## Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
   (a) Let $(X, \mathcal{T})$ be a topological space. Explain why the condition that $X$ is compact is stronger than the assumption that $X$ has a finite open cover.
   (b) Show that every topological space has a finite open cover.
   *Hint:* What is the first axiom of a topology?

2. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Prove that the two following definitions of compactness are equivalent.
   - The subset $A$ is *compact* if it is a compact topological space with respect to the subspace topology $\mathcal{T}_A$.
   - The subset $A$ is *compact* if it satisfies the following property: for any collection of open subsets $\{U_i\}_{i \in I}$ of $X$ such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite subscollection $U_1, U_2, \ldots, U_n$ such that $A \subseteq \bigcup_{i=1}^n U_i$.

3. Give an example of a subsets $A \subseteq B$ of $\mathbb{R}$ such that ...
   (a) $A$ is compact, and $B$ is noncompact
   (b) $B$ is compact, and $A$ is noncompact

4. Determine the connected components of $\mathbb{R}$ with the following topologies.
   (a) the topology induced by the Euclidean metric
   (b) the discrete topology
   (c) the indiscrete topology
   (d) the cofinite topology

## Worksheet problems

(Hand these questions in!)

- Worksheet 16, Problems 1, 2.
- Worksheet 17, Problems 2, 3, 4.

## Assignment questions

(Hand these questions in!)

1. **Definition (Connected components of a topological space).** Let $(X, \mathcal{T}_X)$ be a topological space. A subset $C \subseteq X$ is called a *connected component* of $X$ if
   (i) $C$ is connected;
   (ii) if $C$ is contained in a connected subset $A$, then $C = A$.

In other words, the connected components are the 'maximal' connected subsets of $X$.

(a) Show that any connected component of $X$ is closed. *Hint:* Homework #12 Problem 3.

(b) Let $x \in X$. Show that the set

$$\bigcup_{\substack{A \text{ is a connected set,} \\ x \in A}} A$$

is a connected component of $X$.

(c) Show that (as a set) $X$ is the **disjoint union** of its connected components. In other words, show that every point of $X$ is contained in one, and only one, connected component.

(d) Determine the connected components of $\mathbb{Q}$ (with the Euclidean metric).
(Remember to rigorously justify your answer!)

(e) Deduce from the example of $\mathbb{Q}$ that connected components need not be open.

(f) Suppose that $X$ has the property that every point has a connected neighbourhood. Show that the connected components of $X$ are open.

2. In this problem we will prove the theorem,

> **Theorem (Products of compact spaces).** Let $X$ and $Y$ be nonempty topological spaces. Then $X \times Y$ is compact with respect to the product topology if and only if both $X$ and $Y$ are compact.

Let $X$ and $Y$ be nonempty compact topological spaces. Let $\mathcal{U}$ be any open cover of $X \times Y$ (with the product topology).
For this exercise, we will call a subset $A \subseteq X$ *good* if $A \times Y$ is covered by a finite sub-collection of open sets in $\mathcal{U}$. Our goal is to show that $X$ is good.

(a) Suppose that $A_1, \ldots, A_r$ is a finite collection of good subsets of $X$. Show that their union is good.

(b) Fix $x \in X$. For each $y \in Y$, explain why it is possible to find open sets $U_y \in X$ and $V_y \in Y$ so that $(x, y) \in U_y \times V_y$ and $U_y \times V_y$ is contained in some open set in $\mathcal{U}$.

(c) Explain why there is a finite list of points $y_1, \ldots, y_n \in Y$ so that the sets $\{V_{y_1}, \ldots, V_{y_n}\}$ cover $Y$.

(d) Define

$$U_x = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}.$$

Show that $U_x$ is a good set, and is an open subset of $X$ containing $x$. This shows that every element $x \in X$ is contained in a good open set $U_x$.

(e) Consider the collection of open subsets $\{U_x \mid x \in X\}$ of $X$. Use the fact that $X$ is compact to conclude that $X$ is good.

(f) Now, let $X$ and $Y$ be two nonempty topological spaces. Suppose that their Cartesian product $X \times Y$ is compact with respect to the product topology. Prove that $X$ and $Y$ are compact. *Hint:* Consider the projection maps $\pi_X, \pi_Y$.

3. Prove the following result. This theorem is a major reason we care about compactness!

> **Theorem (Generalized Extreme Value Theorem).** Let $X$ be a nonempty compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). Then $\sup(f(X)) < \infty$, and there exists some $z \in X$ such that $f(z) = \sup(f(X))$. That is, $f$ achieves its supremum on $X$.

*Hint*: See Worksheet #17 Problems 2, 4(a), 4(b), and Homework #5 Problem 2(a).

4. (a) Let $(X, d)$ be a metric space. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in $X$ that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.

   (b) Prove that any compact metric space is sequentially compact.

Combined with Homework #5 Problem 4, this exercise proves:

> **Theorem (Compactness vs sequential compactness in metric spaces).** Let $(X, d)$ be a metric space. Then $X$ is compact if and only if $X$ is sequentially compact.

(Neither direction of this theorem holds, however, for arbitrary topological spaces!)

Combined with Worksheet #6 Problem 4, this exercise proves:

> **Theorem (Compactness in $\mathbb{R}^n$).** Endow $\mathbb{R}^n$ with the Euclidean metric. A subspace $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.