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Dynamic Behavior and Stability of Simple Frictional Systems

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Abstract—Numerous authors have demonstrated that problems arise over existence and uniqueness of solution in quasi-static contact problems involving large coefficients of Coulomb friction. This difficulty was greatly elucidated by a simple two-degree-of-freedom model introduced by Klarbring. In the present paper, the dynamic behavior of Klarbring's model is explored under a wide range of loading conditions. It is demonstrated that the dynamic solution is always unique and deviates from the quasi-static only in a bounded oscillation for sufficiently low friction coefficients. Above the critical coefficient, slip in one of the two directions is found to be unstable so that the system never exists in this state for more than a short period of time compared with the loading rate. In the limit of vanishing mass, these periods become infinitesimal but permit unidirectional state changes with discontinuous displacements. A revised quasi-static algorithm is developed from this limit and is shown to predict the dynamic behavior of the system within a bounded oscillation for large coefficients of friction.

Keywords—Coulomb friction, Stability, Nonuniqueness, Quasi-static, Instantaneous jump.

1. INTRODUCTION

If an elastic system is subjected to loads that vary only slowly in comparison with the period of the lowest natural frequency, it is usually possible to predict its behavior using a *quasi-static* analysis in which the mass of the system is neglected and the structure is assumed to pass through a sequence of equilibrium states. However, if the system has Coulomb friction boundary conditions, the quasi-static analysis can predict multiple solutions if the coefficient of friction is sufficiently high [1]. This is a serious difficulty if numerical methods (e.g., finite elements methods) are used, since there is then no way of guaranteeing that the solution found is the correct one for the physical problem under consideration. There is evidence to suggest that the difficulty can be resolved by reintroducing the mass into the analysis, i.e., by using a full elastodynamic formulation—but this is extremely computationally intensive and introduces its own difficulties associated with the numerical description of the resulting wave motion in the system.

2. KLARBRING'S MODEL

These paradoxical questions are conveniently illustrated and investigated in the context of the simple two degree of freedom system introduced by Klarbring [1,2] and shown in Figure 1. A rigid mass M is supported by a generalized massless elastic support and makes contact with a rigid frictional surface, at which the coefficient of friction is f. A force F (components F_1, F_2) is applied to the body and varies slowly in time in magnitude and/or direction. The body is allowed to translate horizontally and/or vertically, but is not permitted to rotate. We denote the

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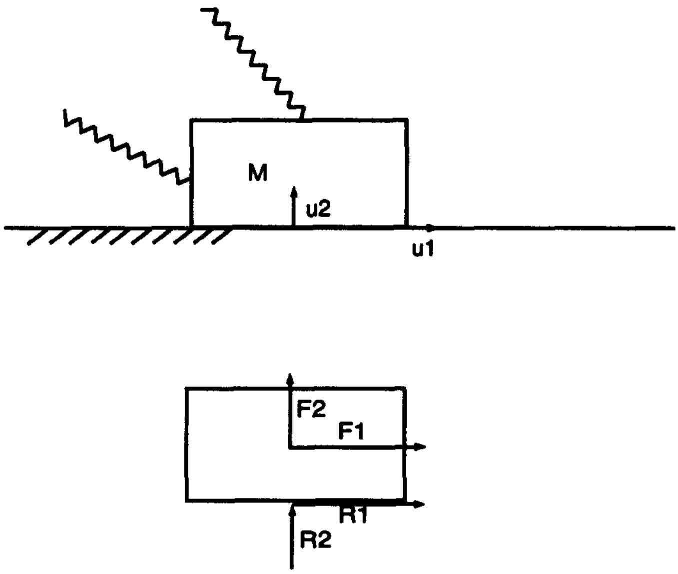


Figure 1. Klarbring's model.

time-varying displacements of the mass by u_1, u_2 and the reaction forces at the contact interface by R_1, R_2, as shown in Figure 1.

2.1. States of the System

We distinguish four possible states for the system at any given time, namely *stick, forward slip, backward slip*, and *separation*. We shall define these states and their governing conditions in the following sections.

(1) **Stick** is the state in which the mass makes contact with the plane and is not moving. In other words,

$$u_2 = 0 \qquad \text{and} \qquad \dot{u}_1 = 0. \tag{1}$$

This condition is possible if and only if the normal reaction at the interface is positive and the tangential reaction is less than the limiting value permitted by Coulomb friction, i.e.,

$$R_2 > 0 \qquad \text{and} \qquad |R_1| < fR_2. \tag{2}$$

(2) **Forward slip** is the state in which the mass remains in contact with the plane but moves to the right, i.e.,

$$u_2 = 0 \qquad \text{and} \qquad \dot{u}_1 > 0. \tag{3}$$

Once again, the normal reaction force must be positive, but in this case, the tangential reaction is *equal* to the limiting friction force—fR_2 (we assume that the static and dynamic coefficient of friction are equal) and opposes the motion, so that

$$R_2 > 0 \qquad \text{and} \qquad R_1 = -fR_2. \tag{4}$$

(3) **Backward slip** is the corresponding state where the mass moves to the *left*, and hence, by similar arguments, we have

$$u_2 = 0, \qquad \dot{u}_1 < 0, \qquad R_2 > 0, \qquad \text{and} \qquad R_1 = fR_2. \tag{5}$$

(4) **Separation**. Finally, the mass may lose contact with the plane, in which case there are no reaction forces and the displacement u_2 must be positive, i.e.,

$$R_1 = 0, \qquad R_2 = 0, \qquad \text{and} \qquad u_2 > 0. \tag{6}$$

Notice, that each of the above states is defined by two equations and one or more inequalities. Two additional equations will be obtained from equilibrium considerations (or more generally from Newton's law), thus permitting the unknown reactions R_1, R_2 and displacements u_1, u_2 to be determined. We then anticipate that the *inequalities* will serve to determined which state is realized at any given time.

2.2. The Quasi-Static Governing Equations

Under the quasi-static assumption, the system is assumed to pass through a sequence of equilibrium states and hence we obtain the two equilibrium equations,

$$F_1 + R_1 - k_{11}u_1 - k_{12}u_2 = 0, \tag{7}$$

$$F_2 + R_2 - k_{21}u_1 - k_{22}u_2 = 0. \tag{8}$$

These equations also serve to define the support stiffness matrix k_{ij}. We note that the reciprocal theorem demands that $k_{12} = k_{21}$, i.e., that the matrix be symmetric. Also the energy stored in the spring must be positive for all conceivable displacements and this requires that

$$k_{11} > 0, \qquad k_{22} > 0, \qquad k_{11}k_{22} > k_{12}^2. \tag{9}$$

The off-diagonal stiffness, k_{12}, can be either positive or negative, but the coordinate direction x_1 can be defined to make $k_{12} > 0$ without loss of generality. We shall, therefore, assume $k_{12} > 0$ for the purpose of illustration. In all the following discussion, the effect of k_{12} being negative is equivalent to an interchange of the definitions of 'forward' and 'backward' slip.

