# LINEAR ELASTOSTATICS

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**Keywords**

Linear elasticity, Hooke's law, stress functions, uniqueness, existence, variational methods, boundary-value problems, singularities, dislocations, asymptotic fields, anisotropic materials.

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**Summary**

Governing equations are developed for the displacements and stresses in a solid with a linear constitutive law under the restriction that strains are small. Alternative variational formulations are introduced which can be used to obtain approximate analytical solutions and which are also used to establish a uniqueness theorem. Techniques for solving boundary-value problems are discussed, including the Airy stress function and Muskhelishvili's complex-variable formulation in two dimensions and the Papkovich-Neuber solution in three dimensions. Particular attention is paid to singular stress fields due to concentrated forces and dislocations and to geometric discontinuities such as crack and notch tips. Techniques for solving two-dimensional problems for generally anisotropic materials are briefly discussed.

## 1. Introduction

The subject of Elasticity is concerned with the determination of the stresses and displacements in a body as a result of applied mechanical or thermal loads, for those cases in which the body reverts to its original state on the removal of the loads. If the loads are applied sufficiently slowly, the particle accelerations will be small and the body will pass through a sequence of equilibrium states. The deformation is then said to be 'quasi-static'. In this chapter, we shall further restrict attention to the case in which the stresses and displacements are linearly proportional to the applied loads and the strains and rotations are small. These restrictions ensure that the principle of linear superposition applies — i.e., if several loads are applied simultaneously, the resulting stresses and displacements will be the sum of those obtained when the loads are applied separately to the same body. This enables us to employ a wide range of series and transform techniques which are not available for non-linear problems.

### 1.1. Notation for position, displacement and strain

We shall define the position of a point in three-dimensional space by the Cartesian coordinates $(x_1, x_2, x_3)$. Latin indices $i, j, k, l$, etc will be taken to refer to any one of the values 1,2,3, so that the symbol $x_i$ can refer to any one of $x_1, x_2, x_3$. The Einstein summation convention is adopted for repeated indices, so that, for example

$$x_i x_i \equiv \sum_{i=1}^{3} x_i x_i = x_1^2 + x_2^2 + x_3^2 = R^2 \ , \tag{1}$$

where $R$ is the distance of the point $(x_1, x_2, x_3)$ from the origin. We can also define position using the *position vector*

$$\boldsymbol{R} = \boldsymbol{e}_i x_i \ , \tag{2}$$

where $\boldsymbol{e}_i$ is the unit vector in direction $x_i$.

Suppose that a given point is located at $\boldsymbol{R} = \boldsymbol{e}_i x_i$ in the undeformed state and moves to the point $\boldsymbol{\rho} = \boldsymbol{e}_i \xi_i$ after deformation. We can then define the *displacement vector* $\boldsymbol{u}$ as

$$\boldsymbol{u} = \boldsymbol{\rho} - \boldsymbol{R} \ , \tag{3}$$

or in terms of components,

$$u_i = \xi_i - x_i \ . \tag{4}$$

When the body is deformed, different points will generally experience different displacements, so $\boldsymbol{u}$ is a function of position. We shall always refer displacements to the undeformed position, so that $u_i$ is a function of $x_1, x_2, x_3$.

### 1.2. Rigid-body displacement

There exists a class of displacements that can occur even if the body is rigid and hence incapable of deformation. An obvious case is a rigid body translation $u_i = C_i$, where $C_i$ are constants (independent of position). We can also permit a small rotation about each of the three axes (recall that in the linear theory rotations are required to be small). The most general rigid-body displacement field can then be written

$$u_i = C_i + D_j \epsilon_{ijk} x_k \ , \tag{5}$$

where $\epsilon_{ijk}$ is the *alternating tensor* which is defined to be 1 if the indices are in cyclic order (e.g. 1,2,3 or 2,3,1), –1 if they are in *reverse* cyclic order (e.g. 2,1,3) and zero if any two indices are the same.

### 1.3. Strain, rotation and dilatation

In the linear theory, the strain components $e_{ij}$ can be defined in terms of displacements as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \ . \tag{6}$$

This leads (for example) to the definitions

$$e_{11} = \frac{\partial u_1}{\partial x_1} \ ; \quad e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{7}$$

for normal and shear strains respectively. We also define the *rotation*

$$\omega_k = \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_j}{\partial x_i}\right) \ . \tag{8}$$

It can be verified by substitution that if the displacement is given by (5), there is no strain ($e_{ij} = 0$) and the rotation $\omega_k = D_k$.

By considering the deformation of an infinitesimal cube of material of initial volume $V$, it can be shown that the proportional change in volume is the sum of the three normal strains — i.e.

$$\frac{\delta V}{V} \equiv e = e_{ii} \ . \tag{9}$$

This quantity is known as the *dilatation* and is denoted by the symbol $e$.

### 1.4. Compatibility of strain

If the strains $e_{ij}$ and rotations $\omega_k$ are known functions of $x_1, x_2, x_3$, equations (6, 8) constitute a set of partial differential equations that can be integrated to obtain the displacements $u_i$. In a formal sense, one can write

$$\boldsymbol{u}_B = \boldsymbol{u}_A + \int_A^B \frac{\partial \boldsymbol{u}}{\partial S} dS \ , \tag{10}$$

where $A, B$ are two points in the body and the integration is performed along any line between $A$ and $B$ that is entirely contained within the body. The displacement $\boldsymbol{u}$ must be a single-valued function of position and hence the integral in equation (10) must be path-independent. Using equations (6, 8) to define the derivatives inside this integral, it can be shown that this requires that the strains satisfy the *compatibility equations*

$$\epsilon_{pks}\frac{\partial}{\partial x_k}\left(\frac{\partial e_{sj}}{\partial x_i} - \frac{\partial e_{si}}{\partial x_j}\right) = 0 \ . \tag{11}$$

Alternatively, these equations can be obtained by eliminating the displacement components between equations (6). Equation (11) can be expanded to give three equations of the form

$$\frac{\partial^2 e_{11}}{\partial {x_2}^2} + \frac{\partial^2 e_{22}}{\partial {x_1}^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} \tag{12}$$

and three of the form

$$\frac{\partial^2 e_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial e_{23}}{\partial x_1} + \frac{\partial e_{31}}{\partial x_2} - \frac{\partial e_{12}}{\partial x_3}\right) \ . \tag{13}$$

The compatibility equations are sufficient to ensure that the integral in (10) is single-valued if the body is simply connected, but if it is multiply connected, they must be supplemented by the explicit statement that the corresponding integral around a closed path surrounding any hole in the body be zero. Explicit forms of these additional conditions in terms of the strain components were developed by E.Cesaro and are known as *Cesaro integrals*.

## 2. Traction and stress

We shall use the term *traction* and the symbol $\boldsymbol{t}$ to define the limiting value of force per unit area applied to a prescribed infinitesimal elementary area, such as a region of the boundary of the body. Since the loaded surface is defined, the traction is a vector $t_i$. To define a component of *stress* $\boldsymbol{\sigma}$ within the body, we need to identify both the plane on which the stress component acts and the direction of the traction on that plane. We define the plane by its outward normal, so that the $x_i$-plane is perpendicular to the direction $x_i$. Notice that this plane can also be defined as the locus of all points $(x_1, x_2, x_3)$ satisfying the equation $x_i = C$ where $C$ is any constant. With this notation, we then define the stress component $\sigma_{ij}$ as the component of traction in the $j$-direction acting on the $x_i$-plane. The resulting components are illustrated in Figure 1.1.

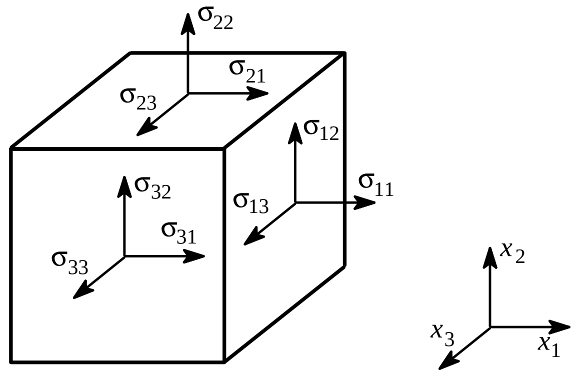


**Figure 1.1:** Notation for stress components.

The equilibrium of moments acting on the block in Figure 1.1 requires that $\sigma_{ij} = \sigma_{ji}$ and hence that the matrix of stress components

$$\boldsymbol{\sigma} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix} \tag{14}$$

is symmetric. Notice that the diagonal elements of the stress matrix define *normal stresses* and the convention implies that tensile normal stresses are positive. The off-diagonal elements define *shear stresses*.

### 2.1. Equilibrium of stresses

The stress components in any continuum are constrained by the requirement that all parts of the body obey Newton's law of motion. Applying this condition to an infinitesimally small rectangular element of material $(\delta x_1 \delta x_2 \delta x_3)$, we obtain

$$\frac{\partial \sigma_{ij}}{\partial x_j} + p_i = \rho \frac{\partial^2 u_i}{\partial t^2} \ , \tag{15}$$

where $p_i$ represents the components of a body force vector $\boldsymbol{p}$ per unit volume, $\rho$ is the density and $t$ is time. The basic postulate of elastostatics is that the loading rate is sufficiently small for the acceleration term $\partial^2 u_i / \partial t^2$ to be neglected, leading to the equilibrium equation

$$\frac{\partial \sigma_{ij}}{\partial x_j} + p_i = 0 \ . \tag{16}$$

## 3. Transformation of coordinates

If $x_1, x_2, x_3$ and $x'_1, x'_2, x'_3$ are two sets of Cartesian coordinates sharing the same origin, we can define a matrix $\boldsymbol{l}$ of *direction cosines* such that $l_{ij}$ is the cosine of the angle between the axes $x_i$ and $x'_j$. It then follows that

$$x'_i = l_{ij} x_j \ ; \quad x_i = l_{ji} x'_j \tag{17}$$

and since the three rows and three columns of $\boldsymbol{l}$ each defines a set of orthogonal unit vectors, we also have

$$l_{ij}l_{ik} = \delta_{jk} \; ; \quad l_{ij}l_{kj} = \delta_{ik} \; . \tag{18}$$

Vectors, such as the displacement $\boldsymbol{u}$ can be transformed to and from the new coordinate system by the relation

$$u'_i = l_{ij}u_j \; ; \quad u_i = l_{ji}u'_j \; . \tag{19}$$

and the strain components $e_{ij}$ transform according to the rules

$$e'_{ij} = l_{ip}l_{jq}e_{pq} \; ; \quad e_{ij} = l_{pi}l_{qj}e'_{pq} \; , \tag{20}$$

which follow from the definitions (6) and (17, 19). The corresponding stress transformation equations are obtained by considering the equilibrium of an infinitesimal tetrahedron whose four surfaces are perpendicular to $x_1, x_2, x_3, x'_i$ respectively. We obtain

$$\sigma'_{ij} = l_{ip}l_{jq}\sigma_{pq} \; ; \quad \sigma_{ij} = l_{pi}l_{qj}\sigma'_{pq} \; , \tag{21}$$

which of course have the same form as (20). Quantities which transform according to equations of this form are known as *Cartesian tensors* of rank 2.

## 4. Hooke's law

Linear elasticity is restricted to materials that obey Hooke's law in the sense that the stress and strain tensors are linearly related. The most general such relation can be written

$$\sigma_{ij} = c_{ijkl}e_{kl} = c_{ijkl}\frac{\partial u_k}{\partial x_l} \; , \tag{22}$$

where $c_{ijkl}$ is a Cartesian tensor of rank 4 known as the *elasticity tensor*. It can be transformed into the coordinate system $x'_1, x'_2, x'_3$ using the relation

$$c'_{ijkl} = l_{ip}l_{jq}l_{kr}l_{ls}c_{pqrs} \; . \tag{23}$$

Equation (22) can be viewed as a set of linear algebraic equations for $e_{kl}$, which can be inverted to give an equation of the form

$$e_{ij} = s_{ijkl}\sigma_{kl} \; , \tag{24}$$

where $s_{ijkl}$ is known as the *compliance tensor*.

Both the elasticity tensor and the compliance tensor must satisfy the symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klij} = c_{ijlk} \; , \tag{25}$$

which follow from (i) the symmetry of the stress and strain tensors (e.g. $\sigma_{ij} = \sigma_{ji}$) and (ii) the reciprocal theorem, which we shall discuss in §6.4 below. Using these conditions, the maximum number of independent constants in $c_{ijkl}$ is reduced to 21. However, the material may have additional structural symmetries in particular coordinate systems which further reduces the number of independent elastic constants. The greatest degree of symmetry arises when the material is *isotropic* so that the elasticity tensor is invariant under all Cartesian coordinate transformations. In this case, only two constants are independent and they can be defined such that

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) \; , \tag{26}$$

where $\lambda, \mu$ are *Lamé's constants*. The elastic constitutive law (22) then takes the form

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \ . \tag{27}$$

An alternative statement of the isotropic constitutive law is

$$e_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu\sigma_{kk}\delta_{ij}}{E} \ , \tag{28}$$

where $E, \nu$ are *Young's modulus* and *Poisson's ratio* respectively. Clearly the two sets of elastic constants are related, since (28) is the inversion of (27). In fact

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)} \ ; \quad \mu = \frac{E}{2(1+\nu)} \ ; \quad E = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)} \ ; \quad \nu = \frac{\lambda}{2(\lambda+\mu)} \ . \tag{29}$$

### 4.1. Equilibrium equations in terms of displacements

Hooke's law (22) and the strain displacement relations (6) can be used to write the equilibrium equations (16) in terms of the displacements, giving

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + p_i = 0 \ . \tag{30}$$

If the material is isotropic, we obtain

$$(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_k \partial x_k} + p_i = 0 \ , \tag{31}$$

using (27) in place of (22).

## 5. Loading and boundary conditions

Suppose that an elastic body occupies the region $\Omega$ and that its boundary is denoted by $\Gamma$. In a typical problem, the body force $p_i$ will be prescribed and the stress and displacement components are required to satisfy equation (16, 30) respectively throughout $\Omega$. In addition, three boundary conditions must be specified at each point on the boundary.

If the local outward normal to the surface is denoted by the unit vector $\boldsymbol{n}$ (i.e. $n_i$ are the direction cosines of the normal), the corresponding traction $t_i$ can be written

$$t_i = n_j\sigma_{ji} \ . \tag{32}$$

The boundary condition at any given point may comprise prescribed values of traction $t_i$ or of displacement $u_i$ or of some combination of the two. For example, if an elastic body is in contact with a plane frictionless rigid body defined by a normal in direction $x_3$, the normal displacement $u_3$ must be zero, and the two shear tractions

$$t_1 = \sigma_{31} \ ; \quad t_2 = \sigma_{32} \tag{33}$$

must be zero. Notice that we cannot prescribe both the traction and the displacement in the same direction at any point, since this would generally lead to an ill-posed problem.

