Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Show that a sequence is Cauchy if and only if it satisfies the following condition: For every $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that $a_n \in B_\epsilon(a_N)$ for all $n \geq N$.

2. Let X be a set, and let d_1 and d_2 be two topologically equivalent metrics on X. In this question we verify that topologically equivalent metrics have the same convergent sequences, and the same continuous functions. Both of these conditions only depend on the open subsets of X, not on the exact metric!
   (a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Show that $(x_n)_{n\in\mathbb{N}}$ converges in (X, d_1) if and only if it converges in (X, d_2).
   (b) Let (Y, d) be a metric space, and let $f : X \to Y$ be a function. Show that $f : (X, d_1) \to (Y, d)$ is continuous if and only if $f : (X, d_2) \to (Y, d)$ is continuous.

3. Consider the space $X = \mathbb{R}$ with the Euclidean metric. Let $a < b$ and $c < d$ be real numbers.
   (a) Prove that the two open intervals (a, b) and (c, d) are homeomorphic.
   (b) Prove that (a, b) is homeomorphic to the open ray (c, ∞).
   (c) Prove that the closed intervals $[a, b]$ and $[c, d]$ are homeomorphic.
   (d) Is $[a, b]$ homeomorphic to the closed ray $[c, \infty)$?
   (e) Prove that the half-open interval $[a, b)$ is homeomorphic to the closed ray $[c, \infty)$.

4. See Assignment Problem 1 for the definition of the diameter of a set.
   (a) Show that a nonempty subset A of a metric space X has diameter 0 if and only if A is a single point.
   (b) Give an example of a metric spaces X and open balls $B_r(x) \subseteq X$ where $\operatorname{diam}(B_r(x)) = 2r$, and an example where $B_r(x) \subseteq X$ where $\operatorname{diam}(B_r(x)) < 2r$.

5. **(Review from real analysis)**
   (a) Suppose that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence of real numbers, and that for some $M \in \mathbb{R}$, $a_n \leq M$ for all n. Show that $\lim_{n\to\infty} a_n \leq M$.
   (b) Let S be a bounded set of real numbers, and L its supremum (i.e. its least upper bound). Show that there exist a sequence of elements $s_n \in S$ such that $\lim_{n\to\infty} = L$.
   (c) Conversely, let S be a bounded set of real numbers, and L its supremum. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of elements of S that converge to a real number s_∞. Show that $s_\infty \leq L$. In particular, if s_∞ is an upper bound, then $s_\infty = L$.
   (d) Let S be a bounded set of real numbers. Show that, if S is closed, it contains its supremum.

Worksheet problems

(Hand these questions in!)

- Worksheet #8 Problems 2, 3(a), (b).

Assignment questions

(Hand these questions in!)

1. **Definition (diameter).** Let (X, d) be a metric space, and let $A \subseteq X$ be a nonempty subset. The *diameter* of the set A is defined to be

$$\operatorname{diam}(A) = \sup_{a_1, a_2 \in A} d(a_1, a_2).$$

We define the diameter of the empty set $\varnothing$ to be 0.

(a) For $x \in X$ and $r > 0$, show that the diameter of ball $B_r(x)$ is at most $2r$.

(b) Verify that, if $A \subseteq B$, then $\operatorname{diam}(A) \leq \operatorname{diam}(B)$.

(c) Show that the diameter of a set A is finite if and only if A is bounded.

(d) Suppose that $A_1, \ldots, A_n$ is a finite collection of sets. Show that if each A_i has finite diameter, then so does the union $\bigcup_{i=1}^n A_i$.

(e) Prove or give a counterexample: $\operatorname{diam}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \operatorname{diam}(A_i)$.

(f) Suppose that A is sequentially compact. Show that there exists a_1, a_2 in A such that $d(a_1, a_2) = \operatorname{diam}(A)$.

(g) For any nonempty set A of a metric space X, show that $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.
Hint: You may assume the results of Warm-Up Problem 5 without proof.

(h) Prove or find a counterexample: $\operatorname{diam}(A) = \operatorname{diam}(\operatorname{Int}(A))$.

2. (a) Let X be a metric space. Suppose that for some $\epsilon > 0$, for all $x \in X$ the closure of the ball $B_\epsilon(x)$ is sequentially compact. Show that X is complete.

(b) Suppose that X is a set with the discrete metric. Deduce that X is complete.

(c) Consider the open interval $(0, 1)$ with the Euclidean metric. Show that, for every $x \in (0, 1)$, there exists some $\epsilon > 0$ so that the closure of the ball $B_\epsilon(x)$ is sequentially compact. Deduce that this condition (where the choice of ϵ is allowed to depend on x) is not strong enough to guarantee completeness.

3. (a) Prove the following theorem.

> **Theorem.** A metric space X is complete if and only if it satisfies the following condition: For every nested sequence of nonempty closed subsets $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ such that $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$, the intersection $\bigcap_n A_n$ is nonempty.

(b) Prove that, given nested sets $A_n \subseteq X$ as described in the theorem statement, the intersection $\bigcap_n A_n$ contains at most one element.

4. **Definition (Uniform Continuity).** Let $f : X \to Y$ be a function of metric spaces. Then f is called *uniformly continuous* if for all $\epsilon > 0$, there exists some $\delta > 0$ so that for all $x \in X$ there is containment $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.

Observe that this definition is stronger than the usual definition of continuity; a uniformly continuous function is always continuous. This definition differs from the usual definition, in that the choice of δ does not depend on the particular point x; we may make a single choice of δ "uniformly" over X. Differentiable functions $\mathbb{R} \to \mathbb{R}$ with bounded derivatives are examples of uniformly continuous functions.

(a) Give an example of a continuous function $\mathbb{R} \to \mathbb{R}$ that is, and an example that is not, uniformly continuous. No justification needed.

(b) Let X be a metric space with the discrete metric. Show that every function of metric spaces $f : X \to Y$ is uniformly continuous.

(c) Show by example that the image of a bounded set under a uniformly continuous function may not be bounded. *Hint:* Use part (b).

(d) Prove that the image of a Cauchy sequence under a uniformly continuous function is Cauchy.

(e) Suppose that $f : X \to Y$ is a uniformly continuous function of metric spaces. Show that, even if X is complete, its image $f(X)$ need not be complete. *Hint:* Use part (b).

(f) Suppose that X and Y are homeomorphic metric spaces via a uniformly continuous homeomorphism with a uniformly continuous inverse. Show that X is complete if and only if Y is.