Notes on and solutions to Gavin's Differential Equations Projects, 1995-1996

by Gavin LaRose (glarose@NebrWesleyan.edu), Nebraska Wesleyan University, May 1996

Project 1: A question of law

To solve this problem, we need to find the velocity as a function of height fallen, given the differential equation suggested by Newton's law ($F_{tot} = ma$, where F_{tot} is the total force on a "body" and m and a are its mass and acceleration, respectively) and the "physical measurements department":

$$m\frac{dv}{dt} = mg - Kv^2. \tag{1.a}$$

The easiest way to solve this is to rewrite the time derivative in terms of distance fallen, s, using the chain rule—

$$m\frac{ds}{dt}\frac{dv}{ds} = mg - Kv^2,$$

or

$$mv\frac{dv}{ds} = mg - Kv^2.$$

This is easily solved by separating variables to obtain

$$\frac{v\,dv}{mg - Kv^2} = \frac{ds}{m}.$$

Integrating both sides and solving for v, we find

$$v = \sqrt{\frac{mg}{K} - Ce^{-2ks/m}}. \tag{1.1}$$

The acceleration due to gravity, g, is $g = 32.2$ ft/s^2 in English units. Because the client's weight is 180 pounds, their mass is 180 pounds/g, or $m = 5.59$ slugs. Finally, from equation (1.1) we know that the terminal velocity is $v_T = \sqrt{mg/K}$, which for a female partner of the firm is 120 mph. Assuming that the weight of this partner is 120 pounds and using that $v_T = 120$ mph$= 176$ ft/s, we have $K = mg/v_T^2 = (120/176^2) = 3.87 \times 10^{-3}$.

Finally, to find C, note that $v(0) = 0$, so $C = mg/K$. The actual velocity for the client falling from each window is then easily found. (In that the difference in height between the first and second story window is 10 feet, it seems reasonable to assume that we are interested in $s = 5$, $s = 15$, $s = 25$, and $s = 45$.)

The harder way to solve the problem (which is the manner that most of my students approached it, in that I didn't give them huge numbers of hints on using the chain rule) is to attack the original time-dependent problem using partial fractions. In this case,

$$\frac{dv}{mg - Kv^2} = \frac{dt}{m},$$

so that, after breaking up the integral on the right-hand side, integrating, and solving for v, we find

$$v = \sqrt{\frac{gm}{K}} \cdot \frac{C\exp(2\sqrt{Kg\,t/m}) - 1}{C\exp(2\sqrt{Kg\,t/m}) + 1}. \tag{1.2}$$

Applying the initial condition $v(0) = 0$, we find that $C = 1$, which is nice. To find distance as a function of time, we have to integrate this again, for the purposes of which it is convenient to divide out a factor of $\exp(\sqrt{Kg\,t/m})$ from the top and bottom first, to obtain

$$v = \sqrt{\frac{gm}{K}} \cdot \frac{\exp(\sqrt{Kg\,t/m}) - \exp(-\sqrt{Kg\,t/m})}{\exp(\sqrt{Kg\,t/m}) + \exp(-\sqrt{Kg\,t/m})}.$$

This is now easily integrated by substitution, giving

$$s = \frac{m}{K}\ln(e^{\sqrt{Kg/m}\,t} + e^{\sqrt{Kg/m}\,t}).$$

Then, to find the actual times at which we arrive at given distances it is convenient to solve for t (or, if we should happen to be getting tired of algebra—unlikely though that might be in the case of the assiduous students with whom we are so often confronted—a numerical root finder can be used to find said values of t). This is straightforward, yielding

$$t = \sqrt{\frac{m}{Kg}}\operatorname{arccosh}(e^{Ks/m}). \tag{1.3}$$

At which point completion of the problem is easy. Alternately, we can recognize that the expression for $v(t)$ in equation (1.2) is equivalent to $v(t) = \sqrt{gm/K} \cdot \tanh(\sqrt{Kg/m} \cdot t)$ so that, substituting (1.3) for t, we recover (after using $\sinh^2(x) - \cosh^2(x) = 1$ and $\cosh(\sqrt{Kg/m} \cdot t) = e^{Ks/m}$ (from (1.3)))

$$v = \sqrt{\frac{gm}{K}}\left(\frac{\sqrt{e^{2Ks/m} - 1}}{e^{ks/m}}\right), \tag{1.4}$$

which is equivalent to (1.1).