### 5.1. Saint-Venant's principle

B.de Saint-Venant first enunciated the concept that if two systems of loading at a local region on a boundary are statically equivalent (i.e. they correspond to the same total force and moment) then their elastic stress fields will approach each other with increasing distance from the loaded region. An equivalent statement, appealing to the concept of superposition, is that a localized region of tractions that are self-equilibrated (i.e. they correspond to *zero* total force and moment) will cause a stress field that decays with increasing distance from the loaded region. This statement is generally known as Saint-Venant's principle. It cannot be proved and in fact there are some important exceptions, notably for the loading of thin-walled structures. For example, if a thin-walled cylindrical shell is pinched by a pair of equal and opposite forces at one end, the effects will penetrate a considerable distance along the axis of the shell. However, the principle can be extremely useful in other situations. For example, if two non-conforming elastic bodies are pressed together, a relatively complex stress field may be developed near the contact region, but at distances that are large compared with the contact area, the fields are well approximated by the solution for a concentrated force.

#### 5.1.1. Weak boundary conditions

Saint-Venant's principle also permits us on occasion to obtain approximate solutions by replacing the true boundary conditions on a part of the boundary by *weak* boundary conditions, which state merely that the tractions in this region should have the same force and moment resultants as those in the actual problem. For example, suppose we seek to determine the stresses in the two-dimensional rectangular body $-a < x_1 < a, -b < x_2 < b$ and that the boundary conditions on $x_1 = a$ are

$$\sigma_{11}(a, x_2) = f_1(x_2) \; ; \quad \sigma_{12}(a, x_2) = f_2(x_2) \; , \tag{34}$$

where $f_1, f_2$ are prescribed functions of $x_2$ in $-b < x_2 < b$. The weak boundary conditions equivalent to (34) are

$$F_1 \equiv \int_{-b}^{b} \Big(\sigma_{11}(a, x_2) - f_1(x_2)\Big) \, dx_2 = 0 \; ; \quad F_2 \equiv \int_{-b}^{b} \Big(\sigma_{12}(a, x_2) - f_2(x_2)\Big) \, dx_2 = 0$$

$$M \equiv \int_{-b}^{b} \Big(\sigma_{11}(a, x_2) - f_1(x_2)\Big) \, x_2 dx_2 = 0 \; , \tag{35}$$

where $F_1, F_2, M$ are the force and moment resultants on the boundary implied by the difference between a candidate stress field and one that exactly satsifies (34). Saint-Venant's principle implies that any solution satisfying (35) will differ significantly from the solution satisfying the *strong* (pointwise) conditions (34) only in a region near $x_1 = a$ of magnitude comparable to the dimension $b$ and hence if $a \gg b$, the solution will be quite accurate in a region distant from this boundary. We shall see in §8.1 below that this device often enables us to obtain closed-form approximations for problems that would otherwise be extremely complex.

### 5.2. Body force

It is important to distinguish between loading of a body by surface tractions and by body force. A body force is an external force that applies in a distributed sense on the internal particles of the body. Thus, it must necessarily involve a physical mechanism that can 'act at a distance'. The commonest case of this kind involves gravitational forces (self weight), but other mechanisms are possible, such as electromagnetic forces.

Another important source of body force arises if the body experiences rotation or translational acceleration. It might be argued that this takes us beyond the field of elastostatics, but a quasi-static solution can still be obtained if the acceleration terms included are only those corresponding to the rigid-body motion. For example, if the body is rotating at constant angular velocity $\Omega$, D'Alembert's principle can be used to convert the corresponding centripetal acceleration into a centrifugal body force $\rho\Omega^2 r$, where $r$ is the distance from the axis of rotation.

If the body forces are prescribed, they can be carried to the right hand side as known functions in equations (16, 30), in which case these become inhomogeneous linear partial differential equations. The solution of these equations can then be constructed as the sum of any particular solution and the general solution of the corresponding homogeneous equation. However, the homogeneous equation is also the equation to be satisfied when there are no body forces. Thus, one strategy for solving problems with body forces is (i) to seek any particular solution of the equilibrium equation (without regard to the boundary conditions on $\Gamma$) and then 'correct' the boundary conditions by superposing an appropriately general solution of the problem without body force.

The particular solution is generally easy to obtain and can often be written down by inspection. For example, for the case of gravitational loading $p_i = -\rho g \delta_{i3}$, a simple particular solution of (31) is

$$u_1 = u_2 = 0 \; ; \quad u_3 = \frac{\rho g x_3^2}{2(\lambda + 2\mu)} \; . \tag{36}$$

For this reason, we shall mostly restrict the following discussion to problems without body force.

### 5.3. Thermal expansion, transformation strains and initial stress

Elastic stresses can also be generated in a body as a result of internal physical processes that tend to change the parameters of the atomic or molecular structure. The simplest example is a change in temperature $\Delta T$, which in the absence of stress would cause the body to expand equally in all three directions, giving the hydrostatic strain components

$$e_{ij} = \alpha \Delta T \delta_{ij} \; , \tag{37}$$

where $\alpha$ is the *coefficient of thermal expansion.* If the temperature is non-uniform, these strains may not satisfy the compatibility equation (11), in which case stresses will be induced so as to restore compatibility. Similar effects can be produced by other physical processes, such as a change in crystal structure as a material transforms from one phase to another. In the absence of stress, these processes (including thermal expansion) would contribute an 'inelastic' strain $e^0_{ij}$ which is additive to the elastic strain given by Hooke's law (24), giving

$$e_{ij} = s_{ijkl}\sigma_{kl} + e^0_{ij} \; . \tag{38}$$

Practical objects are generally manufactured by some inelastic process. For example, a body may be solidified from an initially liquid state, or it may be plastically deformed into its final configuration. These processes typically leave the body in a state of *initial stress* or *residual stress*, meaning the state of stress that would remain in the body if all external loads were removed. Mathematically, there is no way to determine the residual stress without modeling the inelastic manufacturing process from which it derived. Various experimental techniques can be used to estimate the residual stresses in a body once manufactured. For example, X-ray diffraction can be used to estimate the mean atomic spacing and hence the elastic strain at various points on the surface of a body, from which the residual stresses can be deduced using Hooke's law.

If a body contains a non-zero residual stress field before loading, the stresses after loading will simply be the superposition of the residual stresses and the elastic stresses that would be induced in an initially stress-free body by the external loads. In the rest of this chapter, we shall therefore consider only the second of these two components. In other words, we shall assume that the unloaded body is free of stress.

### 6. Strain energy and variational methods

When a body is deformed, the external forces do work. If the deformation is elastic, this work can be recovered on unloading and is therefore stored in the deformed body as *strain energy*. By considering the work done in gradually applying the stress components $\sigma_{ij}$ to an infinitesimal rectangular element, we can show that the *strain energy density* — i.e. the strain energy stored per unit volume — is

$$U_0 = \frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} = \frac{1}{2}s_{ijkl}\sigma_{ij}\sigma_{kl} \tag{39}$$

and the strain energy stored in the entire body $\Omega$ is

$$U = \int_\Omega U_0 d\Omega \; . \tag{40}$$

Notice incidentally that $U_0$ must be positive for all possible states of stress or deformation and this places some inequality restrictions on the tensors $c_{ijkl}, s_{ijkl}$.

The same principle applies to an extended body with a non-uniform stress field. If the external loads are applied sufficiently slowly for accelerations (and hence kinetic energy) to be negligible, the work done during their application must be equal to the total strain energy in the body. This leads to the condition

$$\frac{1}{2}\int_\Omega p_i u_i d\Omega + \frac{1}{2}\int_\Gamma t_i u_i d\Gamma = \int_\Omega U_0 d\Omega \; . \tag{41}$$

We have argued here from the principle of conservation of energy, but this principle is implicit in Hooke's law, which guarantees that the work done on each infinitesimal particle by the body force and by the forces exerted by the surrounding particles is recoverable on unloading. Thus, equation (41) can be derived from the governing equations of elasticity without explicitly invoking conservation of energy. To demonstrate this, we first substitute (32) into the second term and apply the divergence theorem, obtaining

$$\frac{1}{2}\int_\Gamma t_i u_i d\Gamma = \frac{1}{2}\int_\Gamma n_j \sigma_{ji} u_i d\Gamma = \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_j}(\sigma_{ji}u_i) d\Omega \; . \tag{42}$$

Differentiating by parts, we then have

$$\frac{1}{2}\int_\Gamma t_i u_i d\Gamma = \frac{1}{2}\int_\Omega \frac{\partial}{\partial x_j}(\sigma_{ji}u_i) d\Omega = \frac{1}{2}\int_\Omega \frac{\partial \sigma_{ji}}{\partial x_j} u_i d\Omega + \frac{1}{2}\int_\Omega \sigma_{ji}\frac{\partial u_i}{\partial x_j} d\Omega \; . \tag{43}$$

Finally, we use the equilibrium equation (16) in the first term and Hooke's law (22) in the second to obtain

$$\frac{1}{2}\int_\Gamma t_i u_i d\Gamma = -\frac{1}{2}\int_\Omega p_i u_i d\Omega + \frac{1}{2}\int_\Omega c_{ijkl}\frac{\partial u_k}{\partial x_l}\frac{\partial u_i}{\partial x_j} d\Omega \; , \tag{44}$$

from which (41) follows after using (39) in the last term.

#### 6.1. Potential energy of the external forces

We can also construct a *potential energy* of the external forces which we denote by $V$. For a single concentrated force $\boldsymbol{F}$ moving through a displacement $\boldsymbol{u}$ this is defined as

$$V = -\boldsymbol{F}\cdot\boldsymbol{u} = -F_i u_i \; . \tag{45}$$

It follows by superposition that the potential energy of the boundary tractions and body forces is given by

$$V = -\int_{\Gamma_t} t_i u_i d\Gamma - \int_{\Omega} p_i u_i d\Omega \; , \tag{46}$$

where $\Gamma_t$ is that part of the boundary over which the tractions are prescribed. We can then define the *total potential energy* $\Pi$ as the sum of the stored strain energy and the potential energy of the external forces — i.e.

$$\Pi = U + V = \frac{1}{2}\int_{\Omega} c_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}d\Omega - \int_{\Gamma_t} t_i u_i d\Gamma - \int_{\Omega} p_i u_i d\Omega \; . \tag{47}$$

### 6.2. Theorem of minimum total potential energy

Suppose that the displacement field $u_i$ satisfies the equilibrium equations (30) for a particular set of boundary conditions and that we then perturb this state by a small variation $\delta u_i$. The corresponding perturbation in $\Pi$ is

$$\delta\Pi = \int_{\Omega} c_{ijkl}\frac{\partial \delta u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}d\Omega - \int_{\Gamma} t_i \delta u_i d\Gamma - \int_{\Omega} p_i \delta u_i d\Omega \; . \tag{48}$$

Notice that $\delta u_i = 0$ in any region $\Gamma_u$ of $\Gamma$ in which the displacement is prescribed and hence the domain of integration $\Gamma_t$ in the second term on the right-hand side can be replaced by $\Gamma = \Gamma_u + \Gamma_t$.

Substituting for $t_i$ from (32) and then applying the divergence theorem to the second term on the right-hand side of (48), we have

$$\begin{aligned}\int_{\Gamma} t_i \delta u_i d\Gamma &= \int_{\Gamma} \sigma_{ij} n_j \delta u_i d\Gamma = \int_{\Omega} \frac{\partial}{\partial x_j}\left(\sigma_{ij}\delta u_i\right) d\Omega \\ &= \int_{\Omega} \frac{\partial \sigma_{ij}}{\partial x_j}\delta u_i d\Omega + \int_{\Omega} \frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} d\Omega \; .\end{aligned} \tag{49}$$

Finally, using the equilibrium equation (16) in the first term and Hooke's law (22) in the second, we obtain

$$\int_{\Gamma} t_i \delta u_i d\Gamma = -\int_{\Omega} p_i \delta u_i d\Omega + \int_{\Omega} c_{ijkl}\frac{\partial \delta u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}d\Omega \; , \tag{50}$$

and comparing this with (48), we see that $\delta\Pi = 0$. In other words, the equilibrium equation requires that the total potential energy must be stationary with regard to any small variation $\delta u_i$ in the displacement field $u_i$ that is *kinematically admissible* — i.e. consistent with the displacement boundary conditions. A more detailed second order analysis shows that the total potential energy must in fact be a minimum and this is intuitively reasonable, since if some variation $\delta u_i$ could be found which reduced $\Pi$, the surplus energy would take the form of kinetic energy and the system would not remain at rest.

#### 6.2.1. Rayleigh-Ritz approximations and the finite element method

The theorem of minimum total potential energy provides a convenient strategy for the development of approximate solutions to problems where exact solutions are unavailable or overcomplicated. The first step is to define an approximation for the displacement field in the form

$$u_i(x_1, x_2, x_3) = \sum_{n=1}^{m} C_n f_i^{(n)}(x_1, x_2, x_3) \; , \tag{51}$$

where the $f_i^{(n)}$ are a set of approximating functions and $C_n$ are arbitrary constants constituting the *degrees of freedom* in the approximation. The total potential energy is obtained from (47) as a quadratic function of the $C_n$ and the theorem then requires that

$$\frac{\partial \Pi}{\partial C_n} = 0 \; ; \quad n = 1, m \; , \tag{52}$$

which defines $m$ linear equations for the $m$ unknown degrees of freedom $C_n$. The corresponding stress components can then be found by substituting (51) into Hooke's law (22).

If the approximating functions $f_i^{(n)}$ are defined over the entire body $\Omega$, this typically leads to series solutions (e.g. power series or Fourier series) and the method is known as the *Rayleigh-Ritz method*. It is particularly useful in structural mechanics applications, but it is also useful for the challenging problem of the rectangular plate. However, if high accuracy is required it is often more effective to define a set of piecewise continuous functions each of which is zero except over some small region of the body. The body is divided into a set of *elements* and the displacement in each element is described by one or more *shape functions* multiplied by degrees of freedom $C_n$. Typically, the shape functions are defined such that the $C_n$ represent the displacements at specified points or *nodes* within the body. They must also satisfy the condition that the displacement be continuous between one element and the next for all $C_n$. Once the approximation is defined, equation (52) once again provides $m$ linear equations for the $m$ nodal displacements. This is the basis of the *finite element method*. Since the theorem of minimum total potential energy is itself derivable from Hooke's law and the equilibrium equation, an alternative derivation of the finite element method can be obtained by applying approximation theory directly to these equations. To develop a set of $m$ linear equations for the $C_n$, we substitute the approximate form (51) into the equilibrium equations, multiply by $m$ *weight functions*, integrate over the domain $\Omega$ and set the resulting $m$ linear functions of the $C_n$ to zero. The resulting equations will be identical to (52) if the weight functions are chosen to be identical to the shape functions.

