

## Chapter III

### Stirling's Formula

Our object is to prove the famous formula

$$N! \sim \sqrt{2\pi N} \left(\frac{N}{e}\right)^N,$$

which is to say that the ratio of the left hand side divided by the right hand side tends to 1 as  $N$  tends to infinity. We begin by observing that

$$\log N! = \log 1 + \log 2 + \cdots + \log N,$$

which is a right-hand Riemann sum for the integral

$$\int_1^N \log u \, du = N \log N - N + 1.$$

Since the integrand is increasing it follows that  $(N/e)^N < N!$ . To obtain a more precise estimate we employ a trapezoidal approximation to the integral instead of the right-hand Riemann sum. That is, we take  $\frac{1}{2}f(0) + \frac{1}{2}f(1)$  to be our approximation to  $\int_0^1 f(u) \, du$ . If  $f$  is linear in the interval  $0 \leq u \leq 1$  then the approximation is exact, and thus we may expect that the difference between these two quantities can be expressed in terms of  $f''(u)$ , which is a measure of the extent to which  $f$  is not linear.

**Lemma.** *If  $f''(u)$  is continuous throughout the interval  $[0, 1]$  then*

$$\int_0^1 f(u) \, du = \frac{1}{2}f(0) + \frac{1}{2}f(1) - \frac{1}{2} \int_0^1 u(1-u)f''(u) \, du.$$

*Proof.* By integration by parts we see that the left hand side above is

$$= \left[ (u - 1/2)f(u) \right]_0^1 - \int_0^1 (u - 1/2)f'(u) \, du = \frac{1}{2}f(0) + \frac{1}{2}f(1) - \int_0^1 (u - 1/2)f'(u) \, du.$$

By a second integration by parts we see that this latter integral is

$$= \left[ \left( \frac{1}{2}u^2 - \frac{1}{2}u \right) f'(u) \right]_0^1 - \frac{1}{2} \int_0^1 (u^2 - u) f''(u) du = \frac{1}{2} \int_0^1 u(1-u) f''(u) du.$$

On substituting this in the former identity, we obtain the stated identity.

To continue with the proof of Stirling's Formula we let

$$c_n = \int_0^1 \log(n+u) du - \left( \frac{1}{2} \log n + \frac{1}{2} \log(n+1) \right).$$

We take  $f(u) = \log(n+u)$  in the Lemma, and observe that  $f''(u) = -1/(n+u)^2$ . Then by the Lemma we see that

$$c_n = \frac{1}{2} \int_0^1 \frac{u(1-u)}{(n+u)^2} du.$$

Hence

$$0 \leq c_n \leq \frac{1}{2n^2} \int_0^1 u(1-u) du = \frac{1}{12n^2}.$$

On summing this over  $n$ , for  $1 \leq n \leq N-1$ , we find that

$$\frac{1}{2} \log 1 + \log 2 + \cdots + \log(N-1) + \frac{1}{2} \log N = \int_1^N \log u du - \sum_{n=1}^{N-1} c_n.$$

On adding  $\frac{1}{2} \log N$  to both sides we deduce that

$$\log N! = N \log N - N + \frac{1}{2} \log N + 1 - \sum_{n=1}^{N-1} c_n.$$

Put  $C = 1 - \sum_{n=1}^{\infty} c_n$ . The series here is convergent by comparison with the series  $\sum 1/n^2$ . From the above we see that

$$N \log N - N + \frac{1}{2} \log N + C \leq \log N! = N \log N - N + \frac{1}{2} \log N + C + \sum_{n=N}^{\infty} c_n.$$

To estimate the tail of the series, we note that

$$\sum_{n=N}^{\infty} c_n \leq \sum_{n=N}^{\infty} \frac{1}{12n^2} < \frac{1}{12} \sum_{n=N}^{\infty} \frac{1}{n(n-1)} = \sum_{n=N}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{12(N-1)}.$$

Thus on exponentiating, we see that

$$\left( \frac{N}{e} \right)^N \sqrt{N} e^C \leq N! \leq \left( \frac{N}{e} \right)^N \sqrt{N} e^C e^{1/(12(N-1))}.$$

That is,

$$(1) \quad N! = e^C \sqrt{N} \left(\frac{N}{e}\right)^N (1 + O(1/N)),$$

which is a quantitative form of Stirling's formula, apart from the fact that it remains to show that  $e^C = \sqrt{2\pi}$ . This we establish by means of a sneaky trick. For non-negative integers  $n$  let

$$I_n = \int_0^\pi (\sin u)^n du.$$

Thus  $I_0 = \pi$  and  $I_1 = 2$ . We observe that for larger  $n$ ,

$$I_n = \int_0^\pi (\sin u)^{n-2} (1 - \cos^2 u) du = I_{n-2} - \int_0^\pi (\sin u)^{n-2} \cos^2 u du.$$

By integration by parts we see that this latter integral is

$$\left[ \frac{(\sin u)^{n-1}}{n-1} \cos u \right]_0^\pi + \int_0^\pi \frac{(\sin u)^n}{n-1} du = \frac{I_n}{n-1}.$$

On combining this with the former identity we deduce that

$$I_n = \frac{n-1}{n} I_{n-2}.$$

Hence by induction,

$$I_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2} \pi,$$

$$I_{2n+1} = \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 1} 2.$$

On one hand we observe that  $2 \cdot 4 \cdots (2n) = 2^n n!$ , while on the other hand we see that

$$1 \cdot 3 \cdots (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdots (2n)} = \frac{(2n)!}{2^n n!}.$$

Thus the formulæ above can be written as

$$I_{2n} = \frac{(2n)!}{2^{2n} n!^2} \pi,$$

$$I_{2n+1} = \frac{n!^2 2^{2n+1}}{(2n)!(2n+1)}.$$

Since  $0 \leq \sin u \leq 1$  for  $0 \leq u \leq \pi$ , it follows that the numbers  $I_n$  are decreasing. In particular,

$$I_{2n} \geq I_{2n+1} \geq I_{2n+2} = \frac{2n+1}{2n+2} I_{2n}.$$

On dividing through by  $I_{2n}$  it follows that

$$\frac{2n+1}{2n+2} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1.$$

By the "Squeeze Theorem" it follows that  $\lim I_{2n+1}/I_{2n} = 1$ . But by (1) this ratio is

$$\frac{n!^4 2^{4n+1}}{(2n)!^2 (2n+1)\pi} \sim \frac{(n/e)^{4n} n^2 e^{4C} 2^{4n+1}}{(2n/e)^{2n} 2n e^{2C} (2n+1)\pi} \sim \frac{e^{2C}}{2\pi}.$$

Thus this last constant is 1, which is to say that  $e^C = \sqrt{2\pi}$ .

There are other ways to determine this constant, one of them being particularly relevant to our intended use of Stirling's formula to show that the binomial distribution is approximately normal. If we leave the constant in the form  $e^C$  in that application, then the constant is found in the denominator, since the binomial coefficient has one factorial in the numerator and two in the denominator. That is, we would find that when  $k$  is near  $np$  then

$$\binom{n}{k} p^k (1-p)^{n-k} \sim \frac{1}{e^C \sigma} e^{-(k-\mu)^2/(2\sigma^2)}$$

where  $\mu = np$  and  $\sigma = \sqrt{np(1-p)}$ . Here the sum of the left hand side is exactly 1, by the binomial theorem. The sum of the right hand side is approximately

$$\frac{1}{e^C \sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)} dx = \frac{\sqrt{2\pi}\sigma}{e^C \sigma}.$$

Since this constant must be 1, it follows that  $e^C = \sqrt{2\pi}$ .

It should be emphasized that although the ratio of  $N!$  divided by  $\sqrt{2\pi N}(N/e)^N$  tends to 1, the difference between these two quantities is still large, indeed close to  $(N-1)!$  in size. For example,

$$100! = 9.3326215443944 \times 10^{157},$$

while

$$\sqrt{200\pi}(100/e)^{100} = 9.3244847625269 \times 10^{157}.$$

By taking more care with the analysis our error term can be expanded into further (smaller) main terms; for example, it can be shown that

$$N! = \sqrt{2\pi N} \left(\frac{N}{e}\right)^N \left(1 + \frac{1}{12N} + O(1/N^2)\right).$$