Some quick comments—
The points that my students had the most troubles with were—
1. (surprisingly, to me) deducing equation (1.a);
2. doing the partial fractions integration to find $v(t)$;
3. figuring out how to find $s(t)$; and
4. figuring out what K was.

At the risk of stating the obvious, most of them also fuzzed the mass/weight distinction which isn't really so clear as all that when using English units, and had to be reminded of it.

Project 2: Farming the great ranges of the West...

This problem was in some senses easier than the first, but I think perhaps harder to do a good job on. For the first part, the easiest way to make the transition from equation (2.1),

$$\frac{dN}{dt} = RN(1 - \frac{N}{K}) - P(1 - e^{-N^2/\epsilon A^2}), \tag{2.1}$$

to equation (2.2),

$$\frac{du}{d\tau} = ru(1 - \frac{u}{q}) - (1 - e^{-u^2/\epsilon}), \tag{2.2}$$

is to let $N = \beta u$ and $t = \alpha\tau$. Substituting in, the left-hand side is multiplied by β/α and the right-hand side has various terms involving β and α. Simplifying, we find

$$\frac{du}{d\tau} = R\alpha u(1 - u\frac{\beta}{K}) - \frac{P\alpha}{\beta}(1 - e^{-\beta^2 u^2/\epsilon A^2}),$$

so that if $r = AR/P$, $q = K/A$, $\alpha = A/P$, and $\beta = A$, the transformation is effected.

Equation (2.2) is a reasonable model for the fish population as described—the rate of change of the population is "proportional to the number of fish present" due to the ru term, and "limited by the number of fish that are able to be supported by the space on the farm," due to the term $-ru^2/q$. The second term in the equation is predation, which indeed produces "a measurable effect on the fish population whenever there are significant numbers of fish present"—when $|u| \not\ll 1$, $e^{-u^2/\epsilon} \ll 1$ and the second term becomes simply -1, a (constant) measurable effect.

To analyze the behavior of (2.2), it is most convenient to plot the phase-plane for a number of different values of r, q, and ϵ. Further, in that q is given to be "close to 1," we take $q = 1$, and as ϵ is small, we take $\epsilon = .001$. Then, for $r = 1, 3, 5, 15$ and 30, we obtain figure 2.1.

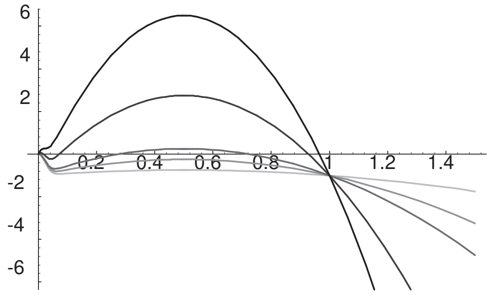

Figure 2.1: $u'(t)$ vs $u(t)$ for r =1, 3, 5, 15, and 30 (light $\rightarrow$ dark curves)

We see that for large values of r, there is a stable steady state $u \approx 1$, and for smaller values of r ($r < 5$ or so) this steady state *does not exist!*. Therefore, if $r \geq 5$ there is a "stable fish population from which harvesting could take place." There is also something interesting going on for small values of u. Looking more closely at this part of the phase plane, we see figure 2.2.

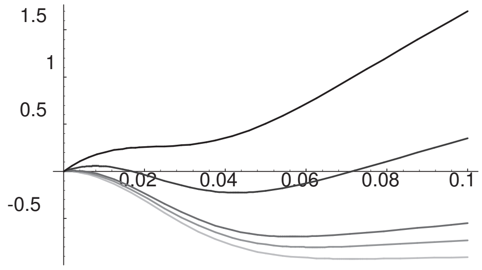

Figure 2.2: $u'(t)$ vs $u(t)$ for r =1, 3, 5, 15, and 30 (light $\rightarrow$ dark curves)

Thus we see that there is a second steady state, which for $r \geq 15$ is very small. In order to obtain the larger steady state, we must start with a fish population that is larger than the unstable steady state between the two stable states. For $r = 5$, this is about $u = .3$ (from figure 2.1). Thus for initial fish populations larger than about $u = .3$ (and $r \geq 5$), the large steady state will be obtained.