### 6.3. Castigliano's second theorem

The strain energy $U$ can be written as a function of the stress components, using the final expression in (39). We obtain

$$U = \frac{1}{2} \int_\Omega s_{ijkl} \sigma_{ij} \sigma_{kl} d\Omega \; . \tag{53}$$

If we now perturb the stress field by a small variation $\delta\sigma_{ij}$, the corresponding perturbation in $U$ will be

$$\delta U = \int_\Omega s_{ijkl} \sigma_{kl} \delta\sigma_{ij} d\Omega = \int_\Omega \frac{\partial u_i}{\partial x_j} \delta\sigma_{ij} d\Omega \; . \tag{54}$$

The divergence theorem gives

$$\int_\Gamma u_i \delta\sigma_{ij} n_j d\Gamma = \int_\Omega \frac{\partial}{\partial x_j} \left( u_i \delta\sigma_{ij} \right) d\Omega = \int_\Omega \frac{\partial u_i}{\partial x_j} \delta\sigma_{ij} d\Omega + \int_\Omega \frac{\partial \delta\sigma_{ij}}{\partial x_j} u_i d\Omega \tag{55}$$

and the second term on the right-hand side must be zero, since the stress perturbation $\delta\sigma_{ij}$ must satisfy the equilibrium equation (16) with no body force. Using (54, 55), we then have

$$\delta U = \int_{\Gamma_u} u_i \delta\sigma_{ij} n_j d\Gamma = \int_{\Gamma_u} u_i \delta t_i d\Gamma \; , \tag{56}$$

where the integral is taken only over $\Gamma_u$, since no perturbation in traction is permitted in $\Gamma_t$ where $t_i$ is prescribed. It follows that the *complementary energy*

$$C = U - \int_{\Gamma_u} u_i t_i d\Gamma \tag{57}$$

must be stationary with respect to all self-equilibrated variations of stress $\delta\sigma_{ij}$. This is *Castigliano's second theorem*. As with the Rayleigh-Ritz method, Castigliano's theorem can be used to obtain approximate solutions to otherwise intractable analytical problems. The first step is to define a self-equilibrated stress field containing an appropriate number of degrees of freedom $C_i$. This can often be done using an appropriate stress function, such as the Airy function of §7.2 or the Prandtl function of §7.4 below. The $C_i$ are then determined by minimizing the complementary energy $C$ of equation (57).

### 6.4. Betti's reciprocal theorem

When an elastic body is loaded sufficiently slowly, the work done by the external forces is equal to the total stored strain energy, which in turn depends on the state of stress $\sigma_{ij}$ in the body through equations (39, 40). The stress depends only on the instantaneous loads applied to the body and not on the history of the loading process, so it follows that the work done by the applied loads must be independent of the sequence in which they are applied.

Suppose that the surface tractions $t_i^{(\alpha)}$ and body forces $p_i^{(\alpha)}$ produce the elastic displacement fields $u_i^{(\alpha)}$, where $\alpha$ takes the values 1 and 2 respectively. We consider two scenarios, one in which the loads $t_i^{(1)}, p_i^{(1)}$ are applied first, followed by $t_i^{(2)}, p_i^{(2)}$ and the other in which the order of loading is reversed.

We first apply $t_i^{(1)}, p_i^{(1)}$, during which the work done will be

$$\frac{1}{2}\int_\Omega p_i^{(1)}u_i^{(1)}d\Omega + \frac{1}{2}\int_\Gamma t_i^{(1)}u_i^{(1)}d\Gamma \ .$$

The loads $t_i^{(1)}, p_i^{(1)}$ are then maintained constant whilst loads $t_i^{(2)}, p_i^{(2)}$ are applied. During this second phase, the work done is

$$\frac{1}{2}\int_\Omega p_i^{(2)}u_i^{(2)}d\Omega + \frac{1}{2}\int_\Gamma t_i^{(2)}u_i^{(2)}d\Gamma + \int_\Omega p_i^{(1)}u_i^{(2)}d\Omega + \int_\Gamma t_i^{(1)}u_i^{(2)}d\Gamma \ ,$$

where the last two terms represent the additional work done by $t_i^{(1)}, p_i^{(1)}$ in moving through the displacements caused by $t_i^{(2)}, p_i^{(2)}$. The total work done is therefore

$$\begin{aligned} W &= \frac{1}{2}\int_\Omega p_i^{(1)}u_i^{(1)}d\Omega + \frac{1}{2}\int_\Gamma t_i^{(1)}u_i^{(1)}d\Gamma + \frac{1}{2}\int_\Omega p_i^{(2)}u_i^{(2)}d\Omega + \frac{1}{2}\int_\Gamma t_i^{(2)}u_i^{(2)}d\Gamma \\ &\quad + \int_\Omega p_i^{(1)}u_i^{(2)}d\Omega + \int_\Gamma t_i^{(1)}u_i^{(2)}d\Gamma \ , \end{aligned} \tag{58}$$

but this must be independent of the order of loading and hence must remain unchanged if we interchange the superscripts $(1), (2)$. This will be the case if and only if

$$\int_\Omega p_i^{(1)}u_i^{(2)}d\Omega + \int_\Gamma t_i^{(1)}u_i^{(2)}d\Gamma = \int_\Omega p_i^{(2)}u_i^{(1)}d\Omega + \int_\Gamma t_i^{(2)}u_i^{(1)}d\Gamma \ . \tag{59}$$

In other words, "*The work done by the loads $t_i^{(1)}, p_i^{(1)}$ moving through the displacements $u_i^{(2)}$ due to $t_i^{(2)}, p_i^{(2)}$ is equal to the work done by the loads $t_i^{(2)}, p_i^{(2)}$ moving through the displacements $u_i^{(1)}$ due to $t_i^{(1)}, p_i^{(1)}$.*" This is Betti's reciprocal theorem.

#### 6.4.1. Applications of Betti's theorem

Betti's theorem establishes a relationship between two different stress and displacement fields for the same elastic body. Generally, one of these fields represents the elasticity problem that we wish to solve, whilst the second is an *auxiliary solution* which in combination with the theorem provides us with information about the problem. Usually the auxiliary solution is taken as a relatively simple state such as uniaxial tension or hydrostatic compression and the theorem then establishes some integral relation for the original problem. For this reason, Betti's theorem is always worth considering when the desired result is defined by an integral, such as the resultant force over some surface or the average displacement.

For example, suppose we choose as auxiliary solution the state of uniform hydrostatic tension

$$\sigma_{ij}^{(2)} = C\delta_{ij} \ , \tag{60}$$

with displacements

$$u_i^{(2)} = \frac{Cx_i}{(3\lambda + 2\mu)} \ , \tag{61}$$

where $C$ is a constant. This field involves no body force ($p_i^{(2)} = 0$) and the traction on all surfaces is one of uniform tension $t_i^{(2)} = Cn_i$. Substituting in (59) and cancelling the common constant $C$, we therefore have

$$\frac{1}{(3\lambda + 2\mu)} \int_\Omega p_i^{(1)} x_i d\Omega + \frac{1}{(3\lambda + 2\mu)} \int_\Gamma t_i^{(1)} x_i d\Gamma = \int_\Gamma u_i^{(1)} n_i d\Gamma \tag{62}$$

and the right-hand side represents the change in volume of the elastic body under the action of the tractions $t_i^{(1)}$ and body forces $p_i^{(1)}$. Thus, using this auxiliary solution, Betti's theorem provides a general expression for the change in volume of a body in terms of the applied loads.

### 6.5. Uniqueness and existence of solution

A typical elasticity problem can be expressed as the search for a displacement field $u_i$ satisfying equations (30) such that the stress components defined through equations (22) and the displacement components satisfy appropriate boundary conditions as discussed in §5. Before embarking upon this search, it is appropriate to ask whether such a solution exists for all legitimate combinations of boundary conditions and if so, whether it is unique.

Suppose the solution is non-unique, so that there exist two distinct stress fields both satisfying the field equations and the same boundary conditions. We can then construct a new stress field by taking the difference between these fields, which is a form of linear superposition. This new field clearly involves no external loading, since the same external loads were included in each of the constituent solutions *ex hyp.* It follows from (41) that $U = 0$, but since $U_0$ must be everywhere positive or zero, it must therefore be zero everywhere, implying that the stress is everywhere zero. Thus, the two solutions must be identical *contra hyp.* and only one solution can exist to a given elasticity problem.

The question of existence of solution is much more challenging and will not be pursued here. A short list of early but seminal contributions to the subject is given by I.S.Sokolnikoff who states that "the matter of existence of solutions has been satisfactorily resolved for domains of great generality." More recently, interest in more general continuum theories including non-linear elasticity has led to the development of new methodologies in the context of functional analysis.

#### 6.5.1. Singularities

In addressing issues of existence and uniqueness, it is critically important to specify the space of functions in which the solution is to be sought, and in particular to define the degree of continuity required or the maximum strength of singularity permitted in the displacement and stress fields. If a concentrated force is applied to the body either at a point on the boundary or at an interior point, equilibrium considerations show that the stresses and hence the strain energy density $U_0$ will increase without limit as we approach this point. More seriously, we find that the integral of $U_0$ over a domain including the loaded point is unbounded, which casts in doubt general theorems that depend on an energy formulation. E.Sternberg and R.A.Eubanks proposed an extension of the uniqueness theorem to cover this case. Equation (41) also implies that a concentrated force will do an infinite amount of work during its application and hence that the displacement under the load is infinite — a result that we shall verify in §11 below.

These difficulties can be avoided, at the cost of some increase of complexity in the boundary-value problem, by replacing the concentrated load by an equivalent finite traction distributed over a small region $\mathcal{A}$. The concentrated load solution can then be regarded as the limit of this more practical problem as $\mathcal{A} \to 0$. Saint-Venant's principle §5.1 implies that only the stresses close to $\mathcal{A}$ will be changed as $\mathcal{A}$ is reduced, so the concentrated force solution also represents an approximation to the stress field distant from the loaded region when $\mathcal{A}$ is finite. Similar arguments can be applied to stronger singularities, such as those due to a concentrated moment.

Stress singularities can also be developed as a result of slope discontinuities in the boundaries of the body, such as a sharp re-entrant notch or the tip of a crack. These are of critical importance in the failure of brittle materials and will be addressed in more detail in §12 below. The resulting singularity is always weaker than that associated with a concentrated force and the strain energy density is integrable, provided this integral is interpreted in the limiting sense

$$U = \lim_{\mathcal{V}\to 0} \int_{\Omega-\mathcal{V}} U_0 d\Omega \; , \tag{63}$$

where $\mathcal{V}$ is a small region including and surrounding the corner. The energy theorems in this section remain valid under this interpretation.

## 7. Two-dimensional problems

We shall consider a problem to be two-dimensional if the stress and displacement components are independent of one of the coordinates, which we take to be $x_3$. For isotropic materials, the problem can be partitioned into a *plane strain* problem and an *antiplane* problem. This partition can also be made for orthotropic materials — i.e. those for which the $x_3$-plane is a plane of material symmetry. For plane strain problems, the out-of-plane displacement $u_3$ is everywhere zero and the only non-zero stress components are $\sigma_{11}, \sigma_{12}, \sigma_{22}, \sigma_{33}$. Also, the condition on $u_3$ implies that $e_{33} = 0$ and hence

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \; , \tag{64}$$

from (28). Using this result in (28), we then have

$$e_{11} = \frac{(1-\nu^2)\sigma_{11}}{E} - \frac{\nu(1+\nu)\sigma_{22}}{E} \; ; \quad e_{22} = \frac{(1-\nu^2)\sigma_{22}}{E} - \frac{\nu(1+\nu)\sigma_{11}}{E} \; ; \quad e_{12} = \frac{(1+\nu)\sigma_{12}}{E} \tag{65}$$

and the boundary value problem can therefore be expressed entirely in terms of the displacements $u_1, u_2$ and stresses $\sigma_{11}, \sigma_{12}, \sigma_{22}$. By contrast, for antiplane problems, these components are all zero and the only non-zero components are $u_3$ and $\sigma_{31}, \sigma_{32}$.

### 7.1. Plane stress

If a body is bounded by the two traction-free parallel planes $x_3 = \pm h/2$, so that $\sigma_{3i}(x_1, x_2, \pm h/2) = 0$, and if the distance $h$ between the planes is small compared with the other dimensions of the body, it can be argued that the stress components $\sigma_{3i}$ will be sufficiently small to be neglected. If the remaining stress components are independent of $x_3$, this defines a state of *plane stress*. As in the case of plane strain, the boundary value problem can be expressed entirely in terms of the displacements $u_1, u_2$ and stresses $\sigma_{11}, \sigma_{12}, \sigma_{22}$, the only difference being that the in-plane strain components are now given by

$$e_{11} = \frac{\sigma_{11}}{E} - \frac{\nu\sigma_{22}}{E} \; ; \quad e_{22} = \frac{\sigma_{22}}{E} - \frac{\nu\sigma_{11}}{E} \; ; \quad e_{12} = \frac{(1+\nu)\sigma_{12}}{E} \; , \tag{66}$$

since $\sigma_{33} = 0$. This equation clearly has the same form as (65), but with different multiplying coefficients, so a plane stress problem can be regarded as equivalent to a plane stress problem for a material with modified elastic properties.

It should be emphasised that the plane stress assumption is an approximation and the true stress field in such cases is usually three-dimensional. An alternative approach is to define a two-dimensional problem in terms of the mean stresses and mean displacements

$$\bar{\sigma}_{ij}(x_1, x_2) = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_{ij}(x_1, x_2, x_3) dx_3 \; ; \quad \bar{u}_i(x_1, x_2) = \frac{1}{h}\int_{-h/2}^{h/2} u_i(x_1, x_2, x_3) dx_3 \; , \tag{67}$$

where $i = 1, 2$. These quantities can be shown to satisfy the plane stress equations exactly. This formulation is known as *generalized plane stress*.

### 7.2. Airy stress function

For the plane strain problem, if there are no body forces ($p_i = 0$), one equilibrium equation is identically satisfied and the remaining two reduce to

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = 0 \; ; \quad \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} = 0 \; . \tag{68}$$

These equations imply the existence of a scalar function $\phi(x_1, x_2)$, such that

$$\sigma_{11} = \frac{\partial^2\phi}{\partial {x_2}^2} \; ; \quad \sigma_{12} = -\frac{\partial^2\phi}{\partial x_2 \partial x_1} \; ; \quad \sigma_{22} = \frac{\partial^2\phi}{\partial {x_1}^2} \; . \tag{69}$$

This is known as the *Airy stress function*. Substituting (69) into (65) and the resulting strain components into (12) shows that $\phi$ must satisfy the *biharmonic equation*

$$\nabla^4\phi \equiv \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = 0 \; . \tag{70}$$

In index notation, we can write

$$\frac{\partial^4\phi}{\partial x_i \partial x_i \partial x_j \partial x_j} = 0 \; , \tag{71}$$

where the indices $i, j$ are summed only over (1,2) since $\phi$ is independent of $x_3$.