2.3. Monotonic Unidirectional Loading

Klarbring [1] considered the special case where the body is initially unloaded ($F_1 = F_2 = 0$) and just makes contact with the plane at the origin ($u_1 = u_2 = 0$). The forces F_1, F_2 are now increased linearly with time, i.e.,

$$F_1 = C_1 t \qquad \text{and} \qquad F_2 = C_2 t. \tag{10}$$

The system may adopt any one of the four states defined in Section 2.1 depending on the values of C_1, C_2.

- **Stick**. In this case, $u_1 = u_2 = 0$, and hence,

  $$R_1 = -C_1 t \qquad \text{and} \qquad R_2 = -C_2 t, \tag{11}$$

  from equations (7) and (8). The reaction forces must satisfy the inequalities (2), which imply

  $$C_2 < 0 \qquad \text{and} \qquad |C_1| < -fC_2, \tag{12}$$

  since $t > 0$. These inequalities constrain the values of C_1, C_2 to the region of $C_1 - C_2$ space labelled 'stick' in Figure 2a.
- **Forward slip**. Substituting equations (3) and (4) into the quasi-static governing equations (7) and (8) and solving for u_1 and R_2, we obtain

  $$u_1 = \frac{fC_2 + C_1}{fk_{21} + k_{11}} t \qquad \text{and} \qquad R_2 = \frac{-k_{11}C_2 + K_{21}C_1}{fk_{21} + k_{11}} t. \tag{13}$$

  Substituting these results into the *inequalities* (3),(4), respectively, noting that $t > 0$, we find that C_1, C_2 must satisfy the conditions,

  $$\frac{fC_2 + C_1}{fk_{21} + k_{11}} > 0 \qquad \text{and} \qquad \frac{-k_{11}C_2 + k_{21}C_1}{fk_{21} + k_{11}} > 0. \tag{14}$$

The implication of these inequalities depends on the sign of the denominator $fk_{21} + k_{11}$. However, k_{11} must be positive (see, Section 2.2) and we have chosen the coordinate system to ensure $k_{21} > 0$. It follows that

$$-\frac{1}{f}C_1 < C_2 < \frac{k_{21}}{k_{11}}C_1, \tag{15}$$

which is the region labelled 'forward slip' in Figure 2a.

- **Backward slip.** A similar solution procedure, using (5) in place of (3),(4) yields

$$\frac{C_1 - fC_2}{k_{11} - fk_{21}} < 0 \quad \text{and} \quad \frac{k_{21}C_1 - k_{11}C_2}{k_{11} - fk_{21}} > 0. \tag{16}$$

As in the case of forward slip, the implication of these two inequalities depends on the sign of the denominator, $k_{11} - fk_{21}$.

(i) If $k_{11} - fk_{21} > 0$, the denominators of the inequalities (16) are positive, and hence, $C_1 - fC_2 < 0$, $k_{21}C_1 - k_{11}C_2 > 0$, i.e.,

$$\frac{1}{f}C_1 < C_2 < \frac{k_{21}}{k_{11}}C_1. \tag{17}$$

This corresponds to the region labelled 'backward slip' in Figure 2a.

(ii) However, if $k_{11} - fk_{21} < 0$ the denominators are *negative*, leading to the conditions

$$\frac{k_{21}}{k_{11}}C_1 < C_2 < \frac{1}{f}C_1. \tag{18}$$

The $C_1 - C_2$ diagram for this case is shown in Figure 2b and the region defined by (18) is labelled 'backward slip'.

- **Separation.** In this case $R_1 = R_2 = 0$ and the quasi-static governing equations (7),(8) can be solved to obtain the vertical displacement

$$u_2 = \frac{k_{11}C_2 - k_{21}C_1}{k_{11}k_{22} - k_{12}k_{21}}t. \tag{19}$$

The denominator is always positive according to (9), so separation occurs only when,

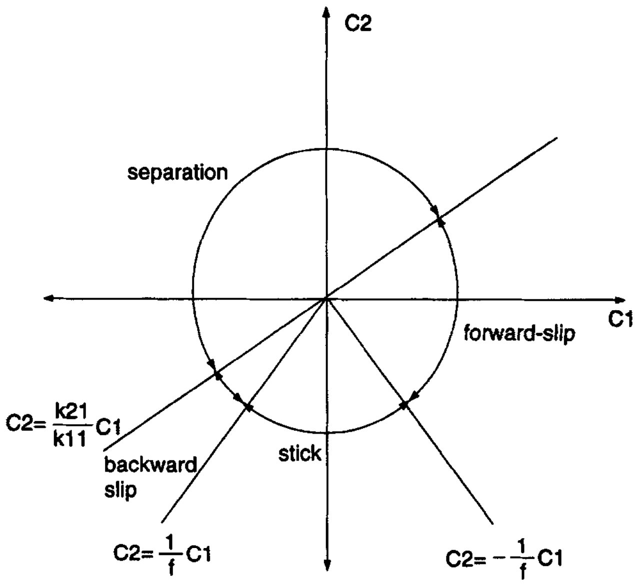
$$C_2 > \frac{k_{21}}{k_{11}}C_1. \tag{20}$$

Referring to Figure 2a, we see that each point in $C_1 - C_2$ space corresponds to one and only one state for the system. By contrast, in Figure 2b, there is a region in which stick, backward slip, and separation are all possible.[1] This region of nonuniqueness arises if and only if the coefficient of friction $f > f^*$, where

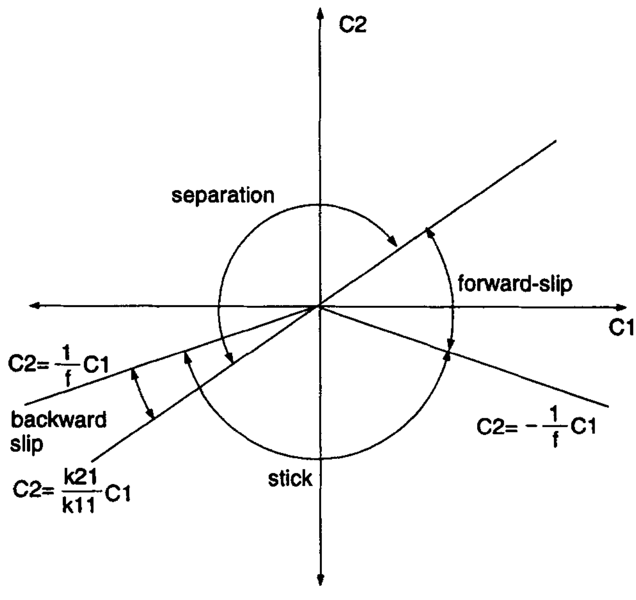
$$f^* = \frac{k_{11}}{|k_{21}|}. \tag{21}$$

Klarbring [2] has shown that similar difficulties are encountered in the more general incremental loading problem if $f > f^*$. In particular, nonuniqueness is then predicted whenever R_1, R_2, and u_2 are simultaneously equal to zero, as must be the case at any transition between contact and separation. He also demonstrated that if the system is instantaneously in a state of backward slip, there are some loading scenarios for which the incremental problem has no solution for $f > f^*$.

