Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Give an intuitive geometric explanation of each of the 3 properties that define a metric.

2. Let $X = \{a, b, c\}$. Which of the following functions define a metric on X?

   (a)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 3 \end{aligned}$$

   (b)
   $$\begin{aligned} d(a,a) = d(b,b) = d(c,c) &= 0 \\ d(a,b) = d(b,a) &= 1 \\ d(a,c) = d(c,a) &= 2 \\ d(b,c) = d(c,b) &= 4 \end{aligned}$$

3. Which of the following functions $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the triangle inequality? Which are metrics?

   (a) $d(x,y) = \dfrac{|y-x|}{2}$

   (b) $d(x,y) = |x-y| + 1$

   (c) $d(x,y) = \min(|y-x|, \pi)$

   (d) $d(x,y) = 1$

   (e) $d(x,y) = |\log(y/x)|$ (for $x, y > 0$)

4. See Assignment Problem 1 for the definition and notation used for the image and preimage of sets under a function f.
   (a) Let X and Y be sets, and $f : X \to Y$ any function. Show that $f^{-1}(Y) = X$, and $f^{-1}(\varnothing) = \varnothing$.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. Compute $f^{-1}(\{0\})$, $f^{-1}(\{4\})$, $f^{-1}(\{-1\})$, $f((0,1))$, and $f^{-1}((0,1))$.

5. Let $f : X \to Y$. See Assignment Problem 1 for the definition of image and preimage.
   (a) Let $A \subseteq X$. Show that $y \in f(A)$ if and only if there is some $a \in A$ such that $f(a) = y$.
   (b) Let $B \subseteq Y$. Show that $x \in f^{-1}(B)$ if and only if $f(x) \in B$.
   (c) Suppose that $A \subseteq A' \subseteq X$. Show that $f(A) \subseteq f(A')$
   (d) Suppose that $B \subseteq B' \subseteq Y$. Show that $f^{-1}(B) \subseteq f^{-1}(B')$.

6. Consider the set $\mathbb{Z}$ with the Euclidean metric (defined by viewing $\mathbb{Z}$ as a subset of the metric space $\mathbb{R}$). What is the ball $B_3(1)$ as a subset of $\mathbb{Z}$? What is the ball $B_{\frac{1}{2}}(1)$?

7. Let (X, d) be a metric space, $r > 0$, and $x \in X$. Show that $x \in B_r(x)$. Conclude in particular that open balls are always non-empty.

8. Let (X, d) be a metric space, and suppose that $r, R \in \mathbb{R}$ satisfy $0 < r \leq R$. Show the containment of the subsets $B_r(x) \subseteq B_R(x)$ of X for any point $x \in X$.

9. Let (X, d) be a metric space, and $r > 0$. For $x, y \in X$, show that $y \in B_r(x)$ if and only if $x \in B_r(y)$.

10. Let (X, d) be a metric space. Let $x_0 \in X$ and $r > 0$. Let's consider the definititon of an open ball in X,
$$B_r(x_0) = \{x \in X \mid d(x, x_0) < r\}.$$
Note that the open ball (by definition) consists entirely of elements of X, it is always a subset of X. Let $Y \subseteq X$ be a susbset of X. We showed on our worksheet that Y inherits its own metric structure from tthe metric on X.
    (a) Suppose we are working with both metric spaces X and Y. Given a point $y_0 \in Y$, we can also view y_0 as a point in X. For $r > 0$, let's write
    $$B_r^Y(y_0) = \{y \in Y \mid d(y, y_0) < r\}$$
    for the open ball around y_0 in the metric space Y, and write
    $$B_r^X(y_0) = \{x \in X \mid d(x, y_0) < r\}$$
    for the open ball around y_0 in tthe metric space X. Explain why $B_r^X(y_0)$ and $B_r^Y(y_0)$ could be different sets.
    (b) Show that $B_r^Y(y_0) = B_r^X(y_0) \cap Y$.
    (c) Describe the ball of radius 2 centered around the point $y_0 = 0$ in the metric space Y, where Y is the subset of the real numbers (with the Euclidean metric)
    - $Y = \mathbb{R}$
    - $Y = [-3, 3]$
    - $Y = \left[0, \frac{1}{2}\right]$
    - $Y = [0, \infty)$
    - $Y = \mathbb{Q}$
    - $Y = \mathbb{Z}$

11. Let $X = \mathbb{R}$ with the usual Euclidean metric $d(x, y) = |x - y|$.
    (a) Let x and $r > 0$ be real numbers. Show that $B_r(x)$ is an open interval in $\mathbb{R}$. What are its endpoints?
    (b) Show that every interval of the real line the form (a, b), $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$ is open, for any $a < b \in \mathbb{R}$.
    (c) Show that the interval $[0, 1] \subseteq \mathbb{R}$ is closed.