A typical plane strain boundary value problem is thus reduced to the search for a function $\phi$ satisfying (70), such that the stress components defined by (69) reduce to the required tractions on the boundary. This method has been applied to a wide range of problems, many of which permit solution in closed form. It is particularly effective when the boundaries of the body are defining surfaces in a convenient coordinate system such as Cartesian or polar coordinates.

Notice that the stress definitions (69) and the governing equation (70) do not contain the elastic properties $E, \nu$. It follows that the stresses in a body of a given shape with prescribed tractions on the boundaries are independent of these properties and in particular that the in-plane stresses $\sigma_{11}, \sigma_{12}, \sigma_{22}$ under the plane stress and plane strain assumptions are the same. This argument fails if there are body forces or if non-zero displacements are prescribed at any of the boundaries.

The Airy function formulation can be extended to problems involving body force provided that the latter are *conservative* — i.e. that $p_i$ can be written as the gradient of a scalar body force potential $V(x_1, x_2)$ as

$$p_i = -\frac{\partial V}{\partial x_i} \ . \tag{72}$$

In-plane stress components satisfying the equilibrium equations (16) can then be defined as

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial {x_2}^2} + V \ ; \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_2 \partial x_1} \ ; \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial {x_1}^2} + V \ . \tag{73}$$

In effect, the expressions (69) are modified by the addition of a two-dimensional hydrostatic stress of magnitude $V$. Using these new definitions in (65, 12), we find that $\phi$ must now satisfy the modified equation

$$\nabla^4 \phi = -\left(\frac{1-2\nu}{1-\nu}\right) \nabla^2 V \ . \tag{74}$$

### 7.2.1. Airy function in polar coordinates

Many two-dimensional problems of practical importance involve bodies defined by surfaces in the polar coordinate system $(r, \theta)$. Examples include stress concentrations due to circular holes and inclusions, curved bars and wedge-shaped regions. It is therefore convenient to give here the expressions for the stress components (in the absence of body force), which are

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \ ; \quad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} \ ; \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \ . \tag{75}$$

Notice that the suffices $r, \theta$ here refer to the directions in which these coordinates increase. The corresponding form of the biharmonic equation (70) is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \phi = 0 \ . \tag{76}$$

J.H.Michell obtained a fairly general Fourier series solution to equation (76) in the form

$$\begin{aligned} \phi \ = \ & A_{01} r^2 + A_{02} r^2 \ln(r) + A_{03} \ln(r) + A_{04} \theta \\ & + (A_{11} r^3 + A_{12} r \ln(r) + A_{14} r^{-1}) \cos\theta + A_{13} r\theta \sin\theta \\ & + (B_{11} r^3 + B_{12} r \ln(r) + B_{14} r^{-1}) \sin\theta + B_{13} r\theta \cos\theta \\ & + \sum_{n=2}^{\infty} (A_{n1} r^{n+2} + A_{n2} r^{-n+2} + A_{n3} r^n + A_{n4} r^{-n}) \cos(n\theta) \\ & + \sum_{n=2}^{\infty} (B_{n1} r^{n+2} + B_{n2} r^{-n+2} + B_{n3} r^n + B_{n4} r^{-n}) \sin(n\theta) \ , \end{aligned} \tag{77}$$

where the coefficients $A_{ij}, B_{ij}$ are arbitrary constants. The solutions to many problems can be obtained by substituting this expression into (75), evaluating the tractions on the boundaries and choosing the coefficients to satisfy the boundary conditions. Notice that some of the terms in (77) are unbounded as $r \to 0$ and hence lead to correspondingly singular stress fields. In most cases these are inappropriate to problems in which the origin is a point in the body, though there are some exceptions, as discussed in §§11.3, 11.4, 12. The remaining bounded terms can also be expressed as polynomials in Cartesian coordinates. For example, the function $r^3\cos\theta = x_1^3 + x_1 x_2^2$.

**Example: Circular hole in a body in uniaxial tension**

As a simple example, consider the problem of a large body in a state of uniaxial tension perturbed by the presence of a small traction-free hole of radius $a$. Taking the origin of both Cartesian and polar coordinates at the centre of the hole, the boundary conditions for this problem can be defined as

$$\sigma_{11} \rightarrow S\ ; \quad \sigma_{12},\ \sigma_{22} \rightarrow 0\ ; \quad r \rightarrow \infty \tag{78}$$

$$\sigma_{rr} = \sigma_{r\theta} = 0\ ; \quad r = a\ . \tag{79}$$

At large values of $r$, equations (78, 69) show that the stress function must take the form

$$\phi = \frac{Sx_2^2}{2} = \frac{Sr^2}{4} - \frac{Sr^2\cos(2\theta)}{4}\ , \tag{80}$$

where we have used the relation $x_2 = r\sin\theta$ to convert the expression into polar coordinates. Comparison with (77) shows that this stress function can be obtained from Michell's solution by choosing $A_{01} = S/4,\ A_{23} = -S/4$ as the only two non-zero coefficients. To describe the perturbation in this stress field due to the hole, we need to superpose appropriate terms from (7) that lead to stresses that decay as $r \rightarrow \infty$. The corresponding stress fields will be singular at the origin, but this is acceptable because the origin is not a point of the body. Clearly only axisymmetric terms and terms varying with $\cos(2\theta)$ are required. Selecting these terms, substituting the resulting stress function into (69, 79) and solving the resulting equations for the unknown coefficients, we find that the complete solution of the problem is defined by the stress function

$$\phi = \frac{Sr^2}{4} - \frac{Sa^2\ln(r)}{2} + S\left(-\frac{r^2}{4} + \frac{a^2}{2} - \frac{a^4}{4r^2}\right)\cos(2\theta)\ . \tag{81}$$

The corresponding stress components are

$$\sigma_{rr} = \frac{S}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{S}{2}\left(\frac{3a^4}{r^4} - \frac{4a^2}{r^2} + 1\right)\cos(2\theta) \tag{82}$$

$$\sigma_{r\theta} = \frac{S}{2}\left(\frac{3a^4}{r^4} - \frac{2a^2}{r^2} - 1\right)\sin(2\theta) \tag{83}$$

$$\sigma_{\theta\theta} = \frac{S}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{S}{2}\left(\frac{3a^4}{r^4} + 1\right)\cos(2\theta)\ . \tag{84}$$

In particular, we note that the maximum tensile stress is $\sigma_{\theta\theta}(a,\pi/2) = 3S$ showing that the hole increases the far-field stress by a *stress concentration factor* of 3.

## 7.3. Complex variable formulation

Two drawbacks to the Airy stress function method are (i) that methods of solution of the resulting boundary value problem for $\phi$ are somewhat *ad hoc* and (ii) it is less convenient for problems where displacement boundary conditions are specified, since the corresponding displacement components cannot be written in terms of simple derivatives of $\phi$. A powerful alternative method for the plane strain problem is to combine the coordinates $x_1, x_2$ into the complex variable $\zeta$ and its conjugate $\bar{\zeta}$, defined as

$$\zeta = x_1 + \imath x_2\ ; \quad \bar{\zeta} = x_1 - \imath x_2\ , \tag{85}$$

where $\imath = \sqrt{-1}$. The inversion of (85) is

$$x_1 = \frac{\zeta + \bar{\zeta}}{2} \; ; \quad x_2 = -\frac{\imath(\zeta - \bar{\zeta})}{2} \; . \tag{86}$$

If $f$ is some function of position, it then follows that

$$\frac{\partial f}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \imath \frac{\partial f}{\partial x_2}\right) \; ; \quad \frac{\partial f}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial f}{\partial x_1} + \imath \frac{\partial f}{\partial x_2}\right) \; . \tag{87}$$

The components of vector quantities are combined as the real and imaginary parts of a complex function — for example the in-plane displacement vector $\boldsymbol{u} = \boldsymbol{i}u_1 + \boldsymbol{j}u_2$ will be written

$$u = u_1 + \imath u_2 \; . \tag{88}$$

One consequence of this relation and (87) is that the gradient operator

$$\boldsymbol{\nabla} f \equiv \boldsymbol{i}\frac{\partial f}{\partial x_1} + \boldsymbol{j}\frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_1} + \imath \frac{\partial f}{\partial x_2} = 2\frac{\partial f}{\partial \bar{\zeta}} \; , \tag{89}$$

from (87). We also have

$$\nabla^2 \phi = \left(\frac{\partial}{\partial x_1} + \imath \frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial x_1} - \imath \frac{\partial}{\partial x_2}\right)\phi = 4\frac{\partial^2 \phi}{\partial \zeta \partial \bar{\zeta}} \; . \tag{90}$$

It follows that the general solution of the Laplace equation

$$\nabla^2 \phi = 4\frac{\partial^2 \phi}{\partial \zeta \partial \bar{\zeta}} = 0 \tag{91}$$

is

$$\phi = f_1(\zeta) + f_2(\bar{\zeta}) \; , \tag{92}$$

where $f_1, f_2$ are arbitrary holomorphic functions of $\zeta$ and $\bar{\zeta}$ only respectively. In the same way, a general biharmonic function (solution of (70)) can be written $\phi = f_1(\zeta) + f_2(\bar{\zeta}) + \bar{\zeta} f_3(\zeta) + \zeta f_4(\bar{\zeta})$.

In this notation, the equilibrium equation (31) takes the form

$$2(\lambda + \mu)\frac{\partial}{\partial \bar{\zeta}}\left(\frac{\partial u}{\partial \zeta} + \frac{\partial \bar{u}}{\partial \bar{\zeta}}\right) + 4\mu \frac{\partial^2 u}{\partial \zeta \partial \bar{\zeta}} + p = 0 \tag{93}$$

and in the absence of body forces ($p = p_1 + \imath p_2 = 0$), a general solution for the displacement can be written

$$2\mu u = -(3 - 4\nu)\chi + \zeta \bar{\chi}' + \bar{\theta} \; , \tag{94}$$

where $\chi, \bar{\theta}$ are arbitrary holomorphic functions of $\zeta, \bar{\zeta}$ only respectively, the overbar represents the complex conjugate function and the prime differentiation with respect to the argument. The stress components can then be obtained by substitution into (27), giving

$$\Theta \equiv \sigma_{11} + \sigma_{22} = -2(\chi' + \bar{\chi}') \tag{95}$$

$$\Phi \equiv \sigma_{11} + 2\imath\sigma_{12} - \sigma_{22} = 2(\zeta \bar{\chi}'' + \bar{\theta}') \; . \tag{96}$$

These expressions can also be written in terms of the single complex function

$$\psi(\zeta, \bar{\zeta}) = \chi + \zeta \bar{\chi}' + \bar{\theta} \; . \tag{97}$$

We then have

$$\Theta = -2\frac{\partial\psi}{\partial\zeta}\ ; \quad \Phi = 2\frac{\partial\psi}{\partial\bar{\zeta}}\ . \tag{98}$$

The complex functions $\chi, \theta$ can be related to the Airy stress function of equation (69). We first integrate the function $\theta$ to obtain a holomorphic function $g$ such that $g' = \theta$. The corresponding Airy function can then be written down as

$$\phi = -\frac{1}{2}\left(g + \bar{g} + \bar{\zeta}\chi + \zeta\bar{\chi}\right) \tag{99}$$

and it can be expressed as a (real) function of $x_1, x_2$ using (85). We can also perform the inverse procedure of determining the complex potentials $\chi, \theta$ equivalent to a given Airy function $\phi$. We first express $\phi$ as a function of $\zeta, \bar{\zeta}$, using (86). Differentiating (99) with respect to $\zeta$ and $\bar{\zeta}$, we then obtain

$$\chi' + \bar{\chi}' = \frac{\partial^2\phi}{\partial\zeta\partial\bar{\zeta}}\ , \tag{100}$$

which is easily solved for $\chi$. Once $\chi$ is known, it can be substituted into (99) and the resulting equation solved for $g$ and hence $\theta$. This is a convenient way of determining the displacements associated with a given Airy stress function.

### 7.3.1. Boundary tractions

The boundary tractions on an arbitrary boundary can be written in the complex form

$$T(s) \equiv T_1 + \imath T_2\ , \tag{101}$$

where $s$ is a curvilinear coordinate along the boundary. Equilibrium of a small triangular element then shows that

$$T_1 ds = \sigma_{11}dx_2 - \sigma_{12}dx_1\ ; \quad T_2 ds = \sigma_{12}dx_2 - \sigma_{22}dx_1 \tag{102}$$

and hence

$$Tds = (\sigma_{11} + \imath\sigma_{12})dx_2 - (\sigma_{12} + \imath\sigma_{22})dx_1 = \frac{1}{2}(\Theta + \Phi)dx_2 - \frac{\imath}{2}(\Theta - \Phi)dx_1\ , \tag{103}$$

using the notation of equation (95).

Substituting for $\Theta, \Phi$ from (98) and rearranging the terms, we then have

$$Tds = \imath\left\{\frac{\partial\psi}{\partial\bar{\zeta}}(dx_1 - \imath dx_2) + \frac{\partial\psi}{\partial\zeta}(dx_1 + \imath dx_2)\right\} = \imath\left\{\frac{\partial\psi}{\partial\bar{\zeta}}d\bar{\zeta} + \frac{\partial\psi}{\partial\zeta}d\zeta\right\} = \imath d\psi\ , \tag{104}$$

or

$$T = \imath\frac{d\psi}{ds}\ . \tag{105}$$

Thus, the boundary values of $\psi$ can be obtained directly from the tractions and in particular, $\psi$ will be constant along any region of boundary that is traction-free.

### 7.3.2. Laurent series and conformal mapping

A general solution for the problem of the annulus $a \leq |\zeta| \leq b$ can be obtained by writing the holomorphic functions $\chi, \theta$ in the form of *Laurent series*

$$\chi = \sum_{-\infty}^{\infty} A_n\zeta^n\ ; \quad \theta = \sum_{-\infty}^{\infty} B_n\zeta^n\ , \tag{106}$$

where $A_n, B_n$ are a set of complex constants. This is essentially the complex-variable equivalent of the Michell solution (77). Alternatively, the complex functions can be obtained directly from the boundary conditions making use of the properties of Cauchy integrals. The solutions for bodies of a wide range of other shapes can then be obtained using the technique of *conformal mapping*. These methods require results from the theory of functions of the complex variable that are beyond the scope of this chapter.

