with the weak topology on $|X/G|$.

We will prove that, if a group G acts simplicially on an abstract simplicial complex X, then the induced action of G of the two-fold barycentric subdivision $\mathrm{sd}^2(X)$ of X is a regular action. To prove this, we introduce a weaker Condition $(\mathcal{S})$ on a group action,

$(\mathcal{S})$ For any $g \in G$ and simplex σ of X, the element g fixes the intersection $\sigma \cap (g \cdot \sigma)$ pointwise.

Equivalently,

$(\mathcal{S}')$ For any $g \in G$, if vertices v and $g \cdot v$ are contained in the same simplex, then $v = (g \cdot v)$.

Exercise 6. (a) Verify that Condition $(\mathcal{R})$ implies Condition $(\mathcal{S}')$.

(b) Verify that Conditions $(\mathcal{S})$ and $(\mathcal{S}')$ are equivalent.

Theorem V. *Let X be an abstract simplicial complex, and G a group.*

- *If G acts simplicially on X, then the induced action of G on the barycentric subdivision $\mathrm{sd}(X)$ of X satisfies Condition $(\mathcal{S})$.*
- *If G acts on X by a simplicial action satisfying Condition $(\mathcal{S})$, then the induced action of G on the barycentric subdivision $\mathrm{sd}(X)$ of X satisfies Condition $(\mathcal{R})$.*

In particular, if G acts simplicially on X, then the induced action of G on the twice-iterated barycentric subdivision $\mathrm{sd}^2(X)$ of X is a regular group action in the sense of Defintion III.

Exercise 7. Prove Theorem V.

Exercise 8. (Bonus) Suppose that a simplicial action of G on X satisfies Condition $(\mathcal{S})$. Show that $|X^G| \cong |X|^G$. Here X^G denotes the subcomplex of simplices fixed pointwise by G, and $|X|^G$ the subspace of $|X|$ fixed pointwise by the induced action of G.