

1 Metric Spaces

Definition 1.1. (Metric; Metric space.) Let X be a set. A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The value $d(x, y)$ is sometimes called the *distance from x to y* .

A set X endowed with a metric d is called a *metric space*, and is denoted (X, d) (or simply X when the metric is clear from context).

Theorem 1.2. (The Euclidean Metric). Define

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

as follows. For $\bar{x} = (x_1, \dots, x_n)$ and $\bar{y} = (y_1, \dots, y_n)$, let

$$\begin{aligned} d(\bar{x}, \bar{y}) &= \|\bar{x} - \bar{y}\| \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}. \end{aligned}$$

Then d is a metric, called the Euclidean metric, and makes $(\mathbb{R}^n, d)$ into a metric space.

Proof. We need to verify that d satisfies the three conditions that define a metric.

Step 1. Verify that d satisfies condition (M1).

Step 2. Verify that d satisfies condition (M2).

Step 3. Explain why, to verify (M3), it's enough to check that

$$(d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}))^2 \geq d(\bar{x}, \bar{z})^2$$

Hint: What is the definition of an *increasing* function?

Step 4. Expand $(d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}))^2 = (||\bar{x} - \bar{y}|| + ||\bar{y} - \bar{z}||)^2$.

Step 5. Expand

$$\begin{aligned} d(\bar{x}, \bar{z})^2 &= (\bar{x} - \bar{z}) \cdot (\bar{x} - \bar{z}) \\ &= ((\bar{x} - \bar{y}) + (\bar{y} - \bar{z})) \cdot ((\bar{x} - \bar{y}) + (\bar{y} - \bar{z})) \end{aligned}$$

Step 6. Conclude that d satisfies (M3).

□

In-class Exercises

1. Determine whether the following functions define metrics on the corresponding sets. Rigorously justify your answers!

(a) Let $X = \mathbb{R}$. Define

$$d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$d(x, y) = (x - y)^2.$$

(b) Let $X = \mathbb{R}^2$. Define

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$d(\bar{x}, \bar{y}) = |x_1 - y_1| + |x_2 - y_2|.$$

Hint: First check, what does Theorem 1.2 say in the case $n = 1$?

(c) Let X be any set. Define

$$d : X \times X \longrightarrow \mathbb{R}$$

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. Show that the restriction $d|_{Y \times Y}$ of d to $Y \times Y \subseteq X \times X$ defines a metric on Y . Conclude that any subset of a metric space inherits a metric space structure.
3. **(Optional)** Let $a < b \in \mathbb{R}$. Let $\mathcal{C}(a, b)$ denote the set of continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Verify whether each of the following functions defines a (well-defined) metric on the set $\mathcal{C}(a, b)$. Be sure to clearly state which properties of continuous functions and integration you are using!

(a) $d_1 : \mathcal{C}(a, b) \times \mathcal{C}(a, b) \longrightarrow \mathbb{R}$

$$d(f, g) = \int_a^b |f(x) - g(x)| \, dx$$

(b) $d_\infty : \mathcal{C}(a, b) \times \mathcal{C}(a, b) \longrightarrow \mathbb{R}$

$$d(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|$$

4. **(Optional)** Let (X, d) be a metric space. Which of the following functions $\tilde{d} : X \times X \rightarrow \mathbb{R}$ defines a new metric space structure on X ?

(a) For any $x, y \in X$, $\tilde{d}(x, y) = c(d(x, y))$ for $c \in \mathbb{R}, c > 0$.

(b) For any $x, y \in X$, $\tilde{d}(x, y) = (d(x, y))^2$.

(c) For any $x, y \in X$, $\tilde{d}(x, y) = \min(d(x, y), 1)$.

(d) For any $x, y \in X$, $\tilde{d}(x, y) = \max(d(x, y), 1)$.

(e) For any $x, y \in X$, $\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$.