### 7.4. Antiplane problems

We shall describe a deformation field as antiplane if the only non-zero displacement and stress components are $u_3, \sigma_{31}, \sigma_{32}$ and these are all independent of $x_3$. In this case, the stress components are

$$\sigma_{31} = \mu \frac{\partial u_3}{\partial x_1} \; ; \quad \sigma_{32} = \mu \frac{\partial u_3}{\partial x_2} \; , \tag{107}$$

from (27) and the only non-trivial equilibrium equation reduces to

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + p_3 = 0 \; . \tag{108}$$

Substituting (107) into (108) we find that $u_3$ must satisfy the two-dimensional Poisson equation

$$\frac{\partial^2 u_3}{\partial {x_1}^2} + \frac{\partial^2 u_3}{\partial {x_2}^2} = -\frac{p_3}{\mu} \; . \tag{109}$$

If there are no body forces ($p_3 = 0$), the stresses can be represented in terms of *Prandtl's stress function* $\varphi$ through

$$\sigma_{31} = \frac{\partial \varphi}{\partial x_2} \; ; \quad \sigma_{32} = -\frac{\partial \varphi}{\partial x_1} \; . \tag{110}$$

If we define a coordinate system local to a point on the boundary such that $n, t$ are respectively normal and tangential to the boundary, the boundary traction will be $\sigma_{3n}$ and this will be zero if the *tangential* gradient

$$\frac{\partial \varphi}{\partial t} = 0 \; . \tag{111}$$

It follows that lines of constant $\varphi$ represent traction-free surfaces and this is very convenient in the solution of boundary-value problems.

Equating corresponding stress components from (107, 110) respectively, we obtain

$$\mu \frac{\partial u_3}{\partial x_1} = \frac{\partial \varphi}{\partial x_2} \; ; \quad \mu \frac{\partial u_3}{\partial x_2} = -\frac{\partial \varphi}{\partial x_1} \; , \tag{112}$$

which are recognizable as examples of the Cauchy-Riemann equations, showing that $\mu u_3, \varphi$ can be combined to form the real and imaginary parts of a holomorphic function of the complex variable $\zeta$. It also follows that $\varphi$ is harmonic

$$\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{\partial^2 \varphi}{\partial {x_2}^2} = 0 \; , \tag{113}$$

when there are no body forces.

## 8. Solution of boundary-value problems

The stress functions introduced in §7 effectively reduce the typical two-dimensional elasticity problem to a boundary-value problem for one or more potentials. For example, the Airy function representation of §7.2 reduces the problem to the determination of a real scalar potential $\phi$ that satisfies the biharmonic equation (70) and for which appropriate derivatives take specified values on the boundaries.

Classical techniques for the solution of such problems for real stress functions generally start with the search for separated-variable solutions of the governing equation. For example, in Cartesian coordinates, it is easily verified that the function

$$\phi(x_1, x_2, \lambda) = \Big[(A + Bx_2)\cosh(\lambda x_2) + (C + Dx_2)\sinh(\lambda x_2)\Big]\cos(\lambda x_1) \tag{114}$$

satisfies (70) for all values of $A, B, C, D$ and $\lambda$. More general biharmonic functions can then be written down as an integral such as

$$\phi(x_1, x_2) = \int_0^\infty \Big[(A(\lambda) + B(\lambda)x_2)\cosh(\lambda x_2) + (C(\lambda) + D(\lambda)x_2)\sinh(\lambda x_2)\Big]\cos(\lambda x_1)d\lambda \;, \tag{115}$$

which can be regarded as the superposition of terms such as (114) with different values of $A, B, C, D$, $\lambda$. The form (115) is of course equivalent to taking the Fourier cosine transform of the problem and it has sufficient degrees of freedom to satisfy the boundary conditions on the two edges ($x_2 = \pm b$) of the rectangular body $a < x_1 < a, -b < x_2 < b$ exactly if the boundary conditions are symmetric about $x_2 = 0$, since there are four functions $A, B, C, D$ to satisfy four boundary conditions (two tractions on each edge). Antisymmetric problems would require the cosine to replaced by a sine and unsymmetric problems can be decomposed into the sum of a symmetric and an unsymmetric problem.

The boundary conditions on the other edges $x_1 = \pm a$ then remain to be satisfied. In some special cases, this can be done by restricting $\lambda$ to specific values leading to a series solution. For example, the function defined by the Fourier series

$$\phi(x_1, x_2) = \sum_{n=0}^{\infty} \left[(A_n + B_n x_2)\cosh\left(\frac{n\pi x_2}{a}\right) + (C_n + D_n x_2)\sinh\left(\frac{n\pi x_2}{a}\right)\right]\cos\left(\frac{n\pi x_1}{a}\right) \tag{116}$$

is symmetrical about each of the planes $x_1 = \pm a$ and will therefore identically satisfy the symmetry boundary conditions

$$\sigma_{12} = 0 \;; \quad u_1 = 0 \;; \quad x_1 = \pm a \;. \tag{117}$$

## 8.1. The corrective problem

To complete the solution, we usually need to solve a boundary-value problem with traction-free conditions on the edges $x_2 = \pm b$ and prescribed non-zero tractions on $x_1 = \pm a$. The classical approach is to solve this 'corrective' problem in two steps. We first superpose the stress function

$$\phi_1 = C_1 x_2^2 + C_2 x_2^3 + C_3(x_1 x_2^3 - 3b^2 x_1 x_2) \;, \tag{118}$$

for which the stress components are

$$\sigma_{11} = 2C_1 + 6C_2 x_2 + 6C_3 x_1 x_2 \;; \quad \sigma_{12} = 3C_3(b^2 - x_2^2) \;; \quad \sigma_{22} = 0 \;, \tag{119}$$

from (69). It is easily verified that $\phi_1$ satisfies the biharmonic equation (70) and it is clear from (119) that it leaves the surfaces $x_2 = \pm b$ traction-free. After this superposition, the three free constants $C_1, C_2, C_3$ can then be used to satisfy the weak (force-resultant) boundary conditions (35) on $x_1 = \pm a$. Notice that if the weak boundary conditions are satisfied at one end $x_1 = a$, they will then automatically be satisfied at the other end $x_1 = -a$. This arises because the Airy stress function ensures that every particle of the body satisfies the equilibrium equations (16) and hence the whole body must also be in equilibrium. Thus, if weak or strong boundary conditions are satisfied over any region $\Gamma_1$ of $\Gamma$, weak boundary conditions must necessarily be satisfied over the remainder of the boundary $\Gamma - \Gamma_1$.

### 8.2. The Saint-Venant problem

Saint-Venant's principle (see §5.1) suggests that the error incurred by satisfying equations (35) instead of the *strong* (pointwise) boundary conditions (34) will decay with distance from $x_2 = a$ and will be small at a distance that is comparable with $b$. To investigate the nature of these decaying fields, we consider the problem of the semi-infinite strip $x_1 > 0, -b < x_2 < b$ subject only to a set of self-equilibrated tractions on the end $x_1 = 0$. We anticipate that the stresses will decay with increasing $x_1$ and hence we start by seeking a stress function of the separated variable form

$$\phi = f(x_2)\exp(-\lambda x_1) \ , \tag{120}$$

where $\lambda$ is an unknown parameter representing the decay rate. Substituting into the biharmonic equation (70), cancelling the common exponential factor, and solving the resulting ordinary differential equation for $f(x_2)$, we obtain

$$f(x_2) = A_1\cos(\lambda x_2) + A_2\sin(\lambda x_2) + A_3x_2\cos(\lambda x_2) + A_4x_2\sin(\lambda x_2) \ . \tag{121}$$

Notice that the first and last terms are even functions of $x_2$, whereas the second and third terms are odd functions. The symmetry of the geometry about $x_2 = 0$ naturally partitions the problem into a symmetric and an antisymmetric problem. We require the resulting stress field to satisfy the traction-free condition on $x_2 = \pm b$ and after substituting (121) into (69), this provides four homogeneous linear algebraic equations (two tractions to be zero on each of two surfaces) for the four constants $A_1, A_2, A_3, A_4$. These equations have a non-trivial solution if and only if the characteristic determinantal equation

$$(\sin(2\lambda b) + 2\lambda b)(\sin(2\lambda b) - 2\lambda b) = 0 \tag{122}$$

is satisfied. This equation has a denumerably infinite set of eigenvalues $\lambda_i$, for each of which there exists a non-trivial eigenfunction

$$\phi_i(x_1, x_2) = C_i f_i(x_2)\exp(-\lambda_i x_1) \ , \tag{123}$$

where $C_i$ is the one remaining free constant (since the degeneracy associated with the determinant being zero generally reduces the system of four algebraic equations to three). We can then construct a more general stress function as an eigenfunction series

$$\phi(x_1, x_2) = \sum_{i=1}^{\infty} C_i f_i(x_2)\exp(-\lambda_i x_1) \ . \tag{124}$$

R.D.Gregory has shown that this function provides a general solution to the problem where the semi-infinite strip is loaded by arbitrary self-equilibrated tractions on the end $x_1 = 0$.

The eigenvalues $\lambda_i$ of (122) are all complex and hence the fields corresponding to each term in (124) exhibit oscillatory decaying behaviour with $x_1$, the decay rate corresponding to the real part of $\lambda_i$. The eigenvalue with the smallest (negative) real part defines the most slowly decaying term and hence the region over which the use of the weak boundary conditions has a significant influence on the solution. The lowest eigenvalues correspond to $\Re(\lambda)b = 2.1$ for the symmetric problem and to $\Re(\lambda)b = 3.7$ for the antisymmetric problem.

## 9. The prismatic bar under shear and torsion

Consider a prismatic bar with axis in the $x_3$-direction, whose cross-section $\Omega$ is defined by a closed curve $\Gamma$ in the $x_1x_2$-plane. If the surface $\Gamma$ is traction-free, the most general state of stress corresponds to the transmission of arbitrary force resultants (three forces and three moments) along the bar. If these resultants are applied at the end $x_3 = 0$ (say), Saint-Venant's principle assures us that the stress field will depend on the exact traction distribution (the strong boundary conditions) only in a region near the end. In this section, we shall discuss the stress state distant from the ends, which therefore has only six degrees of freedom corresponding to the six resultants.

Three of these resultants — the axial force $F_3$ and the two bending moments $M_1, M_2$ cause only the simple state of stress

$$\sigma_{33} = C_1 + C_2 x_1 + C_3 x_2 \; , \tag{125}$$

where the remaining stress components are everywhere zero and $C_1, C_2, C_3$ are constants determined from equilibrium considerations. This stress field clearly satisfies the traction-free boundary condition on $\Gamma$ and the strains derived from Hooke's law (24) are linear functions of the coordinates which therefore satisfy the compatibility equations (11).

### 9.1. Torsion

If the bar transmits only a torque $M_3$, every segment (except near the ends) is loaded in the same way, so the stress field will be independent of $x_3$. However, the bar will twist under the action of the torque, so adjacent transverse planes will experience a relative rigid body rotation, leading to displacements of the form

$$u_1 = -\beta x_3 x_2 \; ; \quad u_2 = \beta x_3 x_1 \; , \tag{126}$$

where $\beta$ is the angle of twist per unit length. If these were the only displacements (i.e. if $u_3 = 0$) the resulting stress components would not generally satisfy the traction-free boundary condition on $\Gamma$. Saint-Venant recognized that cross-sectional planes would all warp to the same shape, so that

$$u_3 = f(x_1, x_2) \; . \tag{127}$$

Substitution of these results into Hooke's law (27) yields

$$\sigma_{31} = \mu \left( \frac{\partial f}{\partial x_1} - \beta x_2 \right) \; ; \quad \sigma_{32} = \mu \left( \frac{\partial f}{\partial x_2} + \beta x_1 \right) \tag{128}$$

as the only non-zero stress components. Substitution into the equilibrium equations then shows that the warping function $f$ must satisfy the two-dimensional Laplace equation

$$\frac{\partial^2 f}{\partial {x_1}^2} + \frac{\partial^2 f}{\partial {x_2}^2} = 0 \; . \tag{129}$$

Solution of the resulting boundary-value problem is greatly facilitated by the use of Prandtl's stress function of equation (110). Equating the corresponding stress components between equations (128) and (110) and eliminating the unknown function $f$, we find that the Prandtl function $\varphi$ must satisfy the equation

$$\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{\partial^2 \varphi}{\partial {x_2}^2} = -2\mu\beta \; . \tag{130}$$

Also, in view of (111) the traction-free boundary $\Gamma$ must be a line of constant $\varphi$ and for a simply-connected bar this constant can be taken as zero without loss of generality, giving

$$\varphi = 0 \; ; \quad \text{on } \Gamma \; . \tag{131}$$

The torque $M_3$ is given by

$$M_3 = \int_\Omega (x_1\sigma_{32} - x_2\sigma_{31})dx_1dx_2 = -\int_\Omega \left(x_1\frac{\partial\varphi}{\partial x_1} + x_2\frac{\partial\varphi}{\partial x_2}\right)dx_1dx_2\ , \tag{132}$$

using (110). Integrating by parts and using (131), we obtain

$$M_3 = 2\int_\Omega \varphi dx_1dx_2\ . \tag{133}$$

The torsion problem is thus reduced to the search for a function $\varphi$ satisfying (130) in the two-dimensional domain $\Omega$ that satisfies the boundary condition (131). This is a standard boundary-value problem in potential theory. One technique is to choose any particular solution of (130) (for example a second degree polynomial in $x_1, x_2$) and superpose a general two-dimensional harmonic function in the form of the real or imaginary part of a holomorphic function of the complex variable $\zeta$ — e.g.

$$\varphi = -\mu\beta x_1^2 + \Re\left(g(\zeta)\right)\ . \tag{134}$$

The function $g(\zeta)$ can then be chosen so as to satisfy the boundary condition $\varphi = 0$ on $\Gamma$.

**Example: Equilateral triangular cross-section**

Solutions for several simple geometries can be obtained in closed form. For example, it is easily verified that the polynomial stress function

$$\varphi = \frac{\mu\beta}{2\sqrt{3}a}(x_1^2 - 3x_2^2)(\sqrt{3}a - 2x_1) \tag{135}$$

satisfies (130) and it clearly goes to zero on the three straight lines $x_1 = \sqrt{3}x_2$, $x_1 = -\sqrt{3}x_2$ and $x_1 = \sqrt{3}a/2$, defining the boundary of an equilateral triangle of side $a$. Substituting (135) into (133) and evaluating the integral, we find that the torque transmitted is related to the twist per unit length $\beta$ through

$$M_3 = \frac{\sqrt{3}\mu\beta a^4}{80}\ . \tag{136}$$

### 9.1.1. Multiply-connected bodies

If the cross-sectional domain $\Omega$ is multiply-connected — i.e. if the bar has one or more axial holes — the boundary $\Gamma$ will comprise two or more closed curves $\Gamma_1, \Gamma_2, ....$ The traction-free boundary condition requires that $\varphi$ be constant around each of these curves, but the values on the separate curves are not necessarily equal, so only one of them can be set to zero. We therefore have one additional unknown constant for each hole in the body. The equations for determining these constants are obtained from the Cesaro integrals of §1.4.