[1]Notice, that if $k_{21} < 0$, it will be found that this region occurs on the right of the diagram and involves stick, forward slip, and separation.



(a) Quasi-static diagram with unique solution ($f < k_{11}/k_{21}$).



(b) Quasi-static diagram with multiple solution ($f < k_{11}/k_{21}$).

Figure 2.

2.4. Dynamic Solution

A real system will generally have one and only one response to any given loading scenario. Even if a number of unstable equilibrium points exist, the probability of the system history passing exactly through such a point is vanishingly small, so that in practical terms, only unique (or 'almost unique') simulation algorithms can be regarded as satisfactory. Thus, the quasi-static algorithm as so far stated is inadequate for $f > f^*$.

In an attempt to resolve this paradox, it seems reasonable to re-introduce the effect of inertia even though the forces are slowly applied, i.e., to compare the quasi-static predictions with those of a full dynamic analysis. As in Section 2.3, we consider the special case where the forces increase linearly with time (equation (10)). The governing equations (7),(8) can be generalized to include dynamic effects by adding inertia terms, yielding

$$C_1t + R_1 - k_{11}u_1 - k_{12}u_2 = M\ddot{u}_1, \tag{22}$$

$$C_2t + R_2 - k_{21}u_1 - k_{22}u_2 = M\ddot{u}_2. \tag{23}$$

There are four unknown quantities—R_1, R_2, $\ddot{u}_1$, and $\ddot{u}_2$ and two equations (22) and (23), but two additional conditions are available in each of the four contact states.

We consider the case where the mass is initially at rest at the origin ($u_1 = u_2 = \dot{u}_1 = \dot{u}_2 = 0$) and determine the loading conditions for which each of the four states is possible for small values of time.

- **Stick**. In this case, there are no accelerations and hence, the conclusions are unchanged from Section 2.3, i.e., for stick to be a possible state, we must have

$$C_2 < 0 \qquad \text{and} \qquad |C_1| < -fC_2. \tag{24}$$

- **Forward slip**. We note that u_2 and its time derivatives are zero in this state. Using this result and the Coulomb law (4) in (22),(23), we obtain

$$C_1t - fR_2 - k_{11}u_1 = M\ddot{u}_1, \tag{25}$$

$$C_2t + R_2 - k_{21}u_1 = 0. \tag{26}$$

Eliminating R_2, we get an ordinary differential equation for u_1

$$M\ddot{u}_1 + (k_{11} + fk_{21})u_1 = (C_1 + fC_2)t, \tag{27}$$

with initial values, $u_1 = 0$ and $\dot{u}_1 = 0$. Since, $k_{11} + fk_{21}$ is positive *ex hypothesis*, the solution can be written[2]

$$u_1(t) = \frac{C_1 + fC_2}{M\omega^3}(\omega t - \sin\omega t), \tag{28}$$

where

$$\omega = \sqrt{\frac{k_{11} + fk_{21}}{M}}. \tag{29}$$

The normal reaction force can be recovered from (26) as

$$R_2(t) = -C_2t + k_{21}\frac{C_1 + fC_2}{M\omega^3}(\omega t - \sin\omega t). \tag{30}$$

For small values of t, equations (28) and (30) reduce to

$$u_1(t) = \frac{C_1 + fC_2}{M}\frac{1}{3!}t^3 + O\left(t^5\right), \tag{31}$$

$$R_2(t) = -C_2t + O\left(t^3\right), \tag{32}$$

and hence, the conditions for forward slip, $\dot{u}_1 > 0$, $R_2 > 0$, are satisfied if and only if

$$-C_1 < fC_2 < 0. \tag{33}$$

- **Backward slip**. The same equations apply to the case of backward slip, except that the frictional force is reversed, which is equivalent to replacing f by $-f$, i.e.,

$$M\ddot{u}_1 + (k_{11} - fk_{21})u_1 = (C_1 - fC_2)t. \tag{34}$$

[2] As long as the forward slip assumption is valid.

We get a similar for u_1 and R_2 when $k_{11} - fk_{21} > 0$. In particular, at small values of t

$$u_1(t) = \frac{C_1 - fC_2}{M}\left(\frac{1}{3!}t^3 + O\left(t^5\right)\right), \tag{35}$$

$$R_2(t) = -C_2 t + O\left(t^3\right). \tag{36}$$

If $k_{11} - fk_{21} < 0$, the solution for u_1, R_2 contains hyperbolic instead of trigonometric functions, but the small time approximation is still given by (35) and (36). Thus, for all values of f, we conclude that backward slip is possible for

$$C_1 < fC_2 < 0. \tag{37}$$

- **Separation.** Since $u_2, \dot{u}_2$ are both initially zero, the separation inequality $u_2(t) > 0$ demands that the first *nonzero* derivative, $\ddot{u}_2$, be positive at $t = 0$. Imposing this condition on equation (23), we find that separation is only possible if

$$C_2 > 0, \tag{38}$$

since $R_2 = 0$ from (6) and $u_1 = u_2 = 0$ at $t = 0$ from the initial conditions.

2.5. Summary

The above results are summarized in Figure 3. The dynamic analysis gives a unique solution for all values of f, and hence, resolves the issue of the multiple solution range in Figure 2b. We also note that the dynamic predictions differ from the quasi-static even in cases where the latter predicts a unique solution. Of course, Figure 3 only describes the state holding at small values of t and as the system evolves we should anticipate transitions to other states. To explore the relationship between the quasi-static and dynamic solutions at larger values of time, a numerical solution has been developed for the equations of motion.

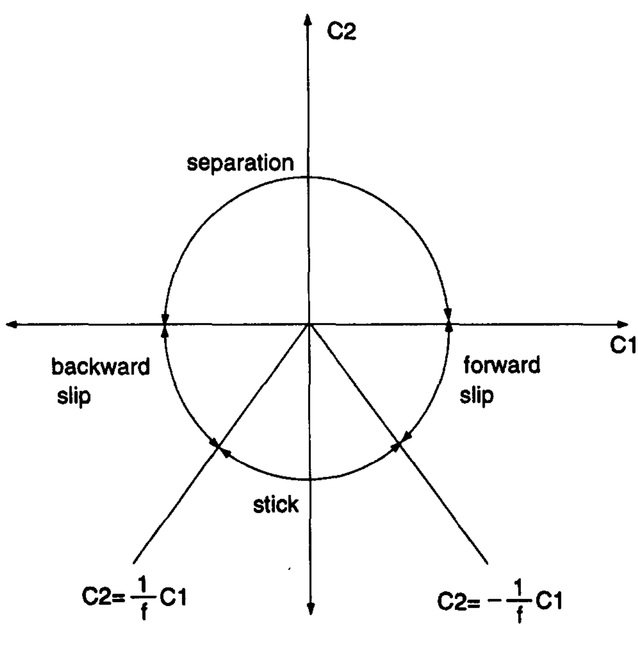


Figure 3. Dynamic diagram.

