

1 Complete metric spaces

Definition 1.1. A metric space (X, d) is called *complete* if every Cauchy sequence in X converges.

Example 1.2. Give an example of a metric space that is not complete.

In-class Exercises

1. Suppose that a metric space (X, d) is sequentially compact. Show that (X, d) is complete.
2. (a) Let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Show that it is contained in some closed and bounded subset of $\mathbb{R}^n$.
(b) Prove the following theorem.

Theorem ($\mathbb{R}^n$ is complete). The space $\mathbb{R}^n$ is complete with respect to the Euclidean metric.

Hint: Review the results from Worksheet #6.

3. Formulate and prove necessary and sufficient conditions on a metric subspace S of $\mathbb{R}^n$ (with the Euclidean metric) for it to be complete.
4. **(Optional)** Let X be a nonempty set with the discrete metric. Under what conditions is X complete?
5. **(Optional)** Let X and Y be metric spaces. Suppose that X is complete.
 - (a) Let $f : X \rightarrow Y$ be a continuous function. Must its image $f(X)$ be complete?
 - (b) Suppose that $f : X \rightarrow Y$ is an isometric embedding. Must $f(X)$ be complete?
6. **(Optional)** Recall that a subset A of a metric space X is *dense* if $\overline{A} = X$. Recall that an *isometry* $f : X \rightarrow Y$ of metric spaces (X, d_X) and (Y, d_Y) is a bijective isometric embedding, that is, a bijective map such that preserves distances in the sense that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Consider the following definition.

Definition (Completion). Let X be a metric space. The *completion* of X is a complete metric space Y along with an isometric embedding $h : X \rightarrow Y$ such that $h(X)$ is dense in Y .

In this question, we will construct the completion of X , and verify that it is unique up to isometry.

- (a) Let A be a dense subset of metric space Z . Show that, if every Cauchy subsequence in A converges in Z , then Z is complete.
- (b) Let (X, d) be a metric space. Let $\tilde{X}$ denote the set of Cauchy sequences $(x_n)_{n \in \mathbb{N}}$ in X . Let Y denote the equivalence classes defined by the equivalence relation on $\tilde{X}$,

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}} \iff d(x_n, y_n) \xrightarrow{n \rightarrow \infty} 0.$$

Verify that this is indeed an equivalence relation, by checking that it is reflexive, symmetric, and transitive.

- (c) For $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ and $\mathbf{y} = (y_n)_{n \in \mathbb{N}}$, let $[\mathbf{x}]$ and $[\mathbf{y}]$ denote the corresponding equivalence classes. Define

$$D : Y \times Y \rightarrow \mathbb{R}$$

$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Show that D is well-defined, that is, its value does not depend on the choice of representative of the equivalence class.

- (d) Show that D defines a metric on Y .
- (e) Define

$$h : X \rightarrow Y$$

$$x \mapsto [(x)_{n \in \mathbb{N}}]$$

sending a point x to the constant sequence $(x)_{n \in \mathbb{N}}$. Show that h is an isometric embedding.

- (f) Show that, for any Cauchy sequence $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ in X , the sequence $(h(x_n))_{n \in \mathbb{N}}$ converges in Y to $[\mathbf{x}]$.
- (g) Conclude from part (f) that $h(X)$ is dense in Y .
- (h) Further conclude that every Cauchy sequence in $h(X)$ must converge in Y , and thus by part (a) the space (Y, D) is complete. This shows that Y is the completion of X .
- (i) Show that this completion is unique, in the sense of the following theorem.

Theorem (The completion of X is unique up to isometry). Let $h : X \rightarrow Y$ and $h' : X \rightarrow Y'$ be isometric embeddings of the metric space (X, d) into complete metric spaces (Y, D) and (Y', D') , respectively, with dense image. Then there is an isometry of $(\overline{h(X)}, D)$ and $(\overline{h'(X)}, D')$ that equals $h' \circ h^{-1}$ when restricted to the subspace $h(X)$.

- (j) Explain how we could define the real numbers $\mathbb{R}$ as the completion of the rationals $\mathbb{Q}$.