Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Consider the sequence of real numbers $1, 2, 3, 4, 5, \ldots$. Which of the following are subsequences?

   (a) $1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \ldots$

   (b) $1, 1, 1, 1, 1, 1, 1, \ldots$

   (c) $2, 4, 6, 8, 10, 12, \ldots$

   (d) $1, 3, 2, 4, 5, 7, 6, 8, 9, 11, \ldots$

2. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Recall that we proved that, if $\lim\limits_{n\to\infty} a_n = a_\infty$, then any subsequence of $(a_n)_{n\in\mathbb{N}}$ also converges to a_∞.

   (a) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence that does not converge. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.

   (b) Suppose that $(a_n)_{n\in\mathbb{N}}$ has a subsequence converging to $a \in X$, and a different subsequence converging to $b \in X$, with $a \neq b$. Prove that $(a_n)_{n\in\mathbb{N}}$ does not converge.

3. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Show that, if $\{a_n \mid n \in \mathbb{N}\}$ is a finite set, then $(a_n)_{n\in\mathbb{N}}$ must have a subsequence that is constant (and, in particular, convergent).

4. Let (X, d) be a metric space, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Suppose that the set $\{a_n \mid n \in \mathbb{N}\}$ is unbounded. Explain why $(a_n)_{n\in\mathbb{N}}$ cannot converge.

5. Find examples of sequences $(a_n)_{n\in\mathbb{N}}$ of real numbers with the following properties.

   (a) $\{a_n \mid n \in \mathbb{N}\}$ is unbounded, but $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence

   (b) $(a_n)_{n\in\mathbb{N}}$ has no convergent subsequences

   (c) $(a_n)_{n\in\mathbb{N}}$ is not an increasing sequence, but it has an increasing subsequence

   (d) $(a_n)_{n\in\mathbb{N}}$ has four subsequences that each converge to a distinct limit point

6. We defined *bounded* as follows.

   > **Definition (Bounded subset.)** Let (X, d) be a metric space. A subset $S \subseteq X$ is called *bounded* if there is some $x_0 \in X$ and some $R \in \mathbb{R}$ with $R > 0$ such that $S \subseteq B_R(x_0)$.

   Show this is equivalent to the following definition.

   > **Definition (Bounded subset.)** Let (X, d) be a metric space, and $S \subseteq X$ a subset. Then S is *bounded*, for **every** $x \in S$, there is some $R_x > 0$ such that $S \subseteq B_{R_x}(x)$.

7. (a) Negate the definition of *bounded* to state what it means for a subset S of a metric space to be *unbounded*.

   (b) Is $\varnothing$ a bounded set?

   (c) Show that any **finite** subset of a metric space is bounded.

8. Let (X, d) be a metric space and let $S \subseteq X$ be a bounded set. Show that any subset of S is bounded.

9. Let (X, d_X) and (Y, d_Y) be bounded metric spaces. Show that $(X \times Y, d_{X\times Y})$ is bounded. What if we only assumed X is bounded?

10. Give examples of subsets of $\mathbb{R}$ (with the Euclidean metric) that satsify the following.

    (a) open, and bounded

    (b) closed, and bounded

    (c) open, and unbounded

    (d) closed, and unbounded

11. Write down a bounded sequence in $\mathbb{R}$. Can you identify a convergent subsequence?

12. Give an example of a metric space (X, d) and a continuous function $f : X \to \mathbb{R}$ such that f has a finite supremum on X, but f does not achieve its supremum at any point $x \in X$.

Worksheet problems

(Hand these questions in!)

- Worksheet #6, Problem 2, 3(a), 3(b), 4.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a continuous function of metric spaces. For each of the following statements, either prove the statement, or state a counterexample. You may state the counterexample without proof.

   (a) If $U \subseteq X$ is open, then $f(U) \subseteq Y$ is open.

   (b) If $C \subseteq X$ is closed, then $f(C) \subseteq Y$ is closed.

   (c) If $C \subseteq X$ is sequentially compact, then $f(C) \subseteq Y$ is sequentially compact.
   Hint: This result could be useful on for the next two assignment problems!

   (d) If $D \subseteq Y$ is sequentially compact, then $f^{-1}(D)$ is sequentially compact.

   (e) If $B \subseteq X$ is bounded, then $f(B) \subseteq Y$ is bounded.

   (f) If $D \subseteq Y$ is bounded, then $f^{-1}(D) \subseteq X$ is bounded.

   Hint: Below are some continuous functions that are useful sources of counterexamples. Subsets of $\mathbb{R}$ have the Euclidean metric unless otherwise noted.

$$(\mathbb{R}, \text{discrete metric}) \longrightarrow (\mathbb{R}, \text{Euclidean}) \qquad x \longmapsto x$$

$$\mathbb{R} \longrightarrow (-1, 1) \qquad x \longmapsto \frac{\arctan(x)}{\pi/2}$$

$$(0, \infty) \longrightarrow (0, \infty) \qquad x \longmapsto \frac{1}{x}$$

$$\mathbb{R} \longrightarrow \mathbb{R} \qquad x \longmapsto 0$$

2. (a) Suppose that B is a nonempty closed and bounded subset of $\mathbb{R}$. Explain why the Least Upper Bound Property of $\mathbb{R}$ implies that the supremum of B exists, and show that B contains its supremum.

   (b) Prove the following theorem. This theorem is one of the important reasons we care about sequential compactness!

   Theorem (Extreme value theorem for metric spaces). Let (X, d) be a metric space and C a nonempty, sequentially compact subset of X. Let $f : X \to \mathbb{R}$ be a continuous function. Then there is a point $c \in C$ so that

   $$f(c) = \sup_{x \in C} f(x).$$

   In other words, the restriction $f|_C$ achieves its supremum.

3. Let X and Y be (nonempty) metric spaces. Let $X \times Y$ be their Cartesian product endowed with the product metric. Let $S \subseteq X$ be a subset, viewed as a metric space itself under the restriction of the metric on X.

   For each of the following statements: prove the statement, or give a counterexample. You may state the counterexample without proof.

   (a) If X is a finite set, then X is sequentially compact.

   (b) If X and Y are sequentially compact, then so is $X \times Y$.

   (c) If $X \times Y$ is sequentially compact, then so are X and Y.

   (d) If X is sequentially compact, then so is S.

   (e) If $A, B \subseteq X$ are sequentially compact, then so is $A \cup B$.

   (f) If some family of subsets $A_i \subseteq X, (i \in I)$, are all sequentially compact, then so is $\bigcup_{i \in I} A_i$.

   (g) If $A, B \subseteq X$ are sequentially compact, then so is $A \cap B$.

4. (a) **Definition (Open cover).** A collection $\{U_i\}_{i \in I}$ of open subsets of a metric space X is an *open cover* of X if $X = \bigcup_{i \in I} U_i$. In other words, every point in X lies in some set U_i.

   Suppose that (X, d) is a sequentially compact metric space. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Prove that (associated to the open cover $\mathcal{U}$) there exists a real number $\delta > 0$ with the following property: for every $x \in X$, there is some associated index $i_x \in I$ such that $B_\delta(x) \subseteq U_{i_x}$.

   (b) **Definition (ϵ-nets of a metric space).** Let (X, d) be a metric space. A subset $A \subseteq X$ is called an ϵ-net if $\{B_\epsilon(a) \mid a \in A\}$ is an open cover of X.

   Suppose that (X, d) is a sequentially compact metric space, and $\epsilon > 0$. Prove that X has a finite ϵ-net.

   (c) Let (X, d) be a sequentially compact metric space, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. Show that there exists some finite collection $U_{i_1}, \ldots, U_{i_n} \in \mathcal{U}$ so that $\{U_{i_1}, \ldots, U_{i_n}\}$ covers X, i.e., so that $X = U_{i_1} \cup \cdots \cup U_{i_n}$.

   We will return to these results later in the course when we study *compactness*.