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Axisymmetric Transonic MHD Flows

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Introduction

Magnetohydrodynamic waves and instabilities of both laboratory and astrophysical plasmas have been studied for many years starting from the assumption of static equilibrium. This assumption is violated in most situations of interest. In fusion research, neutral beam heating and pumped divertor action result in rotation of the plasma with considerable speeds. In astrophysics, plasma flows occur in coronal flux tubes, stellar winds, rotating accretion disks, jets emitted from radio galaxies, etc. In all of those cases, supersonic speeds are a rule rather than an exception. Consequently, it is important to study the stationary states of magnetized plasmas with flow.

A class of explicit self-similar translation symmetric transonic MHD flows has been constructed which exhibits almost all of the intricacies of transonic flow, e.g. the existence of four distinct velocity regimes, which may be elliptic or hyperbolic but cannot be connected by continuous flows, a plethora of different kinds of shocks, limiting line singularities, etc. [1], [2]. This class of solutions is generalised to axisymmetry, so that the issues raised in Ref. [1] with respect to the possibility of crossing limiting line singularities can be studied in relevant geometries for fusion as well as astrophysical plasmas. This approach is supplemented by a numerical study of genuine boundary value problems in stellar wind acceleration, exploiting the Versatile Advection Code [3]. We address the relationship between the specific classes of self-similar flows and the more general flows obtained from large-scale 3D MHD codes such as VAC.

Variational principle

We exploit cylindrical coordinates (R, Z, φ). For $\partial\varphi = 0$ and $\partial t = 0$, mass and flux conservation lead to a stream function $\chi(R, Z)$ for the poloidal velocity and a flux function $\psi(R, Z)$ for the poloidal magnetic field, whereas $\chi = \chi(\psi)$. Axisymmetric stationary ideal MHD equilibria are then derivable from a single variational principle $\delta L = \delta \int \mathcal{L} dV = 0$ with Lagrangian density

$$\mathcal{L}(M^2, \psi, \nabla\psi; R, Z) = \frac{1}{2R^2}(1 - M^2)|\nabla\psi|^2 - \frac{\Pi_1}{M^2} + \frac{\Pi_2}{\gamma M^{2\gamma}} - \frac{\Pi_3}{1 - M^2}\,. \qquad (1)$$

The associated minimizing Euler-Lagrange equations are a PDE for the flux function $\psi(R, Z)$,

$$\nabla \cdot \left(\frac{1 - M^2}{R^2} \nabla\psi \right) + \frac{{\Pi_1}'}{M^2} - \frac{{\Pi_2}'}{\gamma M^{2\gamma}} + \frac{{\Pi_3}'}{1 - M^2} = 0\,, \qquad (2)$$

where $' \equiv \partial/\partial\psi$, and the algebraic Bernoulli equation for the squared poloidal Alfvén Mach number $M^2(R,Z) \equiv \rho v_p^2/B_p^2 = \chi'^2/\rho$:

$$\frac{1}{2R^2}|\nabla\psi|^2 - \frac{\Pi_1}{M^4} + \frac{\Pi_2}{M^{2(\gamma+1)}} + \frac{\Pi_3}{(1-M^2)^2} = 0\,. \tag{3}$$

These equations express force balance across poloidal streamlines and energy conservation along them. Similar equations were obtained for translationally symmetric stationary equilibria [1], [2].

Profiles and flux functions

In contrast with the translationally symmetric case, the governing variational principle contains factors R^2, while the profiles Π_1 and Π_3 are no longer flux functions. In particular,

$$\Pi_1 \equiv \chi'^2 \left(H + \frac{R^2\Omega^2}{2} + \frac{GM_g}{r} \right)\,, \tag{4}$$

$$\Pi_2 \equiv \frac{\gamma}{\gamma-1}\chi'^{2\gamma} S\,, \tag{5}$$

$$\Pi_3 \equiv \frac{\chi'^2}{2}\left(R\Omega - \frac{L}{R} \right)^2\,. \tag{6}$$

where five flux functions H, Ω, S, L, χ' enter. These are the Bernoulli function $H \equiv \frac{1}{2}v^2 + (\gamma/(\gamma-1))\,p/\rho - GM_g/r - R\Omega v_\varphi$ ($\sim$ energy, including the gravitational potential energy for a star of mass M_g), the derivative of the stream function $\chi' \equiv \partial\chi/\partial\psi$, the entropy S, the total specific angular momentum $L \equiv Rv_\varphi - RB_\varphi/\chi'$, and the derivative of the electric field potential $\Omega \equiv R^{-1}(v_\varphi - (\chi'/\rho)B_\varphi)$. By an appropriate choice of the free flux functions, we can construct axisymmetric, stationary flows. Typically, self-similar solutions can be analysed further by analytic means, as in Refs. [1] and [2]. General solutions require a numerical solution procedure. Such a procedure is discussed in the next section.

Versatile Advection Code (VAC)

VAC [3] is a general purpose software package for solving a conservative system of hyperbolic partial differential equations with additional non-hyperbolic source terms. It runs on PCs, workstations, vector and parallel architectures.

The system of PDEs is solved on a finite volume structured grid. Particular systems available are the hydrodynamic (Euler and Navier-Stokes) and the magnetohydrodynamic (ideal and resistive) equations. The source code is dimension independent, so that it can solve 1D, 2D, 2.5D, and 3D problems. The data are analyzed with IDL, AVS, or any other visualization package.

