1 Topological spaces

Definition 1.1. (Topology; Topological space.) Let X be a set. A *topology* $\mathcal{T}$ on X is a collection of subsets of X that satisfies the following properties:

(T1) The sets $\varnothing$ and X are elements of $\mathcal{T}$.

(T2) If $\{U_i\}_{i \in I}$ is any collection of elements of $\mathcal{T}$, then $\displaystyle\bigcup_{i \in I} U_i$ is in $\mathcal{T}$.

(T3) If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

A set X endowed with a topology $\mathcal{T}$ is called a *topological space* and denoted by $(X, \mathcal{T})$, or simply denoted by X when $\mathcal{T}$ is clear from context. The elements of $\mathcal{T}$ are called the *open subsets* of the topological space X.

We have already proved the following result:

Theorem 1.2. (Metrics induce a topology.) *Let (X, d) be a metric space. Then the collection $\mathcal{T}_d$ of all open sets in X forms a topology on X.*

The topology $\mathcal{T}_d$ is called the *topology induced by the metric d.* We will see that not every topology on a set X necessarily arises from a metric structure on X.

Definition 1.3. (Metrizable topologies.) A topology $\mathcal{T}$ on a set X is said to be *metrizable* if there exists a metric d on X such that $\mathcal{T}$ is the set $\mathcal{T}_d$ of open sets for the metric space (X, d).

In-class Exercises

1. Let X be a set.
   (a) Let $\mathcal{T} = \{\varnothing, X\}$. Prove that $\mathcal{T}$ is a topology on X. It is called the *indiscrete topology.*
   (b) Let $\mathcal{T}$ be the collection of all subsets of X. Prove that $\mathcal{T}$ is a topology on X. It is called the *discrete topology.*

2. Let X be a set.
   (a) Show that the discrete topology is metrizable.
   (b) Suppose that X contains at least 2 elements. Show that the indiscrete topology is not metrizable.

3. **(A useful criterion for openness).** Let $(X, \mathcal{T})$ be a topological space. Show that $V \in \mathcal{T}$ (that is, V is open) if and only if for every $x \in V$ there is some set $U_x \subseteq X$ containing x such that $U_x \in \mathcal{T}$ and $U_x \subseteq V$.

4. Consider the following definition.

   Definition 1.4. (Closed subsets of a topological space.) Let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is called *closed* if its complement is open, that is, if $X \setminus C$ is an element of $\mathcal{T}$.

   (a) Verify that X and $\varnothing$ are closed.
   (b) Suppose that B and C are closed subsets of X. Verify that $B \cup C$ is closed.
   (c) Suppose that $\{C_i\}_{i \in I}$ is a collection of closed sets in X. Verify that $\displaystyle\bigcap_{i \in I} C_i$ is closed.

5. **(Optional).**
   (a) Let $X = \{a\}$. Explain why the only topology on X is $\{\varnothing, X\}$.
   (b) Let $X = \{a, b\}$. Find all possible topologies on X.
   (c) Let $X = \{a, b, c\}$. Find all possible topologies on X.
   Hint: Look at the picture on our Canvas homepage.

6. **(Optional).** Verify that the following sets are topologies on $\mathbb{R}$.
   - The topology induced by the Euclidean metric
   - $\mathcal{T} = \{\mathbb{R}, \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, (0,1), \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, \{0,1\}, \{0\}, \{1\}, \varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}, \mathbb{R} \setminus A \text{ is finite}\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \text{ is a union of intervals of the form } [a,b) \text{ for } a,b \in \mathbb{R}\ \} \cup \{\varnothing\}$

7. **(Optional).** Consider the following definitions.

   Definition (Coarser topology; finer topology). Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then the topology $\mathcal{T}_1$ is said to be *coarser* than $\mathcal{T}_2$, and the topology $\mathcal{T}_2$ is said to be *finer* than the topology $\mathcal{T}_1$.

   (a) Let X be a set. Show that the indiscrete topology on X is coarser than any other topology on X.
   (b) Let X be a set. Show that the discrete topology on X is finer than any other topology on X.
   (c) Let X be a set, and let $\mathcal{T}_1$, $\mathcal{T}_2$ be two topologies on X. Show that the identity function
   $$\begin{aligned}(X, \mathcal{T}_2) &\longrightarrow (X, \mathcal{T}_1)\\ x &\longmapsto x\end{aligned}$$
   is continuous if and only if $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.
   (d) Consider all the topologies on $\mathbb{R}$ in Problem 6. For each pair of topologies, determine either that one is coarser than the other, or show that they are not comparable. Which are comparable to the topology induced by the Euclidean metric?