1 Continuous functions on topological spaces

Definition 1.1. (Continuous functions of topological spaces.) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is called *continuous* if ...

A map $f : X \to Y$ is a *homeomorphism* of metric spaces if it is continuous, invertible, and its inverse is continuous.

In-class Exercises

1. Show that the following maps are continuous.
   (a) The constant map
   $$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$
   $$f(x) = y_0$$
   for $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ any topological spaces with $y_0 \in Y$.
   (b) The identity map $id : (X, \mathcal{T}_X) \to (X, \mathcal{T}_X)$ for $(X, \mathcal{T}_X)$ any topological space.
   (c) Any function $f : (X, \text{discrete}) \to (Y, \mathcal{T}_Y)$ for $(Y, \mathcal{T}_Y)$ any topological space.
   (d) Any function $f : (X, \mathcal{T}_X) \to (Y, \text{indiscrete})$ for $(X, \mathcal{T}_X)$ any topological space.

2. Below are two results that you proved for metric spaces. Verify that each of these results holds for abstract topological spaces. This is a good opportunity to review their proofs!
   (a) **Theorem (Equivalent definition of continuity.)** Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then a map $f : X \to Y$ is continuous if and only if for every closed set $C \subseteq Y$, the set $f^{-1}(C)$ is closed.
   (b) **Theorem (Composition of continuous functions.)** Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$, and $(Z, \mathcal{T}_Z)$ be topological spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous maps. Prove that the map $g \circ f : X \to Z$ is continuous.

3. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $S \subseteq X$. Let $f : X \to Y$ be a continuous function. Show that the restriction of f to S,
   $$f|_S : S \to Y,$$
   is continuous with respect to the subspace topology on S.
   Hint: Write $f|_S$ as the composition of f and the inclusion map $\iota_S : S \to X$.

4. An advantage of identifying a basis for a topology is that many topological statements can be reduced to statements about the basis. As an example, prove the following theorem.

   Theorem 1.2. (Equivalent definition of continuity). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{B}_Y$ be a basis for $\mathcal{T}_Y$. Prove that a map $f : X \to Y$ is continuous if and only if for every open set $U \in \mathcal{B}_Y$, the preimage $f^{-1}(U) \subseteq X$ is open.*

5. **(Optional).**

   (a) Let $f : X \to Y$ be a map of two topological spaces, and suppose that Y has the cofinite topology. Prove that f is continuous if and only if the preimage of every point in Y is closed in X.

   (b) Let $f : X \to Y$ be a map of two topological spaces that both have the cofinite topology. Prove that f is continuous if and only if either f is a constant function or the preimage of every point in Y is finite.

6. **(Optional).** Consider the following functions $f : \mathbb{R} \to \mathbb{R}$.

   (a) $f(x) = x$
   (b) $f(x) = 0$
   (c) $f(x) = x^2$
   (d) $f(x) = \cos(x)$
   (e) $f(x) = x + 1$
   (f) $f(x) = -x$

   Determine whether these functions are continuous when both the domain and codomain $\mathbb{R}$ have the topology ...

   - standard (Euclidean) topology
   - $\mathcal{T} = \{\mathbb{R}, \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, (0,1), \varnothing\}$
   - $\mathcal{T} = \{\mathbb{R}, \{0,1\}, \{0\}, \{1\}, \varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R}\}$
   - cofinite topology
   - $\mathcal{T} = \{(-\infty, a) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \in A\} \cup \{\varnothing\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 0 \notin A\} \cup \{\mathbb{R}\}$
   - $\mathcal{T} = \{A \mid A \subseteq \mathbb{R},\ 1 \in A\} \cup \{\varnothing\}$

7. **(Optional).** Let $X = \{a, b, c\}$ be the topological space with the topology

$$\{\varnothing, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}\}.$$

   Let $\mathbb{R}$ be the topological space defined by the usual Euclidean metric. Which of the following functions $f : \mathbb{R} \to X$ are continuous?

   (i) $f(x) = b$ for all $x \in \mathbb{R}$.

   (ii) $f(x) = \begin{cases} a, & x = (-\infty, 0) \\ b, & x = 0 \\ c, & x \in (0, \infty) \end{cases}$

   (iii) $f(x) = \begin{cases} a, & x = 0 \\ b, & x \in (-\infty, 0) \\ c, & x \in (0, \infty) \end{cases}$

   (iv) $f(x) = \begin{cases} a, & x \in (-\infty, 0] \\ b, & x \in (0, \infty) \end{cases}$

8. **(Optional).** Let $f : X \to Y$ be a function of topological spaces. Show that f is continuous if and only if it is continuous when restricted to be a map $f : X \to f(X)$, with $f(X) \subseteq Y$ given the subspace topology.

9. **(Optional).** Let X be a set. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. Show that the identity map

$$id_X : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$
$$id_X(x) = x$$

   is continuous with respect to the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.