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The Final Value Theorem Revisited

Infinite Limits and Irrational Functions

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The final value theorem is an extremely handy result in Laplace transform theory. In many cases, such as in the analysis of proportional-integral-derivative (PID) controllers, it is necessary to determine the asymptotic value of a signal. The final value theorem provides an easy-to-use technique for determining this value without having to first invert the Laplace transform to determine the time signal.

The standard assumptions for the final value theorem [1, p. 34] require that the Laplace transform have all of its poles either in the open-left-half plane (OLHP) or at the origin, with at most a single pole at the origin. In this case, the time function has a finite limit.

Although no limit exists when the Laplace transform has a nonzero pole on the imaginary axis, some textbooks note that the final value theorem can be used when the limit is infinite. For example, in [2, p. 104], (1) given below is used to obtain infinite limits of the closed-loop transfer function for type-0 and type-1 systems with ramp commands as well as for type-1 systems with parabolic commands. Furthermore, [3, p. 96] states that, for poles at the origin, (1) "gives the final value $f(\infty) = \infty$" for a time function $f(t)$. In addition, [4, p. 567], allows poles in the OLHP or at the origin.

The goal of this note is to publicize and prove the "infinite-limit" version of the final value theorem. The version we provide is a slight refinement of the classical literature in that we require that s approach zero through the right-half plane to obtain the correct sign of the infinite limit. We first consider the case of rational Laplace transforms and then state a version that applies to irrational functions.

FINITE-LIMIT CASE

Let $y(t)$ be a signal on $[0, \infty)$, let $\hat{y}(s)$ be its Laplace transform, and define

$$y(\infty) \triangleq \lim_{t\to\infty} y(t)$$

whenever this limit exists. By "exists" we mean that $y(\infty)$ is a real number and $y(t) - y(\infty) \to 0$ as $t \to \infty$. We stress that ∞ and $-\infty$ are not real numbers. For now, we assume that $\hat{y}(s)$ is a proper rational function.

Standard Final Value Theorem

Assume that every pole of $\hat{y}(s)$ is either in the OLHP or at the origin, and assume that $\hat{y}(s)$ has at most a single pole at the origin. Then $y(\infty)$ exists and is given by

$$y(\infty) = \lim_{s\to 0} s\hat{y}(s). \tag{1}$$

Note the "reversal" between t and s in that $t \to \infty$ corresponds to $s \to 0$. Note also the factor of s that precedes $\hat{y}(s)$. A similar reversal occurs in the initial value theorem, which includes a factor of s as well.

As an example, let $\hat{y}(s) = (3s + 2)/(s(s + 1))$. It thus follows from (1) that $y(\infty) = 2$. Indeed, $y(t) = 2 + e^{-t}$.

To see how this result can fail when its hypotheses are not satisfied, consider $y(t) = \sin \omega_0 t$, where $\omega_0 > 0$, so that $\hat{y}(s) = \omega_0/(s^2 + \omega_0^2)$. Since the poles of $\hat{y}(s)$ are not in the OLHP or at the origin, the final value theorem cannot be applied. Although $\lim_{s\to 0} s\hat{y}(s)$ exists for this example, the limiting value 0 is useless since $y(\infty)$ does not exist.

INFINITE-LIMIT CASE

We wish to extend the applicability of (1) beyond the stated conditions on $\hat{y}(s)$. To do this, suppose that $y(\infty)$ does not exist, but assume that $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = -\infty$. Let $y(\infty)$ denote $\pm\infty$ in these cases. For convenience we say that $y(\infty)$ *does not exist but is infinite*.

Note that this definition does not apply to signals such as $y(t) = e^t \sin t$. Alternatively, consider $y(t) = e^t$, so that $y(\infty) = \infty$. Since $\hat{y}(s) = 1/(s - 1)$ it follows that $\lim_{s\to 0} s\hat{y}(s) = 0$, and thus (1) is not satisfied. However, the following result encompasses infinite limits arising from multiple poles at the origin. In the following statement, the notation "$s \downarrow 0$" means that s approaches 0 through the positive numbers.

