Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in a metric space X, and let $a_\infty \in X$. Show that the following statement is equivalent to the statement that $\lim_{n\to\infty} a_n = a_\infty$.

   For every $\epsilon > 0$, the ball $B_\epsilon(a_\infty)$ about a_∞ contains a_n for all but finitely many n.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in a metric space X.
   (a) Let $x \in X$. Negate the definition of convergence to state what it means for the sequence to **not** converge to x.
   (b) Formally state what it means for the sequence $(a_n)_{n\in\mathbb{N}}$ to be non-convergent.

3. Rigorously determine the limits of the following sequences of real numbers, or prove that they do not converge.

   (a) $a_n = 0$ (b) $a_n = \dfrac{1}{n^2}$ (c) $a_n = n$ (d) $a_n = (-1)^n$

4. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers that converge to a_∞ and b_∞, respectively. Prove that the sequence $(a_n + b_n)_{n\in\mathbb{N}}$ converges to $(a_\infty + b_\infty)$.

5. Consider the sequence $\left(\dfrac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Let $\epsilon > 0$ be fixed. Find a number $N \in \mathbb{R}$ so that, for all $m, n \geq N$,

$$\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| < \epsilon.$$

   This shows that the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ is *Cauchy*.

Worksheet Problems

(Hand these questions in!)

- Worksheet 4 Problem 2(b), 3

Assignment questions

(Hand these questions in!)

0. September 17 is National Voter Registration Day!

   If you're eligible to vote this November (or want to find out if you're eligible to vote), you can visit the nonpartisan site

   `https://www.govote.umich.edu/`

   to find out your options for how to register—or how to confirm you're already registered—and how to cast your ballot.

   Whether or not you're eligible to vote, encourage 3 of your eligible friends to register and vote this year!

1. Let $f : X \to Y$ be a function of sets X and Y. Let $A, B \subseteq X$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f(A \cap B) \quad \square \quad f(A) \cap f(B)$ (b) $f(A \cup B) \quad \square \quad f(A) \cup f(B)$

   (c) For $A \subseteq B$, $\quad f(B \setminus A) \quad \square \quad f(B) \setminus f(A)$

2. In this question, we will prove the following result.

   Theorem (Another characterization of closed subsets). Let (X, d) be a metric space, and let $A \subseteq X$. Then A is closed if and only if it satisfies the following condition: If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence of points in A converging to a point $a_\infty \in X$, then the limit a_∞ is contained in A.

   (a) Suppose that $A \subseteq X$ is closed. Let a_∞ be the limit of a convergent sequence $(a_n)_{n\in\mathbb{N}}$ of points in A. Show that $a_\infty \in A$.

   (b) Suppose that $A \subseteq X$ is a subset that contains the limits of every one of its convergent sequences. Prove that A is closed.

3. Prove the following result:

   Theorem (Another definition of continuous functions.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if the following condition holds: given any convergent sequence $(a_n)_{n\in\mathbb{N}}$ in X, then $\big(f(a_n)\big)_{n\in\mathbb{N}}$ converges in Y, and

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right).$$

4. For sets X and Y, let $A, B \subseteq X$ and $C, D \subseteq Y$. Consider the Cartesian product $X \times Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $(A \times C) \cup (B \times D) \quad \square \quad (A \cup B) \times (C \cup D)$

   (b) $(A \times C) \cap (B \times D) \quad \square \quad (A \cap B) \times (C \cap D)$

   (c) $(X \setminus A) \times (Y \setminus C) \quad \square \quad (X \times Y) \setminus (A \times C)$

5. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that $C \subseteq X$ and $D \subseteq Y$ are closed subsets. Prove or find a counterexample: the subset $C \times D \subseteq X \times Y$ is closed with respect to the product metric.

6. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that x_0 is an accumulation point of a subset $S \subseteq X$, and that y_0 is an accumulation point of a subset $T \subseteq Y$. Prove that (x_0, y_0) is an accumulation point of $S \times T$, viewed as a subset of $X \times Y$ with the product metric.