

Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
 - (a) Show that $\mathcal{T}$ is a basis for $\mathcal{T}$.
 - (b) Suppose that $\mathcal{B}$ is a basis for $\mathcal{T}$. Show that any collection of open sets in X containing $\mathcal{B}$ is also a basis for $\mathcal{T}$.
2. Verify that the set of open intervals $(a, b) \subseteq \mathbb{R}$ is a basis for $\mathbb{R}$ (with the Euclidean topology).
3. Let X, Y, Z be sets and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions.
 - (a) Show that if f and g are both surjective, then $g \circ f$ is surjective.
 - (b) Show that if f and g are both injective, then $g \circ f$ is injective.
 - (c) Show by example that, if we only assume one of f and g is surjective, then $g \circ f$ need not be surjective.
 - (d) Show by example that, if we only assume one of f and g is injective, then $g \circ f$ need not be injective.

Worksheet problems

(Hand these questions in!)

- Worksheet #10 Problems 1, 4.
- Worksheet #11 Problems 1.

Assignment questions

(Hand these questions in!)

1. Let $(X, \mathcal{T}_X)$ be a topological space, and let $A, B \subseteq X$.
 - (a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
 - (b) Show by example that $\overline{A \cap B}$ need not equal $\overline{A} \cap \overline{B}$.
 - (c) Show that $\partial(\text{Int}(A)) \subseteq \partial A$, but show by example that these sets need not be equal.
 - (d) Show that $\partial(\overline{A}) \subseteq \partial A$, but show by example that these sets need not be equal.
2. **Definition (Accumulation points of a set).** Let $(X, \mathcal{T})$ be a topological space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every open neighbourhood U of x also contains a point in S distinct from x .
 - (a) Let A' be the set of accumulation points of A . Then $\overline{A} = A \cup A'$.
 - (b) Recall that (by Homework #2 Problem 4(d)), if A is a subset of a metric space X , and x is an accumulation point of A , then every neighbourhood of x contains infinitely many points of A . Is the same statement true for all topological spaces? Give a proof, or state a counterexample.

3. Let $X = \{a, b, c, d\}$ with the topology

$$\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}.$$

State solutions to the following questions. No justification necessary.

- (a) Is X a T_1 -space?
 - (b) Is X Hausdorff?
 - (c) Find the interior, closure, boundary, and set of accumulation points of $\{a, c, d\}$.
 - (d) Find the interior, closure, boundary and set of accumulation points of $\{a, b\}$.
4. **Definition (Cofinite topology).** Let X be a set. The *cofinite* topology on X is the topology $\mathcal{T} = \{U \mid X \setminus U \text{ is finite}\} \cup \{\emptyset\}$.

The word “cofinite” is a contraction of “complements are finite”. We note that if X is a finite set, the cofinite topology is the discrete topology.

- (a) Verify that the cofinite topology is, in fact, a topology.
 - (b) Consider the space $(\mathbb{R}, \text{cofinite})$. State the interior, closure, boundary and set of accumulation points of the sets (i) $A = \{0\}$ and (ii) the interval $B = [0, 1]$.
No justification needed.
5. Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{B}$ be a basis for $\mathcal{T}$. For a subset $A \subseteq X$, prove the following.
- (a) $\text{Int}(A) = \{a \in A \mid \text{there exists a basis element } B \text{ with } a \in B \subseteq A\}$
 - (b) $\overline{A} = \{x \in X \mid \text{any basis element } B \in \mathcal{B} \text{ containing } x \text{ must contain a point of } A\}$
6. (a) In our definition of a basis, we began with a space with a given topology, and defined a basis to be a collection of open subsets satisfying certain properties. In many cases, however, we will wish to topologize our set X by first specifying a basis, and using the basis to define a topology on X . Prove the following theorem, which gives conditions on a collection of subsets $\mathcal{B}$ of X that ensure it will generate a valid topology on X .

Theorem (An extrinsic definition of a basis). Let X be a set and let $\mathcal{B}$ be a collection of subsets of X such that

- $\bigcup_{B \in \mathcal{B}} B = X$
- If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq (B_1 \cap B_2)$.

Let

$$\mathcal{T} = \{U \mid U \text{ is a (possibly empty) union of elements of } \mathcal{B}\}.$$

Then $\mathcal{T}$ is a topology on X , and $\mathcal{B}$ is a basis for $\mathcal{T}$. We say that $\mathcal{T}$ is the topology *generated by* the basis $\mathcal{B}$.

- (b) Let (X, d) be a metric space. Verify that the collection of open balls in X satisfies the criteria of the theorem. This gives a second proof that the collection of open sets that they generate does indeed satisfy the axioms of a topology.