Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. (a) Give an example of a metric space and a subset that is both open and closed. Give an example of a subset that is neither open nor closed.

   (b) Recite the Topologist Scout Oath:

   "On my honour, I will do my best
   to never claim to prove a set is closed by showing that it is not open,
   and to never claim to prove a set is open by showing that it is not closed."

2. Let X be a metric space. Let U, V be subsets of X such that $U \subseteq V$. Let $x \in U$.

   (a) Suppose that x is an interior point of U. Show that x is an interior point of V.

   (b) Suppose that x is an interior point of V. Must it be an interior point of U? Give a proof or a counterexample.

3. Rigorously prove that the following functions $f : \mathbb{R} \to \mathbb{R}$ are continuous. (Here, $\mathbb{R}$ implicitly has the Euclidean metric.)

   (a) $f(x) = 5$ (b) $f(x) = 2x + 3$ (c) $f(x) = x^2$

   (d) $f(x) = g(x) + h(x)$, for continuous functions g and h.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 + 2$. Find the preimages of the following sets, and verify that they are open.

   (a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $(2, 3)$ (d) $(6, \infty)$

5. Let (X, d) be a metric spaces. Show that the identity function

$$\begin{aligned} g : X &\longrightarrow X \\ g(x) &= x \qquad \text{for all } x \in X \end{aligned}$$

   is always continuous.

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $y_0 \in Y$. Show that the constant function

$$\begin{aligned} f : X &\longrightarrow Y \\ f(x) &= y_0 \qquad \text{for all } x \in X \end{aligned}$$

   is always continuous.

Worksheet problems

(Hand these questions in!)

- Worksheet # 3 Problem 3

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function of sets X and Y. Let $C, D \subseteq Y$. For each of the following, determine whether you can replace the symbol $\square$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

   (a) $f^{-1}(C \cup D) \quad \square \quad f^{-1}(C) \cup f^{-1}(D)$ (b) $f^{-1}(C \cap D) \quad \square \quad f^{-1}(C) \cap f^{-1}(D)$

   (c) For $C \subseteq D$, $\quad f^{-1}(D \setminus C) \quad \square \quad f^{-1}(D) \setminus f^{-1}(C)$

2. Let $f : X \to Y$ be a function between metric spaces. We proved that a subset $S \subseteq X$ inherits a metric space structure from the metric on X. Recall that the *restriction* of f to S, often written $f|_S$, is the function

$$f|_S : S \longrightarrow Y$$
$$f|_S(s) = f(s).$$

   (a) Suppose that $U \subseteq X$ is an open subset of X. Verify that $U \cap S$ is an open subset of S.

   (b) Let $B \subseteq Y$ be a subset. Show that $(f|_S)^{-1}(B) = f^{-1}(B) \cap S$.

   (c) Prove the following result.

      Theorem. Let X and Y be metric spaces, and $S \subseteq X$ a metric subspace. If $f : X \to Y$ is a continuous function, then $f|_S : S \to Y$ is continuous.

3. Prove the following theorem.

   Theorem (Equivalent definition of continuity.) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous if and only if it satisfies the following property: for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed.

4. **Definition (Product metric).** Let (X, d_X) and (Y, d_Y) be metric spaces. Then their Cartesian product $X \times Y$ has a metric space structure, defined by the metric

$$d_{X\times Y} : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$$
$$d_{X\times Y}\Big((x_1, y_1), (x_2, y_2)\Big) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}.$$

   We will call $d_{X\times Y}$ the *product metric* on $X \times Y$.

   We remark that when $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$ are Euclidean spaces with the Euclidean metric, then the product metric on $X \times Y \cong \mathbb{R}^{m+n}$ is the usual Euclidean metric.

   (a) Verify that $d_{X\times Y}$ does in fact define a metric on $X \times Y$.

   (b) Prove that if $U \subseteq X$ and $V \subseteq Y$ are open sets, then $U \times V$ is an open subset of $X \times Y$.

   (c) Let $U \subseteq X \times Y$ be an open set, and let $(x, y) \in U$. Show that there is a neighbourhood U_x of x in X and a neighbourhood U_y of y in Y so that $U_x \times U_y \subseteq U$.

(d) **Definition (Projection maps).** For a product of sets $X \times Y$, the maps

$$\begin{aligned} \pi_X &: X \times Y \to X & \qquad \pi_Y &: X \times Y \to Y \\ \pi_X(x, y) &= x & \qquad \pi_Y(x, y) &= y \end{aligned}$$

are called the *projection onto* X and the *projection onto* Y, respectively.

Let (X, d_X) and (Y, d_Y) be metric spaces, and endow their product $X \times Y$ with the product metric $d_{X\times Y}$. Show that the projection map

$$\pi_X : (X \times Y, d_{X\times Y}) \longrightarrow (X, d_X)$$

is continuous. (The same argument, which you do not need to repeat, shows that the map π_Y is continuous).

5. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces. View the product $X \times Y$ as a metric space with the product metric $d_{X\times Y}$ (defined in Problem 4). Suppose that $f_X : Z \to X$ and $f_Y : Z \to Y$ are functions. Show that the function

$$\begin{aligned} F : Z &\longrightarrow X \times Y \\ z &\longmapsto (f_X(z), f_Y(z)) \end{aligned}$$

is continuous if and only if $f_X : Z \to X$ and $f_Y : Z \to Y$ are continuous.