Seeing that the final steady state is larger for large values of r, our preliminary recommendation is for the fish-farmers to use an r that is as large as possible. Supposing that this is $r = 25$, we find after numerically solving equation (2.2) that $u(t)$ is given in figure 2.3.

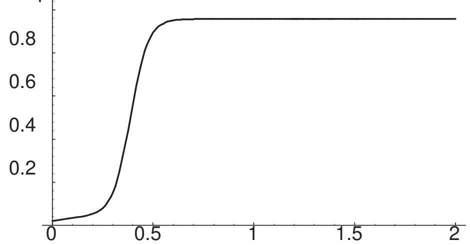

Figure 3: $u(t)$ vs. t, $r = 25$.

Thus, if harvesting is accepted to take the fish population to approximately 80% of its steady state value, it will take approximately 0.2 time units for the population to return to the steady state value, and be ready for reharvesting.

Some quick comments—
The points that my students had the most troubles with were—

1. figuring out how to do the nondimensionalization to obtain equation (2.2) from (2.1);
2. some had trouble with figuring out what the different terms in the equation did, though many were able to deduce this from information they had about population models in the text;
3. many didn't figure out what "analysis of the validity of ([2.2]) as a model for the fish population in a fish farm" meant; and
4. some took a while to figure out that the very best way of analyzing the problem is with a phase plane, and tried direction fields (not as clear) or repeated numerical solution of the differential equation (even worse).

Project 3: defending freedom... or some such

The "freedom" in the title to this one really ought to have been capitalized. This was easily the hardest project, as much because of the required formulation of the differential equation(s) before starting. It may be that more direction should be provided with the early stages of it all, but at the same time I think it's worth making people think about that type of thing too. In any event...

We consider only the two dimensional problem shown in Figure 2 of the problem posed. Ignoring for the moment the fact that it would make better sense to have the defense unit calculate the position of the asteroid and its velocity and launch the defense rocket to intercept it (rather than relying on an internal guidance system), the fact that the rocket is always pointing towards the asteroid means that the tangent to its path will always be the line connecting the instantaneous positions of the rocket and the asteroid, as shown in figure 3.1.

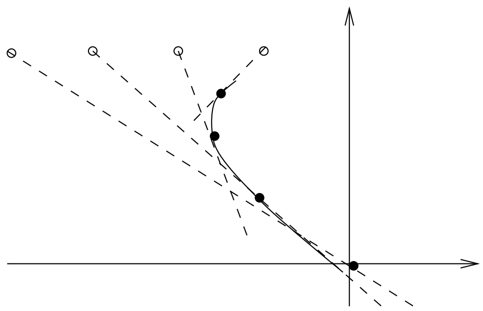

Figure 3.1: Rocket (filled circle) and asteroid (open circle) positions at successive times, showing tangents (dashed) to rocket's path (solid).

If the position of the rocket is given as a function of time by $(x(t), y(t))$ and the velocity of the asteroid is v_a, the slope of the rocket's path is then

$$\frac{dy}{dx} = \frac{y(t) - 5}{x(t) - (-15 + v_a t)}.$$

For parametric equations, $dy/dx = (dy/dt)/(dx/dt)$, so this is equivalent to

$$(x + 15 - v_a t)\frac{dy}{dt} = (y - 5)\frac{dx}{dt}. \tag{3.1}$$

(Note that we have rearranged this to avoid singularities in the equation if the x coordinates of the asteroid and rocket should turn out to be the same...) Further, the velocity of the rocket is specified, say as v_r, so that

$$(x'(t))^2 + (y'(t))^2 = v_r^2. \tag{3.2}$$

Equations (3.1) and (3.2), with initial conditions $(x(0), y(0)) = (0,0)$ constitute a system of equations for the position of the rocket and may be solved numerically. This solution may actually give *two* solutions—both with the path of the rocket always pointing towards the asteroid, but one with the rocket going away from the asteroid and one with it going towards it. For $v_a = 5$ and $v_r = 20$, we obtain figure 3.2 for the case in which the rocket is moving towards the asteroid (the more interesting case).

