Worksheet problems

No problems this week!

Assignment questions

1. (a) For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above, for an arbitrary metric space X with subsets $A, B \subseteq X$. You only need to submit the final answer ($\subseteq, \supseteq, =$, or "none of the above").

   (i) $X \setminus \overline{A} \;\square\; \mathrm{Int}(X \setminus A)$

   (ii) $X \setminus \mathrm{Int}(A) \;\square\; \overline{X \setminus A}$

   (iii) $\mathrm{Int}(A \cup B) \;\square\; \mathrm{Int}(A) \cup \mathrm{Int}(B)$

   (iv) $\mathrm{Int}(A \cap B) \;\square\; \mathrm{Int}(A) \cap \mathrm{Int}(B)$

   (v) $\overline{A \cup B} \;\square\; \overline{A} \cup \overline{B}$

   (vi) $\overline{A \cap B} \;\square\; \overline{A} \cap \overline{B}$

   (b) Give a complete proof of your solution to part (a) (v).

   (c) Give a counterexample to each instance in part (a) where equality fails.

   (d) Prove or give a counterexample: If $A \subseteq X$, then $\partial A = \overline{A} \setminus A$.

   (e) Prove or give a counterexample: If $U \subseteq X$ is open, then $U = \mathrm{Int}(\overline{U})$.

2. Let X and Y be metric spaces. Let $X \times Y$ denote their product, viewed as a metric space with the product metric. Let $A \subseteq X$ and $B \subseteq Y$.

   (a) Determine in general the relationship ($\subseteq, \supseteq, =$, or "none of the above") between $\overline{A} \times \overline{B}$ and $\overline{A \times B}$. Justify your result with proofs and/or counterexamples.

   (b) Do the same for the relationship between $\mathrm{Int}(A) \times \mathrm{Int}(B)$ and $\mathrm{Int}(A \times B)$.

3. Let (X, d) be a metric space. Define the *distance* between nonempty subsets A, B in X by

$$D(A, B) = \inf_{\substack{a \in A \\ b \in B}} d(a, b).$$

   (a) Suppose that A and B are nonempty, sequentially compact subsets of X. Prove that the distance in realized, in the sense that there exists some $a_0 \in A$ and $b_0 \in B$ with $D(A, B) = d(a_0, b_0)$. *Hint:* Homework #5 Problem 2(b).

   (b) Show by example that the distance may not be realized for general nonempty subsets A and B, even if they are closed.

4. **Definition (Equivalent Metrics).** Let X be a set. Two metrics d_1 and d_2 on X are called *equivalent* (or *topologically equivalent*) if they satisfy the following relationship: A subset $U \subseteq X$ is open in the metric space (X, d_1) if and only if it is open in the metric space (X, d_2).

   (Later in the course, we will be able to describe this condition by saying that the two metrics *induce the same topology on* X).

   (a) Prove that the metrics d_1 and d_2 are equivalent if and only if the identity functions

$$\begin{aligned} (X, d_1) &\longrightarrow (X, d_2) \\ x &\longmapsto x \end{aligned} \qquad\qquad \begin{aligned} (X, d_2) &\longrightarrow (X, d_1) \\ x &\longmapsto x \end{aligned}$$

   are both continuous.

(b) Let X be a nonempty finite set. Show that all metrics on X are topologically equivalent.

(c) Consider the natural numbers $\mathbb{N}$. Show that ($\mathbb{N}$, Euclidean) and ($\mathbb{N}$, discrete) are topologically equivalent. Conclude that a bounded metric can be topologically equivalent to an unbounded metric.

(d) Give an example of a set X and two metrics on X that are not topologically equivalent.

(e) For a metric d on the set X, use the notation $B_r^d(x)$ to denote the open ball of radius r about a point x. Show that the metrics d_1 and d_2 are equivalent if and only if open balls "nest" in the following sense: for any point $x \in X$ and radius $r > 0$, there exists $r_1 > 0$ such that

$$B_{r_1}^{d_1}(x) \subseteq B_r^{d_2}(x)$$

and there exists $r_2 > 0$ such that

$$B_{r_2}^{d_2}(x) \subseteq B_r^{d_1}(x).$$

(f) Use your solution to Homework #1 Problem 2 to give an informal argument that the following metrics are $\mathbb{R}^2$ are all equivalent.

$$\begin{aligned} d_1((x_1, y_1), (x_2, y_2)) &= |x_1 - x_2| + |y_1 - y_2| \\ d_2((x_1, y_1), (x_2, y_2)) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ d_\infty((x_1, y_1), (x_2, y_2)) &= \max\Big(|x_1 - x_2|, |y_1 - y_2|\Big) \end{aligned}$$

(In fact, the analogous metrics on $\mathbb{R}^n$ are all equivalent).