3. NUMERICAL SOLUTION

At a given time t, we assume that the instantaneous position $u_1(t), u_2(t)$ and the velocity $\dot{u}_1(t), \dot{u}_2(t)$ of the mass are known. We also assume that we know which of the four states (stick, forward slip, backward slip, and separation) holds instantaneously. The two state equations are used to determine the reaction forces R_1, R_2 and the equations of motion (22) and (23) then yield the accelerations $\ddot{u}_1(t), \ddot{u}_2(t)$. The position and velocity are then updated using the equations

$$u_i(t + \delta t) = u_i(t) + \dot{u}_i(t)\delta t, \tag{39}$$

$$\dot{u}_i(t + \delta t) = \dot{u}_i(t) + \ddot{u}_i(t)\delta t, \tag{40}$$

$i = 1, 2$, where δt is a small increment of time. This procedure enables us to track the motion of the mass as long as it remains in the same state, but we anticipate occasional state changes. To detect these, we continually monitor the quantities appearing in the state *inequalities* and when a violation is detected, an appropriate change is made in the assumed state. For example, if the reaction force R_2 is found to become negative during an increment of forward slip, the state assumption is changed to one of separation. The full set of such state change operations and the resulting updating algorithm are described in more detail in Appendix A. Most of these operations are self-explanatory. However, it is worth noting that the termination of a period of separation is assumed to be governed by inelastic impact conditions, i.e., the normal velocity is instantaneously set to zero and transition occurs either to stick or forward or backward slip depending on the angle of incidence. The rational for this choice is that in more complex systems the effects of elastic recovery will be captured by the dynamics of the spring mass system.

3.1. Results

The numerical code is based on the dynamic solution, so the initial state of the system is equivalent to the dynamic solution of Section 2.4. However, at larger values of time, we might expect the behavior to approach the state predicted by the quasi-static analysis.

3.1.1. Unique solution region

For the purposes of illustration, we consider the system defined by the stiffness matrix

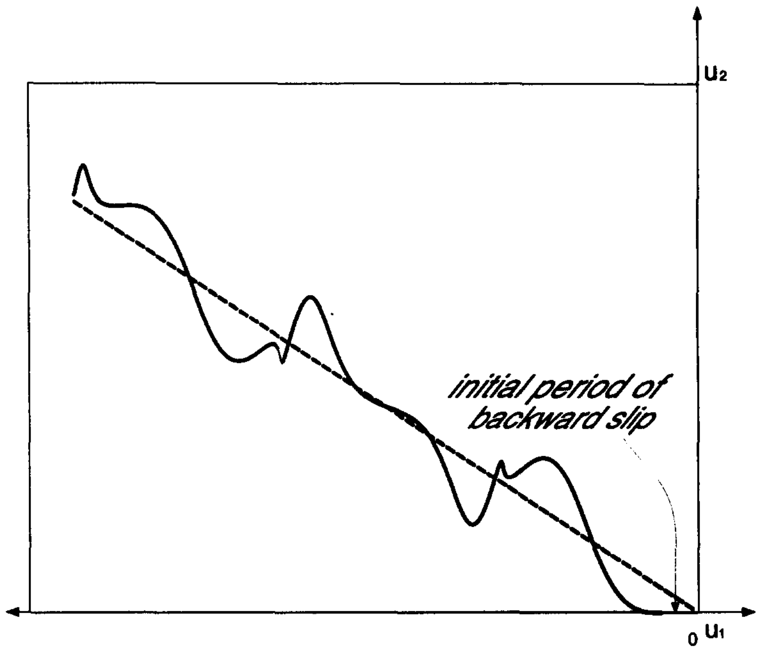
$$k = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \tag{41}$$

and coefficient of friction $f = 0.25$. For this system, $f < k_{11}/k_{21}$, and hence, the quasi-static solution is unique, but there exist two ranges of the loading parameters C_1, C_2 for which the quasi-static and short-time dynamic solutions predict different states. Figure 4a shows the trajectory of the system for the case $C_1 = -1.0$, $C_2 = -0.8$, which lies in the range

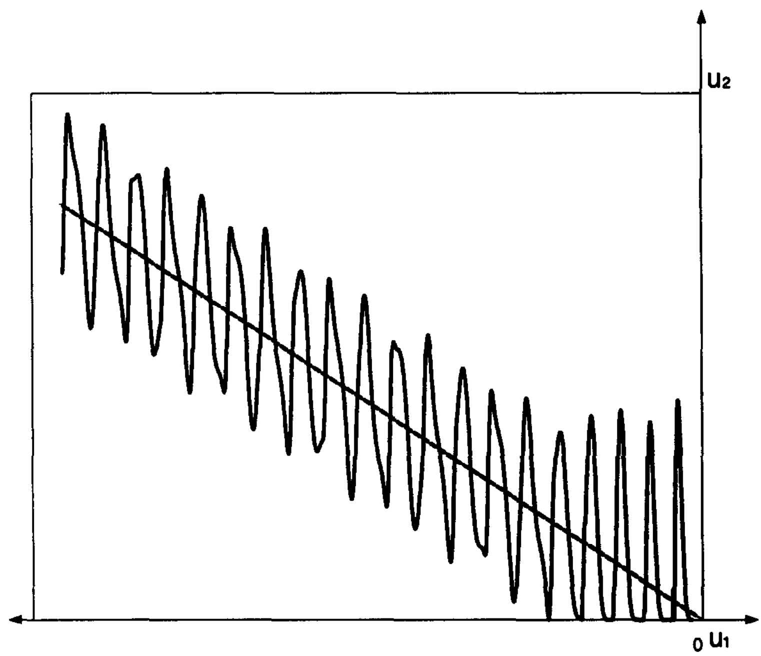
$$\frac{k_{21}}{k_{11}} C_1 < C_2 < 0. \tag{42}$$

The numerical solution starts in a state of backward slip, as predicted in Section 2.4 (Figure 3), but then changes to separation, which remains the state for all subsequent times. The long-term condition of the system involves oscillation about the predicted quasi-static trajectory.

Other cases exhibit essentially similar behavior, but there can be multiple state changes before the system settles into an oscillation about the quasi-static trajectory. For example, Figure 4b shows the trajectory obtained for the loading $C_1 = -1.0$, $C_2 = -0.99$ and exhibits nine state changes during the initial transient. The number of such state changes increases as the operating point approaches the boundary $C_2/C_1 = k_{21}/k_{11}$ in Figures 2a and 3.



(a) Dynamic response from backward slip to separation $C_1 = -1.0$, $C_2 = -0.8$. (—) dynamic, (- - -) quasi-static.



(b) Dynamic response from backward slip to separation $C_1 = -1.0$, $C_2 = -0.99$. (—) dynamic, (- - -) quasi-static.

Figure 4.

One way of interpreting these results is to remark that if we assumed *ab initio* that the state would involve oscillation (driven by the initial conditions) about the quasi-static trajectory, some of these oscillations would carry the state variables outside the permissible range and would therefore involve state changes. As C_2/C_1 approaches k_{21}/k_{11}, the quasi-static trajectory makes an increasingly small angle with the plane, so that larger numbers of these cycles of oscillation would involve negative values of u_2, implying periods of contact.

