

1 Products of topological spaces

Definition 1.1. (The product topology.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the *product topology* $\mathcal{T}_{X \times Y}$ on $X \times Y$ is the collection of subsets of $X \times Y$ generated by the set

$$\mathcal{B} = \{U \times V \mid U \subseteq X \text{ is open, and } V \subseteq Y \text{ is open}\}.$$

This means that $\mathcal{T}_{X \times Y}$ consists of all unions of elements of $\mathcal{B}$.

Proposition 1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. Then the product metric $d_{X \times Y}$ induces the product topology on $X \times Y$.

In-class Exercises

1. Verify that $\mathcal{T}_{X \times Y}$ is indeed a topology on $X \times Y$, and that $\mathcal{B}$ is a basis for this topology.
Hint: One approach is to use Homework #10 Problem 6(a). See Homework #3 Problem 4(b).
2. Prove that the projection map $\pi_X : X \times Y \rightarrow X$ is continuous with respect to the topologies $\mathcal{T}_{X \times Y}$ and $\mathcal{T}_X$. (The same argument shows that the projection map π_Y is continuous).
3. Let $f_1 : X \rightarrow Y_1$ and $f_2 : X \rightarrow Y_2$ be two functions of topological spaces, and define a function

$$\begin{aligned} f : X &\rightarrow Y_1 \times Y_2 \\ f(x) &= (f_1(x), f_2(x)) \end{aligned}$$

Show that f is continuous (with respect to the product topology on $Y_1 \times Y_2$) if and only if both f_1 and f_2 are continuous.

Hint: Worksheet #12 Theorem 1.2.

Hint: Notice $f_i = \pi_i \circ f$, where $\pi_1 : Y_1 \times Y_2 \rightarrow Y_1$, $\pi_2 : Y_1 \times Y_2 \rightarrow Y_2$ are the projection maps.

4. **(Optional).** Prove the following theorem.

Theorem (Equivalent definition of the product topology). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then the product topology $\mathcal{T}_{X \times Y}$ on $X \times Y$ is precisely the collection of subsets of $X \times Y$,

$$\mathcal{T}_{X \times Y} = \left\{ W \mid \begin{array}{l} \text{for each } (x, y) \in W, \text{ there is a some } U \in \mathcal{T}_X \text{ and } V \in \mathcal{T}_Y \\ \text{such that } (x, y) \in (U \times V) \subseteq W \end{array} \right\}.$$

5. **(Optional). (Finite products).** Given a finite product $X = X_1 \times X_2 \times \cdots \times X_n$ of topological spaces, we can define a topology on X by induction, by first taking the product topology on $X_1 \times X_2$, then the product topology on $(X_1 \times X_2) \times X_3$, etc. Show that the resultant topology on X (called the *product topology*) is generated by the basis

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n \mid U_i \subseteq X_i \text{ is open}\}.$$

Repeat the in-class exercises for this topology.

6. **(Optional). (Infinite products).** Let $\{X_i\}_{i \in I}$ be a (possibly infinite) collection of sets, and let $X = \prod_{i \in I} X_i$ be their product. We denote elements of X by $(x_i)_{i \in I}$. Define two topologies on X :

- The *box topology* on X is the topology generated by the basis

$$\mathcal{B}_B = \left\{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ is open} \right\}.$$

- The *product topology* on X is the topology generated by the basis

$$\mathcal{B}_P = \left\{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ is open, } U_i = X_i \text{ for all but finitely many } i \in I \right\}.$$

- Prove that both $\mathcal{B}_B$ and $\mathcal{B}_P$ are bases (in the sense of Worksheet #12, Problem 5). Conclude that the box topology and product topology on X are, in fact, topologies.
- Suppose that I is finite. Show that the box topology and the product topology are equal, and both are the usual product topology on X in the sense of Problem 5.
- Which topology is finer, the box or the product topology? If we consider maps from a topological space into the product X , what can you say about the relationship between continuity of a map with respect to the box topology, and continuity with respect to the product topology? What about for maps out of X ?
- For reasons that are formalized using “category theory” and the concept of a “universal property”, we want our products to satisfy the following statement:

Let $\{Y_i\}_{i \in I}$ be a collection of topological spaces, and $\prod_{i \in I} Y_i$ their product. Let X be any topological space, and let $f_i : X \rightarrow Y_i$ be a collection of functions. Then the function

$$f : X \rightarrow \prod_{i \in I} Y_i$$

$$f(x) = (f_i(x))_{i \in I}$$

is continuous if and only if each function f_i is continuous.

Prove that this property always holds if we put the product topology on $\prod_{i \in I} Y_i$, but that this property may fail if we put the box topology on $\prod_{i \in I} Y_i$. This property is the reason that the product topology is generally considered the “correct” topology on $\prod_{i \in I} Y_i$.

Hint: Consider the function $f : \mathbb{R} \rightarrow \prod_{n \in \mathbb{N}} \mathbb{R}$ given by $f(x) = (x, x, x, \dots)$.

- Let $\{X_n\}_{n \in \mathbb{N}}$ be a countable collection of metric spaces. Show that the product topology on $\prod_{n \in \mathbb{N}} X_n$ is metrizable, but the box topology is not.