Note that the limit $s \downarrow 0$ is consistent with the fact that $\hat{y}(s)$ has poles only in the CLHP and is analytic in the ORHP. Hence, the Laplace transform converges in the ORHP and the limit can be taken along the positive real axis, whereas the limit may not exist when taken from the CLHP.

Extended Final Value Theorem

Assume that every pole of $\hat{y}(s)$ is either in the OLHP or at the origin. Then $y(\infty)$ exists and is given by

$$y(\infty) = \lim_{s \downarrow 0} s\hat{y}(s). \tag{2}$$

In particular, if $s = 0$ is a multiple pole of $\hat{y}(s)$, then $y(\infty)$ does not exist but is infinite.

Proof

Write $\hat{y}(s) = \hat{y}_0(s) + \hat{y}_{\mathrm{AS}}(s)$, where $\hat{y}_0(s)$ is nonzero and has all of its poles at the origin and $\hat{y}_{\mathrm{AS}}(s)$ has all of its poles in the OLHP. Note that $s\hat{y}_{\mathrm{AS}}(s) \to 0$ as $s \to 0$. Next, write $\hat{y}_0(s) = (a_n/s^n) + \cdots + (a_2/s^2) + (a_1/s)$, where n is a positive integer and a_n is nonzero. Hence $s\hat{y}_0(s) = (a_n/s^{n-1}) + \cdots + (a_2/s) + a_1$. If $n = 1$, then $\lim_{s \to 0} s\hat{y}(s) = a_1$, which is the finite-limit case.

If $n \geq 2$, then $\lim_{s \downarrow 0} s\hat{y}(s) = \mathrm{sign}(a_n)\infty$. Taking the inverse Laplace transform, it follows that $y(t) = y_0(t) + y_{\mathrm{AS}}(t)$, where $y_0(t) = [a_n t^{n-1}/(n-1)!] + \cdots + a_2 t + a_1$ and $y_{\mathrm{AS}}(t)$, which is the inverse Laplace transform of $\hat{y}_{\mathrm{AS}}(s)$, satisfies $y_{\mathrm{AS}}(t) \to 0$ as $t \to \infty$. Since $\lim_{t \to \infty} y_0(t) = \mathrm{sign}(a_n)\infty$, it follows that (2) is satisfied. □

To illustrate this result, let $\hat{y}(s) = (s^2 - 2s - 4)/(s^2(s+2))$, which has a pole at $s = -2$ and a double pole at $s = 0$. Then it follows from (2) that $y(\infty) = -\infty$. Indeed, $y(t) = -2t + e^{-2t}$. Note that the limiting value $y(\infty)$ given by (1) has the wrong sign if s approaches 0 through the negative numbers.

The extended final value theorem gives the correct finite or infinite limit when the poles of the Laplace transform are in the OLHP or at the origin. The extended final value theorem does not apply, however, when the Laplace transform has imaginary-but-nonzero poles since, in this case, the limit of the time response does not exist. The extended final value theorem also does not hold for poles in the open-right-half plane, where the limit is infinite.

GENERALIZATION TO IRRATIONAL FUNCTIONS

The standard final value theorem applies when $\hat{y}(s)$ has only poles (that is, isolated singularities) in the OLHP and possibly a simple pole at the origin. Indeed, in this case, $\hat{y}(s)$ can be written as

$$\hat{y}(s) = \frac{\hat{x}(s)}{s},$$

where $\hat{x}(s)$ has only isolated singularities in the OLHP. The corresponding function in the time domain then satisfies

$$y(t) = \int_0^t x(\tau)d\tau.$$

The function $x(t)$ can be obtained by using the standard method to evaluate the residue of $\hat{x}(s)e^{st}$ in the OLHP, which, for $t > 0$, is dominated by exponential functions, and therefore $x(\cdot)$ is absolutely integrable. It follows that

$$y(\infty) = \int_0^\infty x(\tau)d\tau = \hat{x}(0) = \lim_{s \to 0} s\hat{y}(s),$$

that is, the standard final value theorem holds. Consequently, the standard final value theorem need not be restricted to rational functions only.

Consider, for example, $y(t) = (1/2)\mathrm{erf}(2\sqrt{t})$, where erf is the *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\tau^2} d\tau.$$

Note that