Figures 2a and 3 also disagree in their predictions when

$$\frac{k_{21}}{k_{11}}C_1 > C_2 > 0. \tag{43}$$

Figure 5 shows the trajectory for the case $C_1 = 1.0$, $C_2 = 0.7$. The system starts with a period of separation, but reverts to forward slip as predicted by the quasi-static solution. Forward slip then continues indefinitely, though there is some oscillation about the quasi-static trajectory reflected in a periodic variation in slip velocity $\dot{u}_1$ about the (constant) quasi-static value. As in the previous cases discussed, more state changes occur during the transition period if C_2/C_1 approaches the boundary k_{21}/k_{11}.

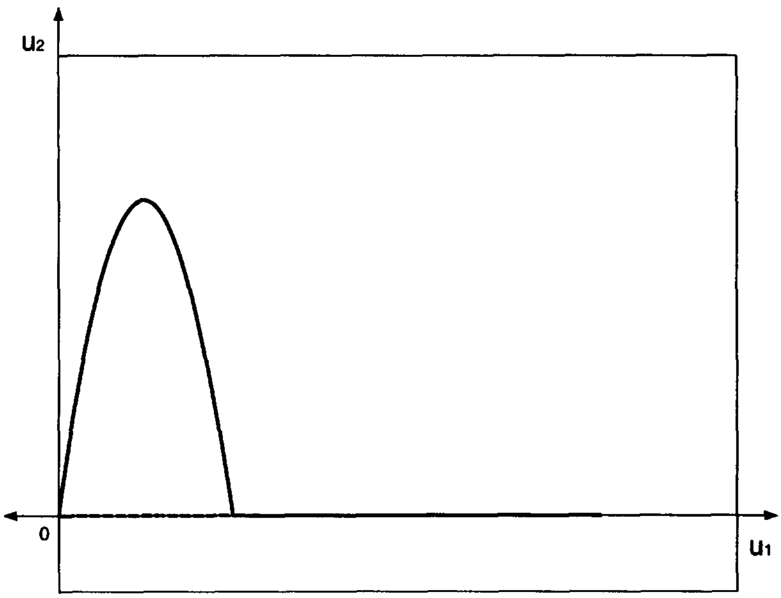


Figure 5. Dynamic response from separation to forward slip $C_1 = 1.0$, $C_2 = 0.7$. (—) dynamic, (- - -) quasi-static.

3.1.2. Multiple solution region

The more interesting case is that in which the coefficient of friction exceeds the critical value ($f > f^*$), since in this case the quasi-static solution is nonunique and the ambiguity can only be resolved by reference to dynamic analysis.

The multiple-solution range in Figure 2b involves the states of stick, backward slip, and separation. If the system starts in a state of stick, the dynamic analysis of Section 2.4 shows that it will never move—this is the one case in Section 2.4 that is *not* restricted to small values of time.

The question then arises as to whether this state of stick is stable, or more generally whether there is a level of initial perturbation that would lead to one of the other quasi-static states being realized as the long-term solution. Tests were made using various combinations of initial nonzero values of $u_1, u_2, \dot{u}_1, \dot{u}_2$.

As an example, we consider the system defined by the stiffness matrix (41), with coefficient of friction $f = 1.25$ which satisfies the criterion for multiple solution. The loading scenario is chosen to bisect the multiple solution sector in Figure 2b. Figure 6 shows the *long-time* state as a function of the initial values of $\dot{u}_1, \dot{u}_2$, for $u_1 = u_2 = 0$. Notice, that the final state is always stick if the initial horizontal velocity $\dot{u}_1$ is of sufficiently large magnitude in either direction. Separation is obtained as a final state only at intermediate levels of $\dot{u}_1$.

The result for large $\dot{u}_1$ is at first sight rather surprising, since it would seem that giving the system a large initial velocity would be more likely to cause it to pass into a state of separation.

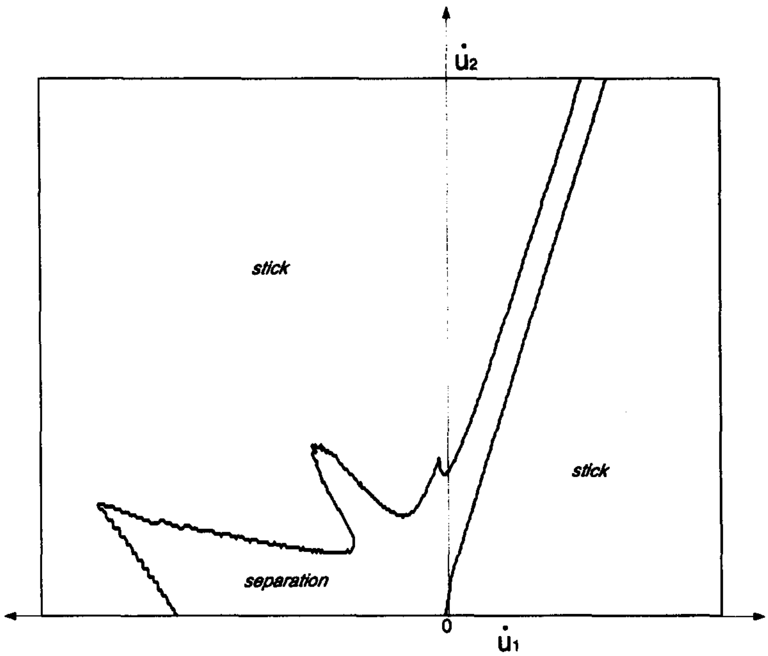


Figure 6. Final state as a function of initial velocities $\dot{u}_1, \dot{u}_2$ for initial conditions $u_1 = u_2 = 0$.

However, we note that the system is linear with regard to the loading rates C_1, C_2, so that a scaled formulation of the problem can be written in terms of the ratio u/C, where $C = \sqrt{C_1^2 + C_2^2}$. Thus, the behavior of the system for large $\dot{u}_1(0)$ is algorithmically equivalent to that for small C. In physical terms, this means that at large $\dot{u}_1(0)$, the response is dominated by the initial perturbation and the contribution of the applied loads C_1t, C_2t is small. In the limit, we obtain the response of the system to initial perturbation in the absence of external load which, not surprisingly, is one of stick.

The backward slip solution

All the regions in Figure 6 correspond to long-term states of separation or stick. No initial conditions were found leading to the third quasi-static solution of backward slip, suggesting that this state might be unstable in some sense.

To explore this hypothesis in more detail, we first note that for linear loading rates the displacements in the quasi-static solution are linear functions of time. For example, in backward slip we have

$$u_1(t) = \frac{(C_1 - fC_2)t}{k_{11} - fk_{21}}. \tag{44}$$

It follows that the corresponding velocity

$$\dot{u}_1(t) = \frac{(C_1 - fC_2)}{k_{11} - fk_{21}}, \tag{45}$$

is constant and there is no acceleration. Thus, the quasi-static solution will be the full dynamic solution of the problem if initial conditions are chosen such that $\dot{u}_1(0)$ is given by equation (45).