## 9.2. Shear and bending

If a transverse shear force with components $F_1, F_2$ is transmitted along the bar, the global equilibrium equations require that

$$\frac{dM_1}{dx_3} = F_2\ ; \quad \frac{dM_2}{dx_3} = -F_1 \tag{137}$$

and hence the bending moments $M_1, M_2$ must be linear functions of $x_3$. It can be shown that the resulting axial stresses $\sigma_{33}$ are still linear functions of $x_1, x_2$, so that (125) is generalized to

$$\sigma_{33} = C_1 + C_2 x_1 + C_3 x_2 + (C_4 x_1 + C_5 x_2) x_3 \ , \tag{138}$$

where the constants $C_1, ...C_5$ can be determined from equilibrium considerations. The in-plane stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ remain zero and the Prandtl stress function of equation (110) must be modified to satisfy the differential equations of equilibrium (16) (with no body force) by writing

$$\sigma_{31} = \frac{\partial \varphi}{\partial x_2} - \frac{C_4 x_1^2}{2} \ ; \quad \sigma_{32} = -\frac{\partial \varphi}{\partial x_1} - \frac{C_5 x_2^2}{2} \ . \tag{139}$$

Substitution in the compatibility equations then show that $\varphi$ must satisfy the governing equation

$$\frac{\partial^2 \varphi}{\partial {x_1}^2} + \frac{\partial^2 \varphi}{\partial {x_2}^2} = \frac{\nu}{(1+\nu)}(C_4 x_2 - C_5 x_1) + C_6 \ , \tag{140}$$

where $C_6$ is an arbitrary constant which will be zero if no torque is transmitted. If $C_4, C_5$ are set to zero, but $C_6$ is retained, the formulation reduces to the Saint-Venant torsion problem of §9.1. Since the terms involving $C_4, C_5$ are known functions, the resulting boundary-value problem is similar to that encountered in §9.1 except that a different particular integral is required for the inhomogeneous equation (140) and $\varphi$ will now be a known function rather than zero on the boundary $\Gamma$. S.P.Timoshenko developed a modified form of this representation so as to reinstate the homogeneous boundary condition on $\Gamma$.

## 10. Three-dimensional problems

For three-dimensional problems, it is more conventional to seek a potential function representation of the displacement vector $u_i$, in which case the governing equation for the corresponding potential is obtained by substitution into (31). For example, if the displacement is defined such that

$$2\mu u_i = 2(1-\nu)\frac{\partial^2 F_i}{\partial x_j \partial x_j} - \frac{\partial^2 F_j}{\partial x_i \partial x_j} \ , \tag{141}$$

substitution into (31) using (29) shows that the vector potential $F_i$ must satisfy the equation

$$\nabla^4 F_i \equiv \frac{\partial^4 F_i}{\partial x_j \partial x_j \partial x_k \partial x_k} = -\frac{p_i}{(1-\nu)} \tag{142}$$

and in particular that $F_i$ is biharmonic if there is no body force. Expressions for the stress components in terms of $F_i$ can be obtained by substituting (141) into the constitutive law (27), giving

$$\sigma_{ij} = \nu \delta_{ij} \frac{\partial^3 F_k}{\partial x_k \partial x_l \partial x_l} + (1-\nu)\frac{\partial^2}{\partial x_k \partial x_k}\left(\frac{\partial F_i}{\partial x_j} + \frac{\partial F_j}{\partial x_i}\right) - \frac{\partial^3 F_k}{\partial x_i \partial x_j \partial x_k} \ . \tag{143}$$

The potential $F_i$ is sometimes known as the *Galerkin vector*. For the special case where the stress and displacement fields are symmetric about the $x_3$-axis, $F_i$ can be reduced to a single biharmonic axisymmetric potential $F_3$. This formulation of axisymmetric problems was first introduced by A.E.H.Love.

### 10.1. The Papkovich-Neuber solution

If a function $f(x_1, x_2, x_3)$ is harmonic, then

$$\frac{\partial^2}{\partial x_j \partial x_j}(x_i f) = 2\frac{\partial f}{\partial x_i} + x_i \frac{\partial^2 f}{\partial x_j \partial x_j} = 2\frac{\partial f}{\partial x_i} \tag{144}$$

and it follows that the function $x_i f$ is biharmonic. This relation enables us to write a fairly general biharmonic function in the form $x_i f + g$ where $f, g$ are two harmonic functions and it can be used to replace the biharmonic functions in the Galerkin solution (141) by harmonic functions, leading to the representation

$$2\mu u_i = -4(1-\nu)\psi_i + \frac{\partial}{\partial x_i}\left(x_j\psi_j + \phi\right) \ , \tag{145}$$

generally known as the Papkovich-Neuber solution. Substitution in (31) using (29) shows that the vector potential $\psi_i$ and the scalar potential $\phi$ must satisfy the equation

$$x_k \frac{\partial^3 \psi_k}{\partial x_i \partial x_j \partial x_j} - (1-4\nu)\frac{\partial^2 \psi_i}{\partial x_k \partial x_k} + \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_j} = -\frac{(1-2\nu)p_i}{(1-\nu)} \tag{146}$$

and a particular solution can be obtained by choosing

$$\frac{\partial^2 \psi_i}{\partial x_j \partial x_j} = \frac{p_i}{2(1-\nu)} \ ; \quad \frac{\partial^2 \phi}{\partial x_j \partial x_j} = -\frac{x_j p_j}{2(1-\nu)} \ . \tag{147}$$

If the body force $p_i$ is conservative, so that we can write

$$p_i = -\frac{\partial V}{\partial x_i} \ , \tag{148}$$

where $V$ is a scalar body force potential, a simpler representation can be obtained by choosing

$$\frac{\partial^2 \psi_i}{\partial x_j \partial x_j} = 0 \ ; \quad \frac{\partial^2 \phi}{\partial x_j \partial x_j} = \frac{(1-2\nu)V}{(1-\nu)} \ . \tag{149}$$

If there are no body forces, the Papkovich-Neuber potentials $\psi_i, \phi$ are both harmonic and this is convenient because it enables one to call on the extensive mathematical literature concerning the theory of harmonic potentials in the solution of boundary-value problems. The expressions for the stress components can then be obtained from (145, 27) as

$$\sigma_{ij} = -2\nu\delta_{ij}\frac{\partial \psi_k}{\partial x_k} + x_k \frac{\partial^2 \psi_k}{\partial x_i \partial x_j} - (1-2\nu)\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right) + \frac{\partial^2 \phi}{\partial x_i \partial x_j} \ . \tag{150}$$

### 10.2. Redundancy and completeness

The perceptive reader will notice that if (148) is substituted into (147), the resulting equations do not reduce to (149). This occurs because the representation (145) is redundant. In other words, more than one set of potentials can be found for a given displacement field and this redundancy can sometimes be used to simplify the resulting expressions. Similar considerations apply to the Galerkin formulation (141). In problems governed by a single scalar harmonic potential (for example, in steady-state heat conduction), a boundary-value problem is well-posed if and only if one boundary condition is specified at every point on the boundary. In three-dimensional elasticity, we have three boundary conditions at each point on the boundary (for example, the three traction components, the three displacement components, or some combination of each). It is natural therefore to expect that a complete representation of the displacement field should be possible using only three harmonic potential functions, rather than the four functions comprising the three components $\psi_i$ and $\phi$. Many authors have attempted to develop such a representation. H.Neuber originally proposed a method for eliminating any one of the four components, but I.S.Sokolnikoff showed that this procedure fails unless the shape of the body $\Omega$ meets certain conditions. One component of $\psi_i$ (say $\psi_j$) can be eliminated only if any straight line segment parallel to the $x_j$-axis which connects two points of $\Omega$ must lie wholly within $\Omega$. The scalar potential $\phi$ can be eliminated only if (i) $\nu \neq 1/4$ and (ii) the body $\Omega$ is 'star shaped', meaning that there exists at least one point in the interior of $\Omega$ such that a straight line drawn from this point to any other point in $\Omega$ lies entirely within $\Omega$.

### 10.3. Solutions for the thick plate and the prismatic bar

Two special geometries meeting the condition for the elimination of the component $\psi_3$ in equation (145) are the thick plate bounded by the parallel planes $x_3 = \pm h/2$ and the prismatic bar whose axis is aligned with the $x_3$-direction. The remaining boundary of the plate or the curve defining the cross-section of the bar can be of quite general form, including cases of multiply-connected bodies.

For problems of this class, it is convenient to make use of an extension of the complex variable formalism introduced in §7.3. We combine the non-zero components $\psi_1, \psi_2$ into the complex harmonic function

$$\psi = \psi_1 + \imath\psi_2 \ , \tag{151}$$

after which the displacement and stress components from (145, 150) can be written

$$2\mu u = 2\frac{\partial\phi}{\partial\bar{\zeta}} - (3-4\nu)\psi + \zeta\frac{\partial\bar{\psi}}{\partial\bar{\zeta}} + \bar{\zeta}\frac{\partial\psi}{\partial\bar{\zeta}} \ ; \quad 2\mu u_3 = \frac{\partial\phi}{\partial x_3} + \frac{1}{2}\left(\bar{\zeta}\frac{\partial\psi}{\partial x_3} + \zeta\frac{\partial\bar{\psi}}{\partial x_3}\right) \ , \tag{152}$$

$$\Theta = -\frac{\partial^2\phi}{\partial {x_3}^2} - 2\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\bar{\psi}}{\partial\bar{\zeta}}\right) - \frac{1}{2}\left(\zeta\frac{\partial^2\bar{\psi}}{\partial {x_3}^2} + \bar{\zeta}\frac{\partial^2\psi}{\partial {x_3}^2}\right) \tag{153}$$

$$\Phi = 4\frac{\partial^2\phi}{\partial\bar{\zeta}^2} - 4(1-2\nu)\frac{\partial\psi}{\partial\bar{\zeta}} + 2\zeta\frac{\partial^2\bar{\psi}}{\partial\bar{\zeta}^2} + 2\bar{\zeta}\frac{\partial^2\psi}{\partial\bar{\zeta}^2} \tag{154}$$

$$\sigma_{33} = \frac{\partial^2\phi}{\partial {x_3}^2} - 2\nu\left(\frac{\partial\psi}{\partial\zeta} + \frac{\partial\bar{\psi}}{\partial\bar{\zeta}}\right) + \frac{1}{2}\left(\bar{\zeta}\frac{\partial^2\psi}{\partial {x_3}^2} + \zeta\frac{\partial^2\bar{\psi}}{\partial {x_3}^2}\right) \tag{155}$$

$$\Psi = 2\frac{\partial^2\phi}{\partial\bar{\zeta}\partial x_3} - (1-2\nu)\frac{\partial\psi}{\partial x_3} + \zeta\frac{\partial^2\bar{\psi}}{\partial\bar{\zeta}\partial x_3} + \bar{\zeta}\frac{\partial^2\psi}{\partial\bar{\zeta}\partial x_3} \ , \tag{156}$$

where

$$\Theta = \sigma_{11} + \sigma_{22} \ ; \quad \Phi = \sigma_{11} + 2\imath\sigma_{12} - \sigma_{22} \ ; \quad \Psi = \sigma_{31} + \imath\sigma_{32} \ . \tag{157}$$

### 10.4. Solutions in spherical harmonics

The Papkovich-Neuber solution (145, 150) reduces the elasticity problem to the determination of several harmonic potential functions with prescribed derivatives on the boundaries. For axisymmetric bodies such as cylinders, spheres, spherical holes and cones (including the half space), appropriate potentials can be expressed in terms of the *spherical harmonics*

$$R^n P_n^m(\cos(\beta))\exp(\imath m\theta) \ ; \quad R^n Q_n^m(\cos(\beta))\exp(\imath m\theta)$$

$$R^{-n-1} P_n^m(\cos(\beta))\exp(\imath m\theta) \ ; \quad R^{-n-1} Q_n^m(\cos(\beta))\exp(\imath m\theta) \ ,$$

where $R, \theta, \beta$ are spherical polar coordinates, $m, n$ are positive integers and $P_n^m, Q_n^m$ are the two solutions of *Legendre's equation*

$$(1-x^2)\frac{d^2 f}{dx^2} - 2x\frac{df}{dx} + \left(n(n+1) - \frac{m^2}{(1-x^2)}\right) f = 0 \ . \tag{158}$$

The same functions can also be expressed in cylindrical polar coordinates $(r, \theta, z)$ or Cartesian coordinates, using the relations

$$R = \sqrt{r^2 + z^2} = \sqrt{x_i x_i} \; ; \quad \theta = \arctan\left(\frac{x_2}{x_1}\right) \; ; \quad \beta = \arccos\left(\frac{z}{R}\right) = \arccos\left(\frac{x_3}{\sqrt{x_i x_i}}\right) \; .$$

The potentials $R^n P_n^m(\cos(\beta)) \exp(\imath m\theta)$ are bounded functions that can be expanded as polynomials in $x_1, x_2, x_3$, whilst $R^{-n-1} P_n^m(\cos(\beta)) \exp(\imath m\theta)$ are singular at the origin $R = 0$. If each of the Papkovich-Neuber potentials is expressed as a series including these terms for all integer values of $m, n$, the resulting field provides a general solution for the problem for the hollow sphere with arbitrary loading on the boundaries. If the loading is also axisymmetric, a sufficiently general solution can be obtained by restricting the series to the terms with $m = 0$ and using only the potentials $\phi, \psi_3$ from Papkovich-Neuber's solution. For the solid sphere, the singular potentials must be excluded. This solution can be used in much the same way as Michell's solution (77) in two dimensions.

The spherical harmonics involving the functions $Q_n^m$ are singular at all points on the $x_3$-axis. These and other functions derived from them can be used to solve problems involving hollow cylinders and cones.

## 11. Concentrated forces and dislocations

### 11.1. The Kelvin problem

Consider *Kelvin's problem* in which a concentrated force $\boldsymbol{F}$ acts at an interior point of an infinite elastic body. There is no length scale in this problem and we can conclude that the stress field is *self-similar*. In other words, contours defining the stresses at some distance $R_1$ from the point of application of the force will have the same shape as those at some other radius $R_2$. It follows that the stress components in spherical polar coordinates can be written in the separated variable form

$$\sigma_{ij} = f(R) g_{ij}(\theta, \beta) \; . \tag{159}$$

If we then consider the equilibrium of the sphere $R < a$, we have

$$F_i + \int_\Gamma n_j \sigma_{ij} d\Gamma = 0 \tag{160}$$

where the elemental surface area $d\Gamma = a^2 \sin\beta d\theta d\beta$. Substituting this result and (159) into (160), we obtain

$$F_i + \int_\Gamma n_j f(a) g_{ij}(\theta, \beta) a^2 \sin\beta d\theta d\beta = 0 \tag{161}$$

and this will be satisfied for all $a$ if and only if $f(R)$ is proportional to $R^{-2}$ and we can wrap the constant of proportionality into $g_{ij}$ without loss of generality, leaving $f(R) = R^{-2}$. Thus the stresses are singular as $R \to 0$. It follows from Hooke's law that the strain components will also vary with $R^{-2}$ and since strains are defined in terms of displacement gradients through (6), the displacement components must be proportional to $R^{-1}$.

