Midterm Exam

Math 490
22 October 2024
Jenny Wilson

Name: ____

Instructions: This exam has 4 questions for a total of 25 points.

The exam is closed-book. No books, cell phones, calculators, or other devices are permitted.

Each student may bring in one double-sided standard-size (8.5 in $\times$ 11 in) sheet of notes, which they must prepare themselves. Notes may be handwritten or typed with font size at least 12.

Scratch paper is available.

Fully justify your answers unless otherwise instructed. You may quote any results proved in class, on a quiz, or on the homeworks without proof. Please include a complete statement of the result you are quoting.

You have 80 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	11	
2	5	
3	4	
4	5	
Total:	25	

1. (11 points) For each of the following: If the statement is true, write "`True`". Otherwise, state a specific counterexample. **No further justification needed.**

   If the statement is not true, you may receive partial credit for writing "`False`" without a counterexample.

   (a) Let $f : X \to Y$ be a function of sets. If the subsets $A, B \subseteq X$ are disjoint, then $f(A)$ and $f(B)$ are disjoint.

   (b) Let $f : X \to Y$ be a function of sets. If the subsets $C, D \subseteq Y$ are disjoint, then $f^{-1}(C)$ and $f^{-1}(D)$ are disjoint.

   (c) Let X be a metric space and $A \subseteq X$. If A has no accumulation points, then it is closed.

   (d) Let X be a metric space and $A \subseteq X$ a nonempty open subset. Then every point of A is an accumulation point of A.

   (e) Let A, B be subsets of a metric space X. Then $\partial(A \cap B) = \partial A \cap \partial B$.

(f) Let A be a bounded subset of a metric space X. Then $\overline{A}$ is bounded.

(g) Let X be a metric subspace and $A \subseteq X$ is subset such that every point of A is an isolated point of A. Then A is closed.

(h) Let $f : X \to Y$ is a continuous function of metric spaces. If $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, then $(f(a_n))_{n\in\mathbb{N}}$ is Cauchy in Y.

(i) Let $f : X \to Y$ be a continuous function of metric spaces. If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence in X, then $(f(a_n))_{n\in\mathbb{N}}$ is convergent in Y.

(j) Let $f : X \to Y$ be a continuous function of metric spaces. If $A \subseteq X$ is closed and bounded, so is $f(A)$.

(k) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous function of Euclidean metric spaces. If $A \subseteq \mathbb{R}^n$ is closed and bounded, so is $f(A)$.

2. (5 points) Let (X, d) be a metric space. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to a point $a_\infty \in X$. Show that the set $A = \{a_n \mid n \in \mathbb{N}\} \cup \{a_\infty\}$ is a closed subset of X.

3. (4 points) We call a function $f : W \to Z$ of metric spaces *open* if the **image** of every open subset of W is open in Z.

   Let X and Y be metric spaces. Let $X \times Y$ be the Cartesian product with the product metric, and let

$$\begin{aligned} \pi_X : X \times Y &\longrightarrow X \\ (x, y) &\longmapsto x \end{aligned}$$

   be the projection map onto X. Show that π_X is an open map.

4. (5 points) Let A be a **dense** subset of a metric space X. Show that, for any **open** subset $U \subseteq X$, there is equality
$$\overline{U \cap A} = \overline{U}.$$