The numerical solution was tested under these conditions. The system does indeed follow the backward slip quasi-static solution, but an arbitrarily small perturbation from this condition is found to grow monotonically with time until eventually the system changes to one of the other two quasi-static states. Even the round-off errors in the computations are sufficient to precipitate this behavior. Also, since the growth of the perturbation is monotonic, only the sign of its initial value is influential in determining which of the two stable states is eventually realized.

Stability analysis

A more general analytical proof can be given of this result. During a period of backward slip, the displacement component u_1 is governed by equation (34)

$$M\ddot{u}_1 + (k_{11} - fk_{21})u_1 = F_1(t) - fF_2(t), \tag{46}$$

where we have reinstated the more general from of the applied forces.

This is a linear ordinary differential equation and its solution can be written as the sum of a particular solution **P** and the general homogeneous solution **H** which will contain two arbitrary constants to enable us to satisfy appropriate initial conditions on $u_1, \dot{u}_1$.

Suppose, the appropriate solution has been found and we now wish to examine the conditions under which a small perturbation on this solution can grow without limit in time. Any such perturbation must satisfy the homogeneous equation. It follows that stability of the system (in backward slip) depends on the existence of a term in **H** that grows with time and is independent of the particular loading scenario, $F_1(t), F_2(t)$.

The solution **H** can be written

$$u_1(t) = Ae^{bt} + Be^{-bt}, \tag{47}$$

where

$$b^2 = \frac{fk_{21} - k_{11}}{M}. \tag{48}$$

Thus, b is pure imaginary for $f < f^*$, causing **H** to be oscillatory in nature, while it is real for $f > f^*$. In the latter case, one of the two terms in (47) grows without limit with t, indicating instability of the solution. In other words, when $f > f^*$, the state of backward slip is always unstable, and hence, cannot be the long-term solution of the problem.

This does not preclude periods of backward slip for $f > f^*$, depending on the loading scenario, but it is worth noting that during any such periods, the system behavior will be very sensitive to the loading conditions, $F_1(t), F_2(t)$.

4. RELATION TO THE QUASI-STATIC SOLUTION

The ultimate objective of this investigation is to use the dynamic analysis to determine the real behavior of the system in the hope of defining a new quasi-static algorithm that captures the important features of the system trajectory in cases where the loading rate is slow in comparison with the time scale of dynamic effects.

For $f < f^*$ we have shown that, even though the dynamic and quasi-static predictions differ qualitatively when t is small, the long-time dynamic solution involves relatively small oscillations about the quasi-static prediction and the two solutions predict the same state. Furthermore, a modest amount of system damping would be sufficient to make the dynamic solution approach the quasi-static asymptotically at large time. The effect of damping is shown in Figure 7. We, therefore, conclude that for $f < f^*$ the quasi-static algorithm gives a good approximation to the behavior of the system for loading rates that are slow in comparison with the periods of the natural frequencies of the system.

For $f > f^*$, the state of backward slip is dynamically unstable and can therefore only persist in the dynamic solution for a limited period of time. Thus, if the loading rate is sufficiently slow, we expect that periods of backward slip can be condensed into instantaneous transitions between the preceding and following states.[3]

Klarbring's analysis of the rate problem [2] shows also that there are more loading rate scenarios for which no continuous transitions are possible from the state of backward slip in the quasi-static

[3]This statement requires some qualification because the exponential growth rate defined in equations (47) and (48) is not the usual dynamic time scale of the system and it can become arbitrarily slow when f is very close to f^*.

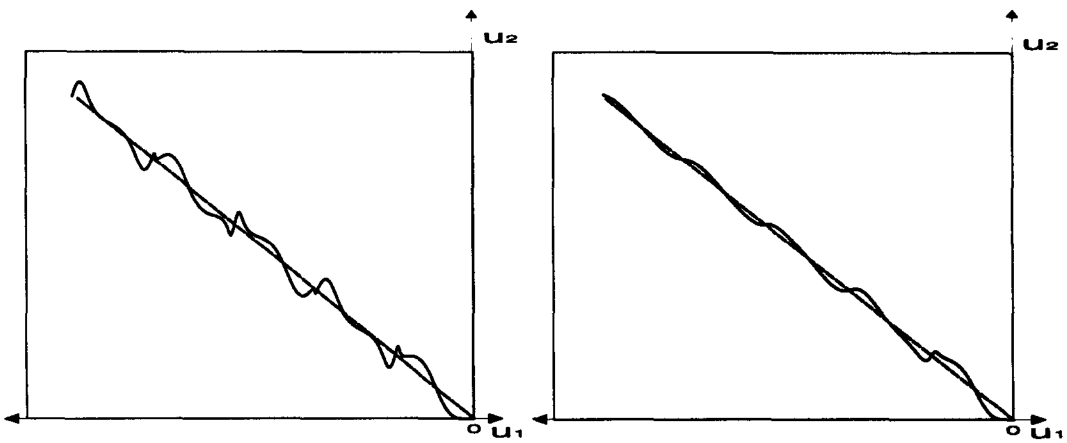


Figure 7. Effect of damping. (—) dynamic, (- - -) quasi-static.

formulation. It is, therefore, tempting to consider a quasi-static algorithm in which backward slip is not admitted as an option when $f > f^*$, particularly since this state tends to occur under loading conditions where one or more of the other quasi-static states is also permissible. However, if the system is in a state of stick and the applied forces are changed in such a way as to cross the boundary on which $R_1 = fR_2 \neq 0$, a transition to backward slip is the only quasi-static possibility that retains continuity in displacement.

The dynamic solution shows that when these conditions are realized, i.e., when limiting friction is exceeded in the backward slip direction from a state of stick—there will be a rapid transition to a state of separation. On the slow time scale of the loading rate, this will appear as an instantaneous jump from stick to separation, involving a discontinuity in displacement. Martins *et al.* [3–5] have demonstrated a similar result for the case where the mass is strictly zero, but damping is introduced to the system and then allowed to approach zero. They note that existence and uniqueness theorems can be established with arbitrary coefficient of friction if the requirement of continuity of displacement is relaxed.

However, it is clearly not sufficient simply to admit discontinuous transitions between states, since there exists a range of loading for which both stick and separation are stable. The dynamic solution shows that the discontinuous transition can only occur in the direction stick-to-separation and that this only occurs when the limiting friction boundary of the stick region is reached.

Some tests were carried out to determine whether the same transition could be precipitated elsewhere in the multiple solution range by a small perturbation. For example, if a small amplitude vibration is imposed on the rigid support, the reaction forces will fluctuate, but a transition will only occur if the vibration is of sufficiently large (finite) magnitude. The magnitude required tends continuously to zero as the operating point approaches the limiting backward slip conditions. Alternatively, if a small normal impulse is applied to the mass, so as to give it a small positive value of $\dot{u}_2$, the system experiences a short period of separation before returning to stick. During the separation period, a small lateral motion also occurs in the negative x_1 direction. Thus, if a series of such impulses were imposed, the mass would creep towards the limiting point and eventually experience the discontinuous transition to separation. This behavior would be predicted however small an impulse were imposed. However, if the impulse is replaced by a more practical force-time history of finite duration, we once again find that there is a limiting magnitude of disturbance required to precipitate motion and this magnitude increases the further we are from the limiting backward slip condition. Thus, we must conclude that the system is strictly stable in the stick condition, but that for practical systems subjected to small but finite disturbances, the discontinuous transition to separation will occur somewhat before the limiting condition is reached.