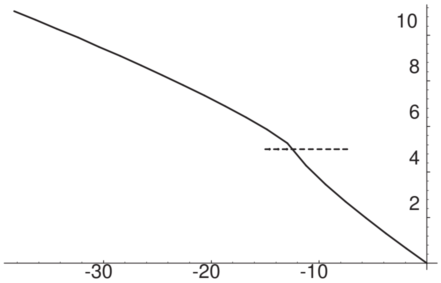

Figure 3.2: Rocket (solid) and asteroid (dashed) positions in space; (0,0) is the defense unit's location.

To verify that there is in fact an interception where the two curves cross, we numerically determine that the rocket reaches $y = 5$ when $t = .6672$, at which time the x coordinates of the asteroid and rocket are also equal, indicating an interception.

Clearly if v_r is very much in excess of v_a the rocket will have no trouble intercepting the asteroid, while if $v_a \gg v_r$ it will be impossible to intercept. However, if $v_a \sim v_r$, we expect that it may or may not be possible to protect the space station. For the case $v_a = 18$, $v_r = 20$, *Mathematica* bails out of the numerical solution of (3.1) and (3.2) after 5000 steps at $t = 0.5754$. Graphically, the paths of the rocket and asteroid to this time are shown in figure 3.3.

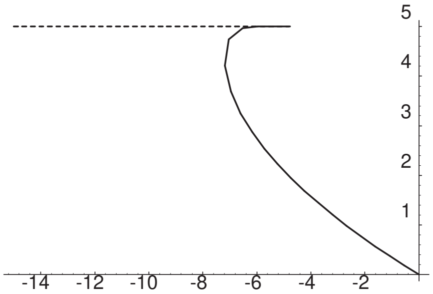

Figure 3.3: Rocket (solid) and asteroid (dashed) positions in space; (0,0) is the defense unit's location.

At this point the rocket is at a y position of $y = 5$ and the rocket and asteroid are at the x positions $x_r = -4.8165$ and $x_a = -4.74$. Therefore, it will take $t_{cr} = (x_a - x_r)/(v_r - v_a)$ seconds for the rocket to catch the asteroid, during which time the asteroid will move from x_a to $x_{cr} = x_a + v_a \cdot t_{cr}$, or $x_{cr} = -4.052$. Therefore, even when the velocities of the asteroid and rocket are as close as $v_a = 18$ and $v_r = 20$ an interception still occurs, but as v_a increases further the point of interception will move closer and closer to the space station. We might wish to declare that the space station is protected if interception occurs while the asteroid is still at least 1 mile from the station, in which case we must have the velocity of the rocket in excess of that of the asteroid, for, as we see here, when the two are close the rocket ends up following the asteroid towards the space station. Therefore v_r must be large enough to catch the asteroid before the edge of our "safe zone," for if $v_a \approx v_r$ we have the unfortunate case in which *both* the asteroid and the rocket hit the station.

Some quick comments—
The points that my students had the most troubles with were—

1. deriving the differential equations (3.1) and (3.2). All groups ended up coming in to talk to me about these;
2. obtaining meaningful numerical solutions to the differential equations that they obtained—in part due to errors using *Mathematica*, and in part because it involved not only solving the differential equations but also dealing with the parametric solutions obtained; and
3. dealing with the numerical difficulties posed by $(i.)$ the possibility that the x positions of the asteroid and rocket could coincide, thus resulting in a singularity if the difference between the two wasn't multiplied

to the right-hand side in equation (3.1), (*ii.*) the case when v_r and v_a were very close in magnitude, and (*iii.*) solving equation (3.2) for either $x'(t)$ or $y'(t)$—this results in a square root with an ambiguous $\pm$ sign; it's safest to leave it in the form (3.2).