Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let X and Y be topological spaces. Let $X \times Y$ be their product (with the product topology) and $\pi_X : X \times Y \to X$ the projection map. Prove that π_X is an open map.

2. Let $(X, \mathcal{T})$ be a topological space. Show that any subset $A = \{x\} \subseteq X$ of a single element is connected.

3. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

   Is X connected?

4. (a) Show that, for $a, b \in \mathbb{R}$, the subsets $\varnothing, \{a\}, (a, b), (a, b], [a, b), [a, b], (a, \infty), [a, \infty), (\infty, b), (\infty, b]$, and $\mathbb{R}$ of $\mathbb{R}$ are all intervals in the sense of Problem a.

   (b) Show that every interval must have one of these forms.

5. Give an example of a subset A of $\mathbb{R}$ (with the standard topology) such that A is not connected, but $\overline{A}$ is connected. (Compare to Assignment Problem 3)

6. Let X be a disconnected topological space. Let $f : X \to Y$ be a continuous function from X to a topological space Y. Show by example that $f(X)$ may be disconnected, or may be connected.

Worksheet problems

(Hand these questions in!)

- Worksheet #14 Problems 1, 3.
- Worksheet #15 Problems 3(a), 4.

Assignment questions

(Hand these questions in!)

1. Let A be a subset of a topological space $(X, \mathcal{T})$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of points in A that converge to a point $a_\infty \in X$. Prove that $a_\infty \in \overline{A}$.

2. Consider $\{0, 1\}$ as a topological space with the discrete topology. Show that a topological space $(X, \mathcal{T})$ is disconnected if and only if there is a continuous **surjective** function $X \to \{0, 1\}$.

3. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A \subseteq X$ be a connected subset. Let B be any subset such that $A \subseteq B \subseteq \overline{A}$. Prove that B is connected.
   Remark: This shows in particular that if A is connected, then so is $\overline{A}$.

4. In this problem, we will prove the following result:

> **Theorem (Connectivity of product spaces).** Let X and Y be nonempty topological spaces. Then the product space $X \times Y$ (with the product topology) is connected if and only if both X and Y are connected.

Hint: See Worksheet #15, Lemma 1.7.

(a) Suppose that $X \times Y$ is nonempty and connected in the product topology $\mathcal{T}_{X\times Y}$. Prove that X and Y are connected.

(b) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are nonempty, connected spaces, and suppose that $(a, b) \in X \times Y$. Prove that $\Big(X \times \{b\}\Big) \cup \Big(\{a\} \times Y\Big)$ is a connected subset of the product $X \times Y$ with the product topology $\mathcal{T}_{X\times Y}$.

(c) Suppose that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are nonempty, connected spaces. Prove that $X \times Y$ is connected in the product topology $\mathcal{T}_{X\times Y}$.

5. (a) Recall the Intermediate Value Theorem from real analysis (which you may use without proof).

   > **Intermediate Value Theorem.** If $f : [a, b] \to \mathbb{R}$ is continuous and d lies between $f(a)$ and $f(b)$ (i.e. either $f(a) \leq d \leq f(b)$ or $f(b) \leq d \leq f(a)$), then there exists $c \in [a, b]$ such that $f(c) = d$.

   Define a subset $A \subseteq \mathbb{R}$ to be an *interval* if whenever $x, y \in A$ with $x < y$, and $x < z < y$ for some $z \in \mathbb{R}$, then $z \in A$. Prove that any interval of $\mathbb{R}$ is connected. *Hint:* Problem 2.

   (b) Prove that any subset of $\mathbb{R}$ that is not an interval is disconnected.

These last two results together prove:

> **Theorem (Connected subsets of $\mathbb{R}$).** A subset of $\mathbb{R}$ is a connected if and only if it is an interval.

6. (a) Prove the following result. *Hint*: See Worksheet #15 Problem 4.

   > **Theorem (Generalized Intermediate Value Theorem).** Let $(X, \mathcal{T}_X)$ be a connected topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where the topology on $\mathbb{R}$ is induced by the Euclidean metric). If $x, y \in X$ and c lies between $f(x)$ and $f(y)$, then there exists $z \in X$ such that $f(z) = c$.

   (b) Prove that any continuous function $f : [0, 1] \to [0, 1]$ has a fixed point. (In other words, show that there is some $x \in [0, 1]$ so that $f(x) = x$).
   Hint: Consider the function

$$
\begin{aligned}
g &: [0,1] \to \mathbb{R} \\
g(x) &= f(x) - x.
\end{aligned}
$$