Klarbring's analysis of the rate problem exposes one other scenario in which the quasi-static algorithm in nonunique—when the contact reactions R_1, R_2 and the normal displacement u_2 are all zero and the loading rate $\dot{F}_1, \dot{F}_2$ is directed into the multiple solution segment of Figure 2b. The reactions and the normal displacements must pass through zero at any continuous transition from contact to separation of vice versa.

It can be shown that the condition $R_1 = R_2 = u_2 = 0$ cannot be reached with a local loading rate in the multiple solution segment. If the system is in separation, a loading rate in this segment *increases* the normal displacement u_2 and if it is in stick or forward slip, this loading rate causes an *increase* in the reaction forces. Thus, Klarbring's multiple solution scenario can only be precipitated if the loading rate is discontinuous at the instant when the condition $R_1 = R_2 = u_2 = 0$ is realized.

There is no difficulty with permitting this level of discontinuity of loading—conditions at the critical point are then exactly analogous with those discussed in Sections 2.3 and 2.4 and the dynamic solution shows that the state realized will be that of stick.

However, in practice, this condition will almost never occur. If the system is in a state of stick and F_1, F_2 are changed in such a way as to reduce R_1, R_2 to zero, the usual behavior is one of

(1) a transition occurs to forward slip and the system then remains in this state until the reactions go to zero, after which a transition occurs to separation: or
(2) the reactions reach the limiting *backward* slip condition and a discontinuous transition to separation occurs as explained above.

The system can also remain in the state of stick to ensure the reactions satisfy inequalities (2) at all times until both reactions are zero, but only if the forces F_1, F_2 are carefully controlled.

If the system is in a state of separation and F_1, F_2 are changed in such a way as to reduce the normal displacement u_2 to zero, the usual transitions are either to forward slip or stick, depending on the local values of $\dot{F}_1, \dot{F}_2$ and governed by Figure 2b.

These considerations enable us to develop an alternative quasi-static algorithm for the case $fk_{12} > k_{11}$ which is elaborated in Appendix B. This algorithm gives a unique solution for all loading scenarios in which $\dot{F}_1, \dot{F}_2$ are continuous functions of t (which excludes entering the multiple-solution segment through the origin as explained above).

Figure 8 compares the behavior predicted by the revised algorithm and the dynamic solution. The system in this figure is the same as that for Figure 6 but the loading trajectory is changed as shown to give an initial period of stick before the limiting condition $R_1 = fR_2$ is passed.

5. CONCLUSIONS

The dynamic model developed in this paper resolves the issues of nonexistence and nonuniqueness for high coefficient of friction exposed in Klarbring's two degree of freedom model. We find that one of the two slip directions becomes an unstable state that is realizable only for relatively short transient periods and that a true quasi-static model of the system will then involve discontinuous motions of the mass when frictional limits in this direction are reached. A revised quasi-static algorithm is proposed for fairly general loading in this range of friction coefficients.

APPENDIX A
DYNAMIC NUMERICAL SOLUTION ALGORITHM

In this section, we summarize the equations used for updating the position and velocity of the mass at each time step and the tests used to determine when a change of state occurs.

- **Stick.** If the system is in a state of stick at time t, we have $u_2(t) = 0$ and $\dot{u}_1 = \dot{u}_2 = \ddot{u}_1 = \ddot{u}_2 = 0$. The reaction forces R_1, R_2 can then be obtained from the equations of motion (22) and (23) in the form

$$R_1 = k_{11}u_1 - F_1 \qquad \text{and} \qquad R_2 = k_{21}u_1 - F_2, \tag{49}$$

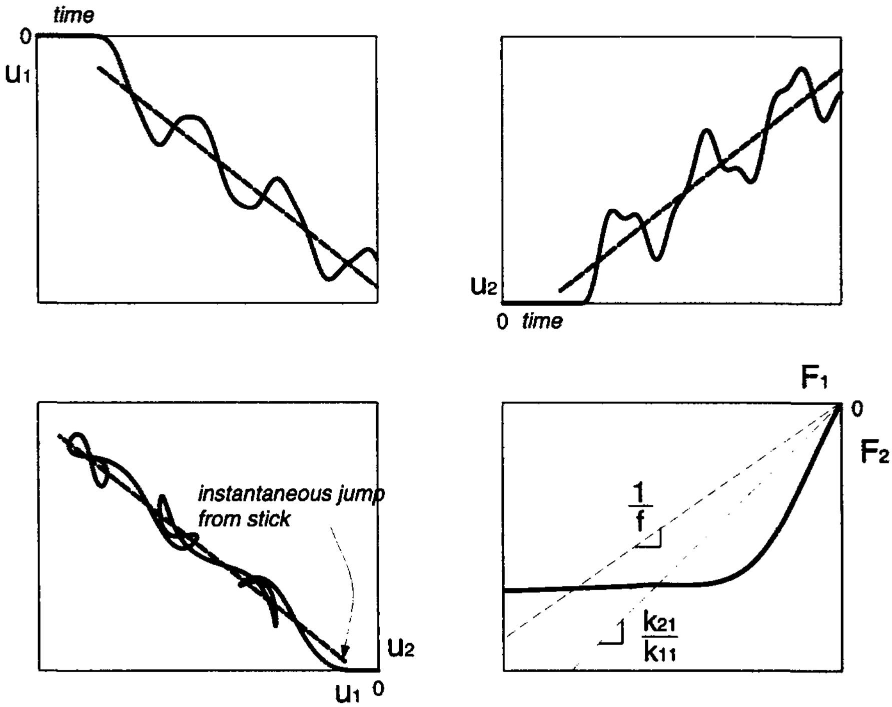


Figure 8. Instantaneous jump predicted by revised quasi-static algorithm (- - -) compared with dynamic solution (—).

where we have reinstated the more general form F_1, F_2 for the applied forces, which can be arbitrary functions of time in the numerical solution. Stick continues as long as $R_2 > 0$ and $|R_1| < fR_2$. If a violation of one of these inequalities is detected in any time step, a state change is made as follows.

(1) $R_2 < 0$. Set $R_2 = 0$ and change state to separation.
(2) $R_2 > 0$, and $R_1 > fR_2$. Set $R_1 = fR_2$ and change state to backward slip.
(3) $R_2 > 0$, and $R_1 < -fR_2$. Set $R_1 = -fR_2$ and change state to forward slip.

- **Forward slip.** During forward slip, we have $u_2(t) = \dot{u}_2(t) = \ddot{u}_2(t) = 0$ and $R_1 = -fR_2$. Substituting into equations (22) and (23) and solving for $\ddot{u}_1, R_2$ we obtain

$$\ddot{u}_1 = -k_{11}u_1 + F_1 - fR_2 \qquad \text{and} \qquad R_2 = k_{21}u_1 - F_2. \tag{50}$$

Forward slip continues as long as $\dot{u}_1 > 0$ and $R_2 > 0$. If a violation of one of these inequalities is detected, a state change is made as follows.

(1) $R_2 < 0$. Set $R_2 = 0$ and change state to separation.
(2) $R_2 > 0$, $\dot{u}_1 < 0$, and $k_{11}u_1 - F_1 - fR_2 < 0$. Set $\dot{u}_1 = 0$ and change state to stick.
(3) $R_2 > 0$, $\dot{u}_1 < 0$, and $k_{11}u_1 - F_1 - fR_2 > 0$. Change state to backward slip.

Notice, that in contrast to the quasi-static solution, a direct transition from forward slip to backward slip is possible without an intervening stick period of finite duration. This will occur if the decelerating forces at the transition are sufficiently large to retain the same sign when the sign of the friction force is changed. This condition is explicitly defined in the given state change algorithm in the interests of rigor. However, the simpler 'quasi-static' algorithm, in which a transition to stick is imposed when $\dot{u}_1 < 0$, can also be used without much loss of accuracy. In cases where the above algorithm predicts a forward to backward slip transition, the simpler algorithm will interpolate a single time increment of stick between the two slip periods and this will have only a local effect on the predicted trajectory.

- **Backward slip**. During backward slip we have $u_2(t) = \dot{u}_2(t) = \ddot{u}_2(t) = 0$ and $R_1 = fR_2$. We obtain

$$\ddot{u}_1 = -k_{11}u_1 + F_1 + fR_2 \qquad \text{and} \qquad R_2 = k_{21}u_1 - F_2, \tag{51}$$

  and backward slip continues as long as $\dot{u}_1 < 0$ and $R_2 > 0$. For
  (1) $R_2 < 0$. Set $R_2 = 0$ and change state to separation.
  (2) $R_2 > 0$, $\dot{u}_1 > 0$, and $k_{11}u_1 - F_1 + fR_2 > 0$; set $\dot{u}_1 = 0$ and change state to stick,
  (3) $R_2 > 0$, $\dot{u}_1 > 0$, and $k_{11}u_1 - F_1 + fR_2 < 0$; change state to forward slip.
- **Separation** continues as long as $u_2 > 0$. We assume an inelastic impact condition, since in more complex problems the effects of elastic recovery will be captured by the dynamics of the spring-mass system. A normal impulse is required to reduce the approach velocity to zero and a proportional frictional impulse will be generated if the system changes to a state of slip. If this impulse is sufficient to cancel the tangential velocity $\dot{u}_1$, a transition to stick will occur. We therefore obtain the following.
  (1) $u_2 < 0$ and $\dot{u}_1 > -f\dot{u}_2$. Change state to forward slip, set $u_2 = \dot{u}_2 = 0$ and $\dot{u}_1 = \dot{u}_1 + f\dot{u}_2$.
  (2) $u_2 < 0$ and $f\dot{u}_2 < \dot{u}_1 < -f\dot{u}_2$. Change state to stick and set $u_2 = \dot{u}_2 = \dot{u}_1 = 0$.
  (3) $u_2 < 0$ and $\dot{u}_1 < f\dot{u}_2$. Change state to backward slip, set $u_2 = \dot{u}_2 = 0$ and $\dot{u}_1 = \dot{u}_1 - f\dot{u}_2$.

  In these expressions, it should be noted that the mass must be approaching the plane for the transition to occur and hence $\dot{u}_2 < 0$.

APPENDIX B

REVISED QUASI-STATIC ALGORITHM FOR $fk_{12} > k_{11}$

Only three states, stick, forward slip, and separation are recognized.

(1) **Stick** is defined as in Appendix A. After each time increment, the reaction forces R_1, R_2 are calculated. In case of violation of any of the inequalities, the following state changes are made.
  (a) $R_2 < 0$. Set $R_2 = 0$ and change state to separation.
  (b) $R_2 > 0$ and $R_1 > fR_2$. Set $R_1 = R_2 = 0$ and change state to separation (discontinuous changes in displacements).
  (c) $R_2 > 0$ and $R_1 < -fR_2$. Set $R_1 = -fR_2$ and change state to forward slip.

(2) **Forward slip**. During forward slip, we have $u_2(t) = \dot{u}_2(t) = 0$ and $R_1 = -fR_2$. Substituting into equations (7) and (8) and solving for u_1, R_1, we obtain

$$u_1 = \frac{fF_2 + F_1}{fk_{21} + k_{11}} \qquad \text{and} \qquad R_2 = \frac{-k_{11}F_2 + k_{21}F_1}{fk_{21} + k_{11}}. \tag{52}$$

  Forward slip continues as long as $\dot{u}_1 > 0$ and $R_2 > 0$. The sign of $\dot{u}_1$ can be determined by comparing the current and previous values of u_1. If a violation of one of these inequalities is detected, a state change is made as follows.
  (a) $R_2 < 0$. Set $R_2 = 0$ and change state to separation.
  (b) $R_2 > 0$, $\dot{u}_1 < 0$. Set $\dot{u}_1 = 0$ and change state to stick.

(3) **Separation** continues as long as $u_2 > 0$. For contact to occur, the immediately preceding value of $\dot{u}_2 < 0$, and hence, $k_{11}\dot{F}_2 < k_{21}\dot{F}_1$ from the generalized form of equation (19). The corresponding state changes are as follows.
  (a) $u_2 < 0$ and $(-1/f)\dot{F}_1 < \dot{F}_2 < (k_{21}/k_{11})\dot{F}_1$. Change state to forward slip, set $u_2 = 0$ and $u_1 = (fF_2 + F_1)/(fk_{21} + k_{11})$.
  (b) $u_2 < 0$ and $(k_{21}/k_{11})\dot{F}_2 < \dot{F}_1 < -f\dot{F}_2$. Change state to stick and set $u_2 = 0$.

REFERENCES

1. A. Klarbring, Contact problems with friction, Doctoral Dissertation, Linköping University, Sweden, (1984).
2. A. Klarbring, Examples of nonuniqueness and nonexistence of solutions to quasi-static contact problems with friction, *Ingenieur-Archiv* **60**, 529–541 (1990).
3. J.A.C. Martins and J.T. Oden, Existence and uniqueness results for dynamic contact problems with nonlinear normal and friction interface laws, *Nonlinear Analysis* **11**, 407–428 (1987).
4. J.A.C. Martins, M.D.P. Monteiro Marques, F. Gastaldi and F.M.F. Simões, A two-degree-of-freedom "quasistatic" frictional contact problem with instantaneous jumps, *Proc. Contact Mechanics Int. Symp.* (Edited by A. Curnier), PPUR, 217–228 (1992).
5. J.A.C. Martins, M.D.P. Monteiro Marques and F. Gastaldi, On an example of nonexistence of solution to a quasistatic frictional contact problem, *European Journal of Mechanics. A/Solids* **13**, 113–133 (1994).