

1 Subdivisions of simplicial complexes

Definition I. Let X be a generalized simplicial complex. A *subdivision* of X is a new generalized simplicial structure X' on the underlying topological space of X such that each simplex of X' is contained in a simplex of X , and every simplex of X is a finite union of simplices of X' .

By this definition, the underlying spaces of X and X' agree as sets and as topological spaces.

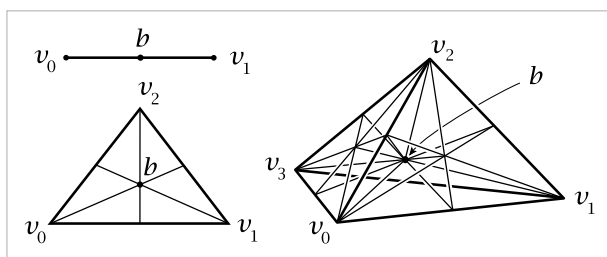
1.1 Barycentric subdivision

The barycentric subdivision of a complex is obtained by introducing a new vertex at the barycentre (i.e. centre of mass) of each simplex and taking the coarsest associated subdivision. Each n -simplex is replaced by a union of $(n + 1)!$ many n -simplices.

Definition II. Given an n -simplex Δ_α^n with vertices $v_0, v_1, \dots, v_n$, the *barycentre* of Δ_α^n is the point

$$\left(\frac{1}{n+1}\right)v_0 + \left(\frac{1}{n+1}\right)v_1 + \dots + \left(\frac{1}{n+1}\right)v_n.$$

We define the barycentric subdivision of a simplex inductively. The subdivision of a 0-simplex $\{v_0\}$ is simply the vertex $\{v_0\}$. The barycentric subdivision of the n -simplex $\{v_0, v_1, \dots, v_n\}$ is constructed by placing a new vertex b at the barycentre and subdividing into a union of n -simplices of the form $\{b, w_0, \dots, w_{n-1}\}$, where $\{w_0, \dots, w_{n-1}\}$ is an $(n - 1)$ -simplex in the barycentric subdivision of a face $\{v_0, \dots, \hat{v}_i, \dots, v_n\}$ of the simplex. The illustration (from Hatcher) shows the cases $n = 1, 2, 3$.



If X is a generalized simplicial complex, its *barycentric subdivision*—which we will denote $\text{sd}(X)$ —is a generalized simplicial complex constructed by inductively gluing in, in place of each n -simplex of X , a barycentrically subdivided n -simplex.

Exercise 1. (Bonus) Verify the details of Definition II: check that the construction of $\text{sd}(X)$ yields a valid generalized simplicial complex structure, that is homeomorphic to X , and that it is a subdivision of X in the sense of Definition I.

Exercise 2. (Bonus) Let X be a generalized simplicial complex. Prove or disprove: The barycentric subdivision of the k -skeleton of X is the k -skeleton of $\text{sd}(X)$.

Proposition III. Let X be a simplicial complex. Let $\mathcal{S} = (S, \subseteq)$ be the set of simplices of X , ordered by inclusion. Then the realization of the order complex $|\mathcal{S}|$ is canonically isomorphic to the barycentric subdivision of X .

In light of Proposition III, we can define the barycentric subdivision of an abstract simplicial complex as the order complex on its poset of simplices under inclusion.

Definition IV. Let X be an abstract simplicial complex. The *barycentric subdivision* $\text{sd}(X)$ of X is an abstract simplicial complex defined as follows. The vertex set of $\text{sd}(X)$ is the set of simplices $S(X)$ of X . A collection of simplices $\{\sigma_0, \sigma_1, \dots, \sigma_p\}$ of X form a p -simplex of $\text{sd}(X)$ precisely when they form a chain under inclusion.

Exercise 3. Prove Proposition III.

Exercise 4. (Bonus) Let $f : X \rightarrow Y$ be a simplicial map of abstract simplicial complexes. Show that f induces a simplicial map $f_* : \text{sd}(X) \rightarrow \text{sd}(Y)$ on the barycentric subdivisions of X and Y . Prove moreover that barycentric subdivision defines an endofunctor on the category of abstract simplicial complexes and simplicial maps.

Exercise 5. Verify that Definitions II and IV are compatible: given an abstract simplicial complex X , the realization of its barycentric subdivision $|\text{sd}(X)|$ is the (geometric) barycentric subdivision of its realization $\text{sd}(|X|)$.

Exercise 6. Let $\mathcal{P}$ be the poset of nonempty subsets of $[n+1]$, ordered by inclusion. Show that its geometric realization $|\mathcal{P}|$ is the barycentric subdivision of an n -simplex. Verify that it has $(n+1)!$ top-dimensional simplices.

Exercise 7. (Bonus) Let X be a simplicial complex, and L a (not necessarily full) subcomplex. Prove that $\text{sd}(L)$ is a full subcomplex of $\text{sd}(X)$.

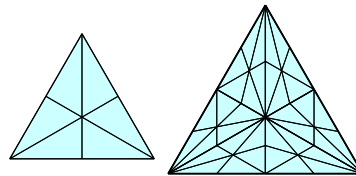
The following proposition shows that a generalized simplicial complex structure can be refined to a simplicial complex structure by repeated barycentric subdivision.

Proposition V. Let X be a generalized simplicial complex.

- The barycentric subdivision $\text{sd}(X)$ of X is regular, that is, $\text{sd}(X)$ has the property that every n -simplex has $(n+1)$ distinct vertices.
- Suppose X is regular. Then its barycentric subdivision $\text{sd}(X)$ is a simplicial complex.

Thus, the two-fold barycentric subdivision $\text{sd}^2(X)$ of any generalized simplicial complex X is a simplicial complex.

The following figure (from Wikipedia) shows the first and second subdivision of a 2-simplex.



Exercise 8. (a) Prove Proposition V.

- (b) Show by example that a single barycentric subdivision of a generalized simplicial complex may not be a simplicial complex.

Exercise 9. (Bonus) Describe the barycentric subdivision of the complex of injective words (see Worksheet #11, Definition III). Verify that it is a simplicial complex.

Exercise 10. (Bonus) Formulate a definition of a barycentric subdivision of a poset $(P, \leq)$, so that the order complex of the barycentric subdivision is the barycentric subdivision of the order complex of $(P, \leq)$. *Hint:* See Kozlov Section 10.3.5.

Exercise 11. (Bonus) Let X be a simplicial complex and $\text{sd}(X)$ its barycentric subdivision. Discuss: can you characterize links of simplices in $\text{sd}(X)$ in terms of the combinatorics of X ?