Warm-up questions

(These warm-up questions are optional, and won't be graded.)

1. Let $(X, \mathcal{T})$ be a topological space.
   (a) Let $(X, \mathcal{T})$ be a topological space. Explain why the condition that X is compact is stronger than the assumption that X has a finite open cover.
   (b) Show that every topological space has a finite open cover.
   Hint: What is the first axiom of a topology?

2. Let $(X, \mathcal{T})$ be a topological space, and $A \subseteq X$ a subset. Prove that the two following definitions of compactness are equivalent.
   - The subset A is *compact* if it is a compact topological space with respect to the subspace topology $\mathcal{T}_A$.
   - The subset A is *compact* if it satisfies the following property: for any collection of open subsets $\{U_i\}_{i \in I}$ of X such that $A \subseteq \bigcup_{i \in I} U_i$, there is a finite subscollection $U_1, U_2, \ldots, U_n$ such that $A \subseteq \bigcup_{i=1}^{n} U_i$.

3. Give an example of a subsets $A \subseteq B$ of $\mathbb{R}$ such that ...
   (a) A is compact, and B is noncompact
   (b) B is compact, and A is noncompact

4. Determine the connected components of $\mathbb{R}$ with the following topologies.
   (a) the topology induced by the Euclidean metric
   (b) the discrete topology
   (c) the indiscrete topology
   (d) the cofinite topology

Worksheet problems

(Hand these questions in!)

- Worksheet 16, Problems 1, 3.
- Worksheet 17, Problems 2, 4(a), 4(b)

Assignment questions

(Hand these questions in!)

0. **(Optional).** Submit your Math 490 course evaluation!
1. Prove the following result. This theorem is a major reason we care about compactness!

> **Theorem (Generalized Extreme Value Theorem).** Let X be a nonempty compact topological space, and let $f : X \to \mathbb{R}$ be a continuous function (where $\mathbb{R}$ has the standard topology). Then $\sup(f(X)) < \infty$, and there exists some $z \in X$ such that $f(z) = \sup(f(X))$. That is, f achieves its supremum on X.

Hint: See Worksheet #17 Problem 2, 4(a), 4(b), and Homework #6 Problem 5(a).

2. (a) Let (X, d) be a metric space. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence in X that contains no convergent subsequence. Prove that, for every $x \in X$, there is some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x)$ contains only finitely many points of the sequence.

   (b) Prove that any compact metric space is sequentially compact.

Combined with Homework #5 Problem 5, this exercise proves:

> **Theorem (Compactness vs sequential compactness in metric spaces).** Let (X, d) be a metric space. Then X is compact if and only if X is sequentially compact.

(Neither direction of this theorem holds, however, for arbitrary topological spaces!)

Combined with Homework #5 Problem 4, this exercise proves:

> **Theorem (Compactness in $\mathbb{R}^n$).** Endow $\mathbb{R}^n$ with the Euclidean metric. A subspace $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.