If we then use equations (39, 40) to calculate the strain energy in the sphere of radius $a$, the resulting integral

$$U = \int_0^\pi \int_0^{2\pi} \int_0^a s_{ijkl} \sigma_{ij} \sigma_{kl} R^2 dR \sin\beta d\theta d\beta = \int_0^\pi \int_0^{2\pi} \int_0^a s_{ijkl} g_{ij} g_{kl} R^{-2} dR \sin\beta d\theta d\beta \tag{162}$$

is unbounded. This causes the concentrated force problem to be ill-posed in the context of variational methods, as explained in §6.5.1. However, in the spirit of Saint-Venant's principle (§5.1), the resulting stress fields can be regarded as a limiting form of the effect of a localized distribution of body force at a distance that is large compared with the extent of the loaded region. Also, concentrated forces can be used as Green's functions to describe distributions of external forces, in which case the unacceptable singularity is removed by the corresponding convolution integral.

The solution of Kelvin's problem is very easily obtained using the Papkovich-Neuber solution of equations (145, 150). We first note that the stresses are obtained by one differentiation of the vector potential $\psi_i$, so in order to obtain stress components varying with $R^{-2}$, we need to start with a potential of the form $R^{-1}f(\theta,\beta)$. The potential is also required to be harmonic and this can be achieved by setting $m=n=0$ in the singular $P$-series spherical harmonic of §10.4, giving

$$\psi_i = \frac{C_i}{R} = \frac{C_i}{\sqrt{x_j x_j}} \ , \tag{163}$$

where $C_i$ are arbitrary constants which can be determined by (i) finding the stress components by substituting (163) into (150), (ii) substituting these into the equilibrium equation (161), and (iii) solving the resulting system of equations. We obtain

$$C_i = -\frac{F_i}{8\pi(1-\nu)} \tag{164}$$

and the corresponding stress components in Cartesian coordinates are

$$\sigma_{ij} = -\frac{1}{8\pi(1-\nu)}\left(\frac{(1-2\nu)(F_i x_j + F_j x_i - F_k x_k \delta_{ij})}{(x_l x_l)^{3/2}} + \frac{3F_k x_i x_j x_k}{(x_l x_l)^{5/2}}\right) \ . \tag{165}$$

This solution can be used as a Green's function to obtain a particular solution — i.e. a solution that satisfies the governing equations in the the body $\Omega$, but not necessarily the boundary conditions on the surface $\Gamma$ — for the case where there exists a general distribution of body force $p_i(x_1,x_2,x_3)$. The external force applied in some small region $\delta x_1\delta x_2\delta x_3$ is $p_i\delta x_1\delta x_2\delta x_3$ and by superposition, we can sum the effect of the entire body force distribution to obtain

$$\begin{aligned}\sigma_{ij} \;\; = \;\; & -\frac{1}{8\pi(1-\nu)}\int_\Omega\left(\frac{(1-2\nu)(p_i(x_j-\xi_j)+p_j(x_i-\xi_i)-p_k(x_k-\xi_k)\delta_{ij})}{[(x_l-\xi_l)(x_l-\xi_l)]^{3/2}}\right.\\ & \left.+\frac{3p_k(x_i-\xi_i)(x_j-\xi_j)(x_k-\xi_k)}{[(x_l-\xi_l)(x_l-\xi_l)]^{5/2}}\right)d\xi_1 d\xi_2 d\xi_3 \ .\end{aligned} \tag{166}$$

Once this integral has been evaluated, the corresponding tractions on the boundary $\Gamma$ from (32) can be compared with the required values and a corrective problem can be defined which when superposed on (166) gives the solution of the original problem. The corrective problem involves no body force, so this technique essentially reduces a body force problem to one without body force.

### 11.2. The problems of Boussinesq and Cerrutti

A related problem of considerable interest is that in which a concentrated force $F_i$ is applied at a point $P$ on the boundary of the body. We can choose the inward normal to the body at $P$ to be the $x_3$-direction without loss of generality. If we focus attention on a region very close to $P$, the remaining boundaries of the body are relatively distant and if the local region of the boundary has a smoothly turning tangent, we can to a first approximation represent the body as the *half space* $x_3>0$. If the boundary has some curvature, this approximation may still involve errors that are singular as the point of application of the force is approached, but the singularity is weaker than that in the half-space approximation. If the force is normal to the plane surface $x_3=0$ (and hence the only non-zero component is $F_3$), the resulting stress field is axisymmetric. The solution to this problem was first given by J.Boussinesq. The companion problem in which the force is tangential to the surface (for example applied in the $x_1$-direction) was solved by V.Cerrutti.

As in §11.1, considerations of self-similarity show that the stress field for both Boussinesq and Cerrutti problems must have the form (159) with $f(R) = R^{-2}$. However, the solution is not represented by equation (165), since this does not leave the unloaded parts of the boundary $x_3 = 0$ traction-free. Solutions can be obtained by starting from the stress field (165) and superposing additional singular potentials to satisfy this boundary condition. An alternative approach is to establish relationships between the Papkovich-Neuber potentials that guarantee that appropriate traction components on the surface are zero, after which a single potential with the correct power law dependence on $R$ provides the solution. Yet another approach due to H.M.Westergaard starts from the observation that equation (165) *does* give zero tractions on the surface $x_3 = 0$ (except at the origin) in the special case where Poisson's ratio $\nu = 0.5$. Westergaard then uses a transformation known as the 'twinned-gradient method' to generalize the resulting stress field to other values of Poisson's ratio.

As with the Kelvin solution, the solutions due to Boussinesq and Cerrutti can be used as Green's functions to write integral expressions for the half space subjected to general distributions of surface traction. The same solutions can also be used to develop integral equation formulations for problems involving bodies of more general shape and form the basis of the *Boundary Element Method* for the numerical analysis of elastic bodies.

### 11.3. Two-dimensional point force solutions

Corresponding solutions can be developed for two-dimensional bodies in plane strain, using the Airy stress function of §7.2. Similarity and equilibrium considerations show that the stress components must vary with $r^{-1}$ and if the force acts in the direction $\theta = 0$, symmetry requires that $\phi$ be an even function of $\theta$. Solutions satisfying these conditions can be obtained from Michell's solution (77) by selecting the terms

$$\phi = A_{12} r \ln(r) \cos\theta + A_{13} r\theta \sin\theta \ . \tag{167}$$

We first consider the special case where $A_{12} = 0$ and the only non-zero constant is $A_{13}$. The corresponding stress components are then

$$\sigma_{rr} = \frac{2A_{13}\cos\theta}{r} \ ; \quad \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \tag{168}$$

and we observe that all surfaces $\theta = \alpha$ are traction-free. In particular, if we apply this solution to the half plane $-\pi/2 \leq \theta \leq \pi/2$, the surface will be traction-free except for a concentrated force at the origin. By considering the equilibrium of a semi-circular region, we find that the magnitude of this force is $\pi A_{13}$ and that it acts in the direction $\theta = -\pi$. More generally, we find that if a force $F$ of arbitrary direction is applied to the half plane at the origin, the resulting stress field is

$$\sigma_{rr} = -\frac{2F\cos\theta}{\pi r} \ ; \quad \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \ , \tag{169}$$

where the angle $\theta$ is measured from the line of action of the force. This is known as the *Flamant solution*.

The same stress function can be used to determine the stresses due to a point force acting at the apex of a wedge of arbitrary angle and it might be thought that we can proceed to the solution for a force applied to an interior point of an infinite body (the two-dimensional Kelvin problem) by selecting a wedge of subtended angle $2\pi$. However, in this case, we find that the Flamant stress field corresponds to elastic displacements that are discontinuous across the two faces of the wedge, which is unacceptable in a continuous body.

We therefore return to the more general stress function (167) and find the corresponding displacements by expressing the Airy function in terms of complex potentials using (99, 100) and then substituting these potentials into (94). We obtain

$$\begin{aligned} 2\mu u_r &= [2(1-\nu)A_{12}+(1-2\nu)A_{13}]\,\theta\sin\theta+[(1-2\nu)A_{12}+2(1-\nu)A_{13}]\ln(r)\cos\theta \\ 2\mu u_\theta &= [2(1-\nu)A_{12}+(1-2\nu)A_{13}]\,\theta\cos\theta-(A_{12}+A_{13})\sin\theta \\ &\quad -[(1-2\nu)A_{12}+2(1-\nu)A_{13}]\ln(r)\sin\theta\;, \end{aligned} \tag{170}$$

where an arbitrary rigid-body displacement has been omitted. The terms $\theta\sin\theta, \theta\cos\theta$ in these expressions are multivalued, since $\theta+2n\pi$ defines the same physical point for all integer values $n$. For the wedge, this difficulty can be overcome by defining a unique principal value for $\theta$, but this cannot be done for the infinite space without introducing a line along which at least one of the displacements is discontinuous. For the two-dimensional Kelvin problem, we must therefore eliminate the multivalued terms by setting $2(1-\nu)A_{12}+(1-2\nu)A_{13}=0$. The remaining constant is then determined from equilibrium considerations and we obtain

$$\phi=\frac{F}{2\pi}\left[\frac{(1-2\nu)r\ln(r)\cos\theta}{2(1-\nu)}-r\theta\sin\theta\right]\;, \tag{171}$$

with stress components

$$\sigma_{rr}=-\frac{(3-2\nu)F\cos\theta}{4\pi(1-\nu)r}\;;\quad \sigma_{r\theta}=\frac{(1-2\nu)F\sin\theta}{4\pi(1-\nu)r}\;;\quad \sigma_{\theta\theta}=\frac{(1-2\nu)F\cos\theta}{4\pi(1-\nu)r}\;, \tag{172}$$

where the angle $\theta$ is measured from the line of action of the force $F$.

### 11.4. Dislocations

The terms involving multi-valued displacements in equation (170) can be used to develop the solution for a *dislocation* at the origin. We consider the situation in which a cut is made along the semi-infinite line $\theta=0$, thus converting the infinite plane into a wedge of subtended angle $2\pi$. The two plane faces of the wedge are then forced apart by tractions chosen so as to ensure that a gap is opened up of uniform magnitude $\delta$. This condition is satisfied by demanding that

$$u_\theta(r,0)-u_\theta(r,2\pi)=\delta\;. \tag{173}$$

We then insert a thin strip of the same material of thickness $\delta$ into the gap and join up the surfaces, leaving a state of stress in the body. This is analogous to the defect in a crystal caused by the insertion of an additional row of atoms up to some interior point and is known as an *edge dislocation*. If we also impose the condition that no external force be applied at the origin, we find that $A_{13}=0$ and equations (173, 170) then give $A_{12}=-\mu\delta/2\pi(1-\nu)$ and

$$\phi=-\frac{\mu\delta}{2\pi(1-\nu)}r\ln(r)\cos\theta=-\frac{\mu\delta}{4\pi(1-\nu)}x_1\ln(x_1^2+x_2^2)\;. \tag{174}$$

The corresponding stresses are

$$\sigma_{rr}=\sigma_{\theta\theta}=-\frac{\mu\delta\cos\theta}{2\pi(1-\nu)r}\;;\quad \sigma_{r\theta}=-\frac{\mu\delta\sin\theta}{2\pi(1-\nu)r}\;. \tag{175}$$

or in Cartesian coordinates

$$\sigma_{11} = \frac{\mu\delta x_1(x_2^2 - x_1^2)}{2\pi(1-\nu)(x_1^2+x_2^2)^2} \; ; \quad \sigma_{12} = \frac{\mu\delta x_2(x_2^2 - x_1^2)}{2\pi(1-\nu)(x_1^2+x_2^2)^2}$$

$$\sigma_{22} = -\frac{\mu\delta x_1(x_1^2 + 3x_2^2)}{2\pi(1-\nu)(x_1^2+x_2^2)^2} \; . \tag{176}$$

In this solution, the choice of principal value and hence the location of the discontinuity in displacement is arbitrary. For example, the same stress function could be applied using a principal value for $\theta$ in the range $\pi/2 < \theta < 5\pi/2$. We would then find a discontinuity of magnitude $\delta$ in the tangential displacement $u_r$ at $\theta = \pi/2$, thus describing a *glide dislocation* of magnitude $\delta$ along this line.

Apart from giving a continuum approximation to the stress field due to a crystal dislocation, this solution can be used as a Green's function for the solution of two-dimensional crack problems. The technique is to place an as yet unknown distribution of climb and glide dislocations along the line of the crack and write the resulting stress field as a convolution integral. For example, if the crack is defined by the set of points $(\xi_1, \xi_2) \in \mathcal{S}$ and $s$ is a curvilinear coordinate defining position along $\mathcal{S}$, we can write

$$\begin{aligned} \phi(x_1,x_2) &= \phi_0(x_1,x_2) \\ + \int_S \Big[(x_1-\xi_1)B_1(s) &+ (x_2-\xi_2)B_2(s)\Big] \ln\Big[(x_1-\xi_1)^2 + (x_2-\xi_2)^2\Big]\, ds \end{aligned} \tag{177}$$

where $B_1, B_2$ are two unknown functions $s$ and $\phi_0(x_1, x_2)$ is a stress function characterizing the stresses due to the external loads in the absence of the crack. If the crack is open, the surfaces will be traction-free. This provides two conditions for each $s$ leading to a well-posed integral equation for $B_1(s), B_2(s)$. More general crack face conditions, such as a crack-bridging traction or frictional contact conditions can also be incorporated.

A similar technique can be used to treat two-dimensional problems in which a region $\Omega_P$ of an otherwise elastic body $\Omega$ experiences plastic deformation. In this case, dislocations are distributed over the domain $\Omega_P$ and their distribution is determined from the yield criterion and the corresponding flow rule. This has significant advantages in numerical implementations, since only the plastic zone needs to be discretized. Similar techniques have been applied to the analysis of crystal plasticity, where the evolution of the distribution is described by an appropriate rate law.

