1 Open and closed sets

Going forward, we will implicitly assume $\mathbb{R}^n$ has the Euclidean metric unless otherwise stated.

Definition 1.1. (Open ball of radius r about x_0.) Let (X, d) be a metric space, and $x_0 \in X$. Let $r \in \mathbb{R}$, $r > 0$. We define the *open ball of radius r about x_0* as the subset of X

$$B_r(x_0) = \{x \in X \mid d(x_0, x) < r\} \subseteq X.$$

Example 1.2. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$. What is $B_1(0)$? What is $B_2(-6)$?

Example 1.3. Let $X = \mathbb{R}^2$ with the usual Euclidean metric. Draw $B_2(1, 0)$.

Definition 1.4. (Interior points; open sets in a metric space.) Let (X, d) be a metric space, and let $U \subseteq X$ be a subset of X. A point $x \in U$ is called an *interior point of U* if there is some radius $r_x \in \mathbb{R}$, $r_x > 0$, so that $B_{r_x}(x) \subseteq U$.

The set $U \subseteq X$ is called *open* if every point $x \in U$ is an interior point of U.

A set $A \subseteq X$ fails to be open if ...

Definition 1.5. (Neighbourhood of a point x.) Let (X, d) be a metric space, and $x \in X$. Then any open set U in X containing x is called an *open neighbourhood of x*, or simply a *neighbourhood of x*.

Example 1.6. Show that the interval $[0, 1) \subseteq \mathbb{R}$ is not open.

Example 1.7. Show that the interval $(0, 1) \subseteq \mathbb{R}$ is open.

Proposition 1.8. *Let (X, d) be a metric space, $x_0 \in X$ and $0 < r \in \mathbb{R}$. Then the ball $B_r(x_0)$ is an open subset of X.*

Proof.

Example 1.9. Let (X, d) be a metric space. Is $\varnothing$ open?

Definition 1.10. (Closed sets in a metric space.) A subset $C \subseteq X$ is *closed* if its complement $X \setminus C$ is open.

Warning: Despite the English connotations of the words 'closed' and 'open', mathematically, 'closed' does **not** mean "not open". 'Open' does not mean "not closed"!

In-class Exercises

1. Consider $\mathbb{R}$ with the Euclidean metric. Find an example of a subset of $\mathbb{R}$ that is ...
   (a) open and not closed,
   (b) closed and not open,
   (c) both open and closed,
   (d) neither open nor closed.

2. Let $\{U_i\}_{i \in I}$ denote a collection of open sets in a metric space (X, d).
   (a) Prove that the union $\bigcup_{i \in I} U_i$ is an open set. Do not assume that I is necessarily finite, or countable!
   (b) Show by example that the intersection $\bigcap_{i \in I} U_i$ may not be open. (This means, give an example of a metric space (X, d) and a collection of open sets $U_i \subseteq X$, and prove that $\bigcap_{i \in I} U_i$ is not open).
   (c) Now assume we have a **finite** collection $\{U_i\}_{i=1}^n$ of open sets in a metric space. Prove that the intersection $\bigcap_{i=1}^n U_i$ is open.

3. Prove the following result.

   Theorem 1.11. *Let (X, d) be a metric space. A subset U of X is open if and only if U can be expressed as a union of open balls in X.*

4. Consider the set $Y = [0, 2]$, as a subset of $X = \mathbb{R}$, with the Euclidean metric. Show that the set $[0, 1)$ is open as a subset of Y, but **not** open as a subset of X.

   Warning: This problem shows that (when we're working with both a metric space and a metric subspace) it is not enough to say a subset is "open". We need to say "open" in which metric space!

5. **(Optional).** Let (X, d) be a metric space. Let Y be a metric subspace of X, that is, Y is a subset of X that we view as a metric space with the restriction of the metric d. Let $U \subseteq Y$ be an open subset of Y. Show that there exists some open subset W of X such that $U = Y \cap W$. Conversely, if W is an open subset of X, show that $W \cap Y$ is an open subset of Y.

6. **(Optional).** Let X be a non-empty set. A function $f : X \to \mathbb{R}$ is called *bounded* if there is some number $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in X$. Let $\mathcal{B}(X, \mathbb{R})$ denote the set of bounded functions from X to $\mathbb{R}$.

   (a) Show that the function

   $$\begin{aligned} d_\infty : \mathcal{B}(X,\mathbb{R}) \times \mathcal{B}(X,\mathbb{R}) &\longrightarrow \mathbb{R} \\ d_\infty(f,g) &= \sup_{x \in X} |f(x) - g(x)| \end{aligned}$$

   is well-defined, that is, the suprema always exist.

   (b) Show that the function d_∞ defines a metric on $\mathcal{B}(X, \mathbb{R})$.

   (c) Explain why the following metric on $\mathbb{R}^n$ is a special case of this construction.

   $$\begin{aligned} d_\infty : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ d(\overline{x}, \overline{y}) &= \max_{1 \leq i \leq n} |x_i - y_i| \end{aligned}$$

   where $\overline{x} = (x_1, \ldots, x_n)$ and $\overline{y} = (y_1, \ldots, y_n)$.

7. **(Optional).**

   (a) Rigorously verify that the sets $\{1\}$, $[1, \infty)$, and $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\} \cup \{0\}$ are all closed subsets of $\mathbb{R}$ (with the Euclidean metric).

   (b) Consider $\mathbb{R}$ with the Euclidean metric. Is the subset $\mathbb{Q} \subseteq \mathbb{R}$ open? Is it closed?

   (c) Let $\mathcal{C}(0, 2)$ be the set of continuous functions from the closed interval $[0, 2]$ to $\mathbb{R}$, with the metric

   $$\begin{aligned} d_\infty : \mathcal{C}(0,2) \times \mathcal{C}(0,2) &\longrightarrow \mathbb{R} \\ d(f,g) &= \sup_{x \in [0,2]} |f(x) - g(x)| \end{aligned}$$

   Determine whether the subset $\{f(x) \in \mathcal{C}(0, 2) \mid f(1) = 0\}$ is closed, open, neither, or both.

8. **(Optional).** Let X be a **finite** set, and let d be any metric on X. What can you say about which subsets of X are open? Which subsets of X are closed?

9. **(Optional).** Let (X, d) be a metric space. Fix $x_0 \in X$ and $r > 0$ in $\mathbb{R}$. Show that the set $\{x \mid d(x_0, x) \leq r\}$ is closed.