Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Determine which of the following subsets of of $\mathbb{R}^2$ can be expressed as the Cartesian product of two subsets of $\mathbb{R}$.

   (a) $\{(x,y) \mid x \in \mathbb{Q}\}$
   (b) $\{(x,y) \mid x,y \in \mathbb{Q}\}$
   (c) $\{(x,y) \mid x > y\}$
   (d) $\{(x,y) \mid 0 < y \leq 1\}$
   (e) $\{(x,y) \mid x^2+y^2 < 1\}$
   (f) $\{(x,y) \mid x^2+y^2 = 1\}$
   (g) $\{(x,y) \mid x = 3\}$
   (h) $\{(x,y) \mid x+y = 3\}$

2. (a) For which values of r is the square $(-r,r) \times (-r,r) \subseteq \mathbb{R}^2$ contained in the unit ball $\{(x,y) \mid x^2+y^2 < 1\}$?
   (b) For which values of r is the r-ball $\{(x,y) \mid x^2+y^2 < r^2\}$ contained in the square $(-1,1) \times (-1,1) \subseteq \mathbb{R}^2$?

3. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. Find examples of subsets A of $\mathbb{R}$ with the following properties.
   (a) $\text{Int}(A) = \varnothing$
   (b) $\text{Int}(A) = \mathbb{R}$
   (c) $\text{Int}(A) = A$
   (d) $\text{Int}(A)$ is strictly contained in A.
   (e) $\overline{A} = \varnothing$
   (f) $\overline{A} = \mathbb{R}$
   (g) $\overline{A} = A$
   (h) A is strictly contained in $\overline{A}$.
   (i) $\partial(A) = \varnothing$
   (j) A has a nonempty boundary, and A contains its boundary ∂A.
   (k) A has a nonempty boundary, and A contains no points in its boundary
   (l) A has a nonempty boundary, and A contains some but not all of the points in its boundary.
   (m) A has a nonempty boundary, and $A = \partial A$.
   (n) A is a **proper** subset of ∂A.
   (o) $\partial(A) = \mathbb{R}$

Worksheet problems

(Hand these questions in!)

- Worksheet # 5 Problem 2

Assignment questions

(Hand these questions in!)

1. Consider the following definition.

   Definition (Boundary of a set A.) Let (X, d) be a metric space, and let $A \subseteq X$. Then the *boundary* of A, denoted ∂A, is the set $\overline{A} \setminus \mathrm{Int}(A)$.

   Let (X, d) be a metric space, and let $A \subseteq X$.

   (a) Prove that $\mathrm{Int}(A) = \overline{A} \setminus \partial A$.

   (b) Prove that $\partial A = \overline{A} \cap (\overline{X \setminus A})$.

   (c) Conclude from part (b) that ∂A is closed.

   (d) Additionally conclude from part (b) that $\partial A = \partial(X \setminus A)$.

   (e) Prove the following characteriziation of points in the boundary:

   Theorem (An equivalent definition of ∂A). Let (X, d) be a metric space, and let $A \subseteq X$. Then $x \in \partial A$ if and only if every ball $B_r(x)$ about x contains at least one point of A, and at least one point of $X \setminus A$.

   (f) Deduce that we can classify every point of X in one of three mutually exclusive categories:

   (i) interior points of A;

   (ii) interior points of $X \setminus A$;

   (iii) points in the (common) boundary of A and $X \setminus A$.

2. Consider the real numbers $\mathbb{R}$ with the Euclidean metric. For each of the following subsets $A \subseteq \mathbb{R}$, find (with brief justification) the interior $\mathrm{Int}(A)$, closure $\overline{A}$, boundary ∂A, and set of accumulation points A' of A.

   (a) A is the whole real number line $\mathbb{R}$

   (b) A is the half-open interval $(0, 1]$.

   (c) A is the integers $\mathbb{Z}$.

   (d) A is the set $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$.

   (e) A is the set of rational numbers $\mathbb{Q}$.

3. **Definition (Dense subset).** Let (X, d) be a metric space and $A \subseteq X$ a subset. Then A is called *dense* in X if it satisfies the following equivalent conditions.

   (i) $\overline{A} = X$.

   (ii) Every point of X is either contained in A, or is an accumulation point of A.

   (iii) For any x in X, there is a sequence of points in A converging to x.

   (iv) The complement $X \setminus A$ contains no nonempty open subsets of X.

   (v) A intersects every nonempty open subset U of X (this means $U \cap A \neq \varnothing$).

   (a) Prove that the conditions are, in fact, equivalent.

   (b) Let $f : X \to Y$ be a continuous function of metric spaces. If $A \subseteq X$ is a dense subset, show that its image $f(A)$ is dense in $f(X)$.

(c) Show by example that the preimage of a dense set of Y under a continuous map need not be dense. *Hint*: Do Problem 4 first.

4. **Definition (The discrete metric).** Given a set X, the *discrete metric on* X is the metric $d : X \times X \to \mathbb{R}$ defined by

$$d(x, x') = \begin{cases} 0, & x = x' \\ 1, & x \neq x' \end{cases} \qquad \text{for all } x, x' \in X.$$

We verified on Worksheet #1 that d is in fact a metric. Let (X, d) be a metric space with the discrete metric.

(a) Show that every subset of X is both open and closed.

(b) Let (Y, d_Y) be any metric space. Prove that **every** function $f : X \to Y$ is continuous.

(c) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Give necessary and sufficient conditions (with proof) for this sequence to be convergent.

(d) Show that for every subset A of X, $\mathrm{Int}(A) = A = \overline{A}$ and $\partial A = \varnothing$.

(e) Show that every subset A of X has no accumulation points.

(f) What subsets of X are dense in X?