Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $X = \{a, b, c, d\}$ with the topology $\mathcal{T} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$.
   (a) Which elements of X are limits of the constant sequence $x_n = d$? The constant sequence $x_n = a$? The constant sequence $x_n = b$?
   (b) Give an example of a sequence in X that does not converge.
2. Let $X = \{0, 1\}$. Find a topology on X for which the following sequence converges: $0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \cdots$
3. Suppose that $(X, \mathcal{T})$ is a topological space, and that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that converges to $a_\infty \in X$. Prove that any subsequence of $(a_n)_{n\in\mathbb{N}}$ converges to $a_\infty \in X$.
4. Let X be a topological space with the indiscrete topology. Prove that any sequence of points in X converges to every point in X.

Worksheet problems

(Hand these questions in!)

- Worksheet #12 Problem 4.
- Worksheet #13 Problems 2, 3.

Assignment questions

(Hand these questions in!)

1. Prove the following result.

   Lemma (The pasting lemma). Let $(X, \mathcal{T}_X)$ be a topological space, and suppose $X = A \cup B$ for two **closed** subsets $A, B \subseteq X$. Let $(Y, \mathcal{T}_Y)$ be a topological space and $f : X \to Y$ a function. If $f|_A$ and $f|_B$ are continuous (with respect to the subspace topologies on A and B), then f is continuous.

2. **Definition (path).** Let X be a topological space, and let $I = [0, 1] \subseteq \mathbb{R}$ be the unit interval with the standard topology. A *path* in X is a continuous function $\gamma : I \to X$. For points $x, y \in X$, we say that a path γ is a *path from x to y* if $\gamma(0) = x$ and $\gamma(1) = y$.

   Note that the path γ does not need to be injective. For example, the constant path $\gamma(t) = x$ is a path from x to x.
   For these problems—as always—you may assume any standard results about which functions between Euclidean spaces, with the standard topology, are continuous.
   (a) Given points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, construct a path from x to y.
   (b) Let x, y be points in the space $\mathbb{R}$ with the cofinite topology. Construct (with justification) a path from x to y.

(c) Let $X = \{a, b, c, d\}$ be a topological space with the topology

$$\mathcal{T} = \Big\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\Big\}.$$

Construct (with justification) a path in X from a to d.

3. Let X be a topological space.

(a) Let $x, y \in X$ and suppose that there exists a path from x to y. Show that there exists a path from y to x.

(b) Let $x, y, z \in X$. Show that, if there exists a path from x to y, and a path from y to z, then there exists a path from x to z. *Hint:* Assignment Problem 1.

Remark: Although we will not formally define this term, we remark that this problem shows that, for a topological space X, the condition "there exists a path from x to y" defines an *equivalence relation* on the points of X.

4. (a) Suppose that $(X, \mathcal{T}_X)$ is a topological space, and that $(Y, \mathcal{T}_Y)$ is a **Hausdorff** topological space. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Suppose that $A \subseteq X$ is a subset such that

$$f(a) = g(a) \qquad \text{for all } a \in A.$$

Prove that

$$f(x) = g(x) \qquad \text{for all } x \in \overline{A}.$$

This says that the values of a continuous function on $\overline{A}$ are completely determined by its values on A.

(b) Suppose you have a function of topological spaces $f : \mathbb{R} \to Y$, where $\mathbb{R}$ has the standard topology and Y is Hausdorff. In a sentence, explain why the previous problem implies that a continuous function f is completely determined by its values on $\mathbb{Q}$.

5. Let $f : X \to Y$ be a continuous map of topological spaces. If X is Hausdorff, does it follow that $f(X)$ (viewed as a subspace of Y with the subspace topology) must be Hausdorff? Give a proof, or find a counterexample.

6. For each of the following sequences: find the set of all limits, or determine that the sequence does not converge. No justification needed.

- Let $X = \{a, b, c, d\}$ have the topology $\{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c, d\}\}$.

  (i) $a,\ b,\ a,\ b,\ a,\ b,\ a,\ b,\ \cdots$ (ii) $c,\ c,\ c,\ c,\ c,\ c,\ c,\ c,\ c, \ldots$

- Let $\mathbb{R}$ have the topology $\mathcal{T} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\varnothing\} \cup \{\mathbb{R}\}$.

  (iii) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ (iv) $(n)_{n\in\mathbb{N}}$ (v) $(-n)_{n\in\mathbb{N}}$

- Let $\mathbb{R}$ have the topology $\mathcal{T} = \{\varnothing\} \cup \{U \subseteq \mathbb{R} \mid 0 \in U\}$.

  (vi) $0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots$ (vii) $1,\ 1,\ 1,\ 1,\ 1,\ 1,\ \cdots$