A modular approach ensures the compatibility between the different code segments. As a result, several spatial and temporal (explicit, semi-implicit and fully implicit) discretizations are applicable to all physics modules.

1D transonic stellar outflows

With VAC, we can calculate steady-state transonic solutions to the MHD equations which mimick thermally and/or magneto-centrifugally driven stellar outflows. Under polytropic $p = \rho^\gamma/\gamma$ conditions, analytic solutions are available for equatorial outflow from a magnetized, rotating star. This Weber-Davis solution has three critical points: the Alfvén point r_A where the radial velocity v_r equals the radial Alfvén speed A_r, and the slow r_s and fast r_f critical points.

Using fully implicit time-stepping, we start from a hydrodynamic, non-rotating, spherically symmetric, isothermal outflow, and relax all assumptions to allow for magnetic fields, rotation and polytropic flows. The obtained solution (Fig. 1) checks precisely with the predicted analytic form. Plotted are the Alfvén, slow and fast Mach numbers as a function of radial distance. The positions of the critical points are indicated. Parameters model solar wind conditions.

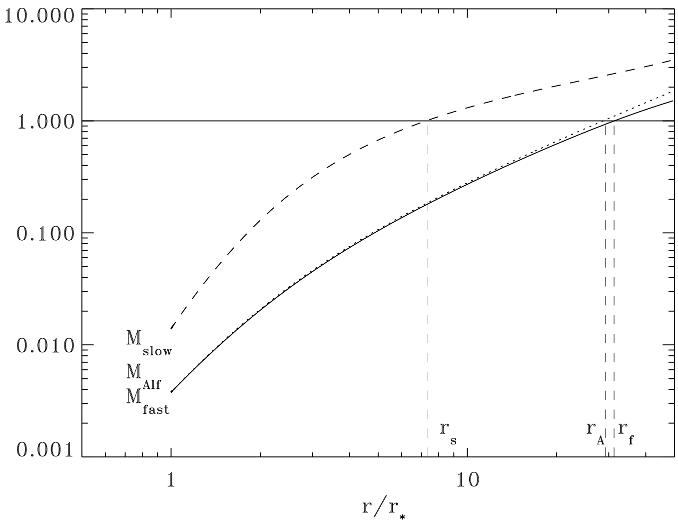

Figure 1: *Alfvén, slow, and fast Mach numbers as a function of radial distance.*

Axisymmetric transonic MHD winds

Extrapolating the Weber-Davis solution out of the equatorial plane yields polytropic, axisymmetric and transonic outflow from magnetized, rotating stars. One obtains critical surfaces, and a continuous acceleration from sub-slow flow at the surface to super-fast flow at large distances. A solution for a purely radial stellar magnetic field is shown in Fig. 2.

Poloidal streamlines of constant $\chi(R, Z)$ and poloidal field lines of constant $\psi(R, Z)$ are parallel. Specific angular momentum is conserved along them, $L(\psi) = Rv_\varphi - RB_\varphi B_p/\rho v_p$. The polytropic condition implies $S = 1/\gamma$. We verify $\Omega(\psi) = [\, v_\varphi - (v_p/B_p)B_\varphi\,]/R$ and the Bernoulli function $H(\psi) = [\, v_R^2 + v_z^2 + v_\varphi^2\,]/2 + \rho^{\gamma-1}/(\gamma-1) - GM_g/r - R\Omega v_\varphi$.

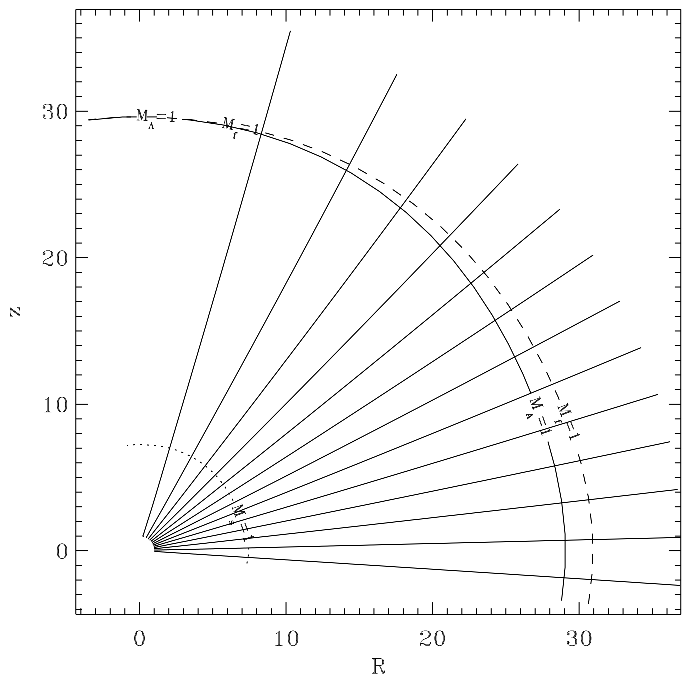

Figure 2: *Radial outflow from a rotating star.*

Conclusions

In analogy with the translationally symmetric, transonic MHD flows, we find a single variational principle governing all axisymmetric transonic MHD flows. Our analysis addresses the relationship between specific classes of self-similar flows which can be studied analytically, and the general flow solutions obtained by a typical large-scale 3D MHD code. VAC is such a general-purpose code, and we demonstrated its capacity to model stellar winds. Different flow regimes, shocks, and limiting line singularities can then be explored by a combined analytical-numerical approach.

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