## 12. Asymptotic fields at singular points

One of the success stories of linear elastostatics in the last century has been its rôle as the basis of Linear Elastic Fracture Mechanics (LEFM) which is now widely used in design against brittle fracture. The method depends on the fact that the stresses in a cracked body increase without limit as we approach the crack tip. M.L.Williams developed a method for expanding the stress field in the vicinity of a crack tip or a sharp notch as an asymptotic series in which each term has power-law dependence on distance $r$ from the tip. It then follows that at sufficiently small values of $r$, only the first (i.e. most singular) term in this series makes a significant contribution to the stress field. The form of the local field is therefore the same for all problems involving sharp cracks and the multiplier on the most singular term, known as the *stress intensity factor* $K$ is a measure of the severity of stress state. Brittle failure is predicted when $K$ reaches some critical value $K_c$, known as the *fracture toughness* which can be determined from an appropriate laboratory experiment.

### 12.1. The eigenvalue problem

To explain the mathematics of the method, we first consider the semi-infinite wedge of Figure 2, bounded by the lines $\theta = \pm\alpha$ in polar coordinates $(r, \theta)$.

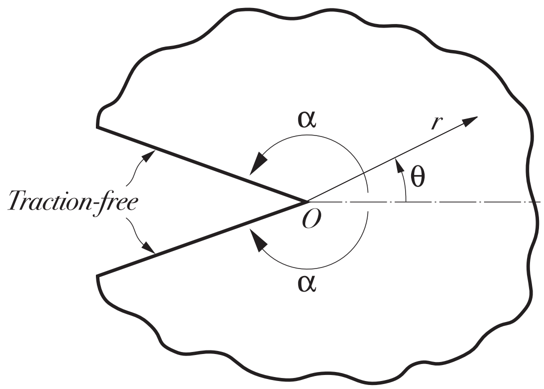


**Figure 2:** The semi-infinite wedge.

We seek a stress field of the separated variable form

$$\boldsymbol{\sigma} = r^{\lambda-1}\boldsymbol{f}(\theta) \ , \tag{178}$$

which in view of (75) implies that the Airy stress function $\phi$ takes the form

$$\phi = r^{\lambda+1}g(\theta) \ , \tag{179}$$

where $g$ is a scalar function of $\theta$ only. In the Michell solution (77), the Fourier terms were restricted to integer values of $n$ to guarantee that the stress components are single-valued functions of $\theta$, but this is not necessary for the wedge since we can define a principal value of a multivalued function in the range $-\alpha < \theta < \alpha$.

The most general function of the form (179) is

$$\phi = r^{\lambda+1}\{A_1\cos(\lambda+1)\theta + A_2\cos(\lambda-1)\theta + A_3\sin(\lambda+1)\theta + A_4\sin(\lambda-1)\theta\} \ , \tag{180}$$

and the corresponding stress components from (75) are

$$\begin{aligned}
\sigma_{rr} &= r^{\lambda-1}\{-A_1\lambda(\lambda+1)\cos(\lambda+1)\theta - A_2\lambda(\lambda-3)\cos(\lambda-1)\theta \\
&\quad -A_3\lambda(\lambda+1)\sin(\lambda+1)\theta - A_4\lambda(\lambda-3)\sin(\lambda-1)\theta\} \\
\sigma_{r\theta} &= r^{\lambda-1}\{A_1\lambda(\lambda+1)\sin(\lambda+1)\theta + A_2\lambda(\lambda-1)\sin(\lambda-1)\theta \\
&\quad -A_3\lambda(\lambda+1)\cos(\lambda+1)\theta - A_4\lambda(\lambda-1)\cos(\lambda-1)\theta\} \\
\sigma_{\theta\theta} &= r^{\lambda-1}\{A_1\lambda(\lambda+1)\cos(\lambda+1)\theta + A_2\lambda(\lambda+1)\cos(\lambda-1)\theta \\
&\quad +A_3\lambda(\lambda+1)\sin(\lambda+1)\theta + A_4\lambda(\lambda+1)\sin(\lambda-1)\theta\} \ ,
\end{aligned} \tag{181}$$

where $A_1, A_2, A_3, A_4$ are arbitrary constants. If $\lambda < 1$, the stress field will be singular as $r \to 0$ and the admissible strength of this singularity is limited by the requirement that the strain energy (40) in a finite region including the corner should be bounded. This condition can be shown by integration to require that $\lambda > 0$.

Substituting (181) into the traction-free boundary conditions

$$\sigma_{\theta r} = \sigma_{\theta\theta} = 0\ ; \quad \theta = \pm\alpha, \ \text{ all } r\ , \tag{182}$$

we obtain four homogeneous linear algebraic equations for $A_1, A_2, A_3, A_4$ which have a non-trivial solution if and only if

$$\{\lambda(\sin 2\alpha) + \sin(2\lambda\alpha)\}\,\{\lambda\sin(2\alpha) - \sin(2\lambda\alpha)\} = 0\ . \tag{183}$$

This procedure and the resulting characteristic equation is clearly very similar to that used for the semi-infinite strip problem in §8.2. As in that case, we construct a more general solution as a series

$$\phi(r,\theta) = \sum_{i=1}^{\infty} C_i r^{\lambda_i+1} g_i(\theta)\ , \tag{184}$$

where $\lambda_i$ are the eigenvalues of (183) and $g_i$ are the corresponding eigenfunctions. R.D.Gregory has shown that this defines the general solution to the problem where the wedge of Figure 2 is finite with prescribed tractions on a distant circular boundary $r = a$. It follows rigorously that sufficiently near the corner (i.e. for sufficiently small $r$), the stress field is well-approximated by the single term corresponding to the eigenvalue $\lambda_1$ with smallest real part and the severity of the stress field (which will therefore characterize the conditions for failure) is determined by the single coefficient $C_1$.

The wedge of Figure 2 represents a crack tip if we set $\alpha = \pi$, in which case the admissible eigenvalues of equation (183) are

$$\lambda = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, ..., \tag{185}$$

each of which is repeated, so that there are two eigenfunctions (one symmetric and one antisymmetric with respect to $\theta = 0$) corresponding to each value of $\lambda$. The dominant stress field is that associated with $\lambda = 1/2$. The symmetric eigenfunction involves purely normal tractions $\sigma_{\theta\theta}$ on the plane $\theta = 0$ and the antisymmetric involves purely shear tractions $\sigma_{\theta r}$. The corresponding stress intensity factors are conventionally defined such that

$$K_I = \lim_{r\to 0} \sigma_{\theta\theta}(r,0)\sqrt{2\pi r}\ ; \quad K_{II} = \lim_{r\to 0} \sigma_{\theta r}(r,0)\sqrt{2\pi r}\ . \tag{186}$$

### 12.2. Other geometries

Similar techniques can be applied to other geometric features producing singular or discontinuous stress fields, such as sharp re-entrant notches, terminal points on an interface between dissimilar materials, or the transition between contact and separation at a contact interface. When the local stress field is singular, generalized stress intensity factors can be defined as the multiplier on the first (dominant) term and can then be used in combination with appropriate experimental measurements to predict the conditions for local failure. Knowledge of the form of the local fields is also extremely useful for numerical (e.g. finite element) solution of such problems, since this is a good guide to the degree of mesh refinement required for convergence of solution and can also be used to develop an appropriate 'singular element' to patch into the corner.

## 13. Anisotropic materials

So far we have restricted attention to isotropic materials, but with increasing interest in elasticity at very small scales where crystal structure plays a rôle, there is increasing emphasis on anisotropy.

Other important applications of anisotropic elasticity include engineered materials such as fiber-reinforced composites.

Two distinct strategies have been developed for the solution of two-dimensional problems involving anisotropic materials. S.G.Lekhnitskii defines the stress field in terms of the Airy and Prandtl stress functions, thereby satisfying the equilibrium equations. He then uses the generalized Hooke's law (22) to find the strains and substitutes the resulting expressions into the compatibility equations (11), obtaining coupled partial differential equations that can be combined to define a sixth order partial differential equation which is then solved by factorization.

By contrast, A.N.Stroh seeks a representation of displacement as a function of a modified form of the complex variable, chosen so as to ensure that the resulting stress fields satisfy the equilibrium equations.

### 13.1. The Stroh formalism

We consider the two-dimensional case in which the displacements $u_i$ and the stress components $\sigma_{ij}$ are functions of $x_1, x_2$ only and independent of $x_3$. Following J.D.Eshelby, Stroh investigated the conditions under which equation (30) admits a two-dimensional solution of the form

$$u_k = \Re\left\{a_k f(z)\right\} \ , \tag{187}$$

where $f$ is any holomorphic function of the modified complex variable

$$z = x_1 + px_2 \tag{188}$$

and $p$ is a complex scalar parameter. Substituting into (30) with no body force ($p_i = 0$), we obtain

$$\Big(c_{i1k1} + p(c_{i2k1} + c_{i1k2}) + p^2 c_{i2k2}\Big)\, a_k f'(z) = 0 \tag{189}$$

and this equation is satisfied for any function $f$ and all $x_1, x_2$ provided

$$\Big(c_{i1k1} + p(c_{i2k1} + c_{i1k2}) + p^2 c_{i2k2}\Big)\, a_k = 0 \ . \tag{190}$$

This constitutes three homogeneous algebraic equations for the three constants $a_k$ which will have only the trivial solution $a_k = 0$ unless the determinant

$$\Big|c_{i1k1} + p(c_{i2k1} + c_{i1k2}) + p^2 c_{i2k2}\Big| = 0 \ . \tag{191}$$

In this special case, the three equations are not linearly independent and a non-trivial solution exists. The determinantal equation (191) expands to a sixth degree polynomial in $p$ which for physically realistic material properties always has three pairs of complex conjugate roots, those with positive imaginary part being denoted $p^{(1)}, p^{(2)}, p^{(3)}$ respectively. If these roots (eigenvalues) are distinct, a general solution of the two-dimensional problem can then be constructed by superposition in the form

$$u_k = \Re\left\{\sum_{\alpha=1}^{3} a_k^{(\alpha)} f^{(\alpha)}\left(z^{(\alpha)}\right)\right\} \ , \tag{192}$$

where $z^{(\alpha)} = x_1 + p^{(\alpha)} x_2$ and $\boldsymbol{a}^{(\alpha)}$ is the eigenvector of (190) corresponding to the eigenvalue $p^{(\alpha)}$.

## 13.2. The Lekhnitskii formalism

Lekhnitskii chooses to satisfy the equilibrium equations without body force by representing the stress components in terms of the Airy and Prandtl stress functions $\phi$, $\varphi$ of equations (69, 110) respectively — i.e.

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial {x_2}^2} \; ; \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_2 \partial x_1} \; ; \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial {x_1}^2} \; ; \quad \sigma_{31} = \frac{\partial \varphi}{\partial x_2} \; ; \quad \sigma_{32} = -\frac{\partial \varphi}{\partial x_1} \; . \tag{193}$$

The remaining stress component $\sigma_{33}$ can be eliminated by noting that $u_3$ is independent of $x_3$, so $e_{33} = 0$. To achieve this elimination, it is convenient to use a condensed form of Hooke's law that recognizes the symmetry of the stress and strain components. We define the matrix $\boldsymbol{a}$ through the equation

$$\{e_{11}, e_{22}, e_{33}, 2e_{23}, 2e_{31}, 2e_{12}\} = \left| \begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{array} \right| \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{array} \right\} , \tag{194}$$

where the coefficients $a_{ij}$ can be defined in terms of $s_{ijkl}$ by comparison with (24). The condition $e_{33} = 0$ then permits us to eliminate $\sigma_{33}$, obtaining

$$\{e_{11}, e_{22}, 2e_{23}, 2e_{31}, 2e_{12}\} = \left| \begin{array}{ccccc} b_{11} & b_{12} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{24} & b_{25} & b_{26} \\ b_{41} & b_{42} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{64} & b_{65} & b_{66} \end{array} \right| \left\{ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{array} \right\} \tag{195}$$

where the $5 \times 5$ matrix $\boldsymbol{b}$ is obtained from $\boldsymbol{a}$ by the relations

$$b_{ij} = a_{ij} - \frac{a_{i3} a_{j3}}{a_{33}} \tag{196}$$

with $i, j = 1, 2, 4, 5, 6$.

Since no displacement components vary with $x_3$, the six compatibility conditions (11) reduce to

$$\frac{\partial^2 e_{11}}{\partial {x_2}^2} + \frac{\partial^2 e_{22}}{\partial {x_1}^2} = 2\frac{\partial^2 e_{12}}{\partial x_1 \partial x_2} \; ; \qquad \frac{\partial e_{23}}{\partial x_1} = \frac{\partial e_{13}}{\partial x_2} \; . \tag{197}$$

Substituting for the strains, using (193, 194, 196), we obtain

$$\mathcal{L}_4 \phi + \mathcal{L}_3 \varphi = 0 \; ; \quad \mathcal{L}_3 \phi + \mathcal{L}_2 \varphi = 0 \; , \tag{198}$$

where the differential operators $\mathcal{L}_i$ are defined as

$$\begin{aligned} \mathcal{L}_4 &= b_{22}\frac{\partial^4}{\partial x_1^4} - 2b_{26}\frac{\partial^4}{\partial x_1^3 \partial x_2} + 2(b_{12} + b_{66})\frac{\partial^4}{\partial x_1^2 \partial x_2^2} - 2b_{16}\frac{\partial^4}{\partial x_1 \partial x_2^3} + b_{11}\frac{\partial^4}{\partial x_2^4} \\ \mathcal{L}_3 &= -b_{24}\frac{\partial^3}{\partial x_1^3} + (b_{25} + b_{46})\frac{\partial^3}{\partial x_1^2 \partial x_2} - (b_{14} + b_{56})\frac{\partial^3}{\partial x_1 \partial x_2^2} + b_{15}\frac{\partial^3}{\partial x_2^3} \\ \mathcal{L}_2 &= b_{44}\frac{\partial^2}{\partial x_1^2} - 2b_{45}\frac{\partial^2}{\partial x_1 \partial x_2} + b_{55}\frac{\partial^2}{\partial x_2^2} \; . \end{aligned} \tag{199}$$

Eliminating either $\varphi$ or $\phi$ between (198), we conclude that both functions must satisfy the sixth order equation

$$(\mathcal{L}_4\mathcal{L}_2 - \mathcal{L}_3^2)f = 0 \ . \tag{200}$$

Particular solutions of this equation can be obtained by substituting $f = f(x_1 + px_2)$ and canceling the common factor $f^{VI}(x_1 + px_2)$, leaving a sixth order algebraic equation for $p$. As in the Stroh formalism, it can be shown that the roots of this equation always occur in three complex conjugate pairs. If the roots are distinct, the general solution can then be obtained by writing

$$\phi = \Re\left\{\sum_{\alpha=1}^{3} f^{(\alpha)}\left(z^{(\alpha)}\right)\right\} \ , \tag{201}$$

where $z^{(\alpha)} = x_1 + p^{(\alpha)}x_2$ and $p^{(\alpha)}$ are the three roots with positive real part. Once $\phi$ is written in this way, $\varphi$ can be recovered by backsubstitution into (198).

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