12. Let (X, d) be a metric space, and let $U \subseteq X$ be a subset. Does the set U necessarily need to be either open or closed? Can it be neither? Can it be both?

13. Let $X = \mathbb{R}$. Find the set of accumulation points and the set of isolated points (defined in Assignment Problem 5) for each of the following subsets of X.
    (a) $S = \{0\}$ (b) $S = (0, 1)$ (c) $S = \mathbb{Q}$ (d) $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$

Worksheet problems

(Hand these questions in!)

- Worksheet #1 Problem 1(a), 1(c)
- Worksheet #2 Problem 2, 3.

Assignment questions

(Hand these questions in!)

1. Let $f : X \to Y$ be a function between sets X and Y. Given a set $A \subseteq X$, its *image* under f is the subset of Y, denoted $f(A)$, defined to be

$$f(A) = \{f(a) \mid a \in A\} \subseteq Y.$$

This states that $f(A)$ is the set of all points in the codomain Y to which f maps some point of A. Given a set $C \subseteq Y$, its *preimage* is the subset of X, denoted $f^{-1}(C)$, defined to be

$$f^{-1}(C) = \{x \mid f(x) \in C\} \subseteq X.$$

This is the set of all points in the domain X that f maps to an element of C. Note that this definition makes sense (and we use the notation $f^{-1}(C)$) even if the function f is not invertible, and an inverse f^{-1} does not exist as a well-defined function of Y.

Let $f : X \to Y$, and let $A \subseteq X$ and $C \subseteq Y$. For each of the following, determine whether you can replace the symbol $\Box$ with $\subseteq, \supseteq, =$, or none of the above. Justify your answer by giving a proof of any set-containment or set-equality you claim. If set-equality does not hold in general, give a counterexample.

(a) $A \quad \Box \quad f^{-1}(f(A)))$ (b) $C \quad \Box \quad f(f^{-1}(C)))$

2. Let $X = \mathbb{R}^2$. Sketch the balls $B_1(0,0)$ and $B_2(0,0)$ for each of the following metrics on $\mathbb{R}^2$. No further justification needed. Denote $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$.
   (a) $d(\overline{x}, \overline{y}) = ||\overline{x} - \overline{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
   (b) $d(\overline{x}, \overline{y}) = |x_1 - y_1| + |x_2 - y_2|$
   (c) $d(\overline{x}, \overline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
   (d) $d(\overline{x}, \overline{y}) = \begin{cases} 0, & \overline{x} = \overline{y} \\ 1, & \overline{x} \neq \overline{y} \end{cases}$

3. Prove the following.

Proposition (Equivalent definition of interior point). For a subset V of a metric space (X, d), a point $x \in V$ is an interior point of V if and only if there exists an open neighbourhood U of x that is contained in V.

4. (a) Prove *DeMorgan's Laws*: Let X be a set and let $\{A_i\}_{i \in I}$ be a collection of subsets of X.

$$(i) \quad X \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} (X \setminus A_i) \qquad (ii) \quad X \setminus \left(\bigcap_{i \in I} A_i \right) = \bigcup_{i \in I} (X \setminus A_i)$$

Hint: Remember that a good way to prove two sets B and C are equal is to prove that $B \subseteq C$ and that $C \subseteq B$!

   (b) Let (X, d) be a metric space, and let $\{C_i\}_{i \in I}$ be a collection of closed sets in X. Note that I need not be finite, or countable! Prove that $\bigcap_{i \in I} C_i$ is a closed subset of X.

(c) Now let (X, d) be a metric space, and let $\{C_i\}_{i \in I}$ be a **finite** collection ($I = \{1, 2, \ldots, n\}$) of closed sets in X. Prove that $\bigcup_{i \in I} C_i$ is a closed subset of X.

5. Consider the following definition.

   Definition (Accumulation points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if for every $r > 0$ the ball $B_r(x)$ around x contains at least one point of S distinct from x.
   Note that x may or may not itself be an element of S.

   (a) An element $s \in S$ that is not an accumulation point of S is called an *isolated point* of S. Negate the definition of an accumulation point to give a precise statement of what it means to be an isolated point.

   (b) Prove that the following definition of accumulation point is equivalent to the one above. In other words, show that a point $x \in X$ is an accumulation point of a set $S \subseteq X$ if and only if it satisfies the following property.

   Alternative Definition (Accumulation points of a set.) Let (X, d) be a metric space, and let $S \subseteq X$ be a set. A point $x \in X$ is called an *accumulation point* of S if every open subset U of X containing x also contains a point in S distinct from x.

   (c) Let (X, d) be a metric space and let $S \subseteq X$ be a **closed** subset. Let x be an accumulation point of S. Show that x is contained in S.

   (d) Let (X, d) be a metric space and let $S \subseteq X$ be any subset. Let x be an accumulation point of S, and let $B_r(x)$ be a ball centered around x of some radius $r > 0$. Show that $B_r(x)$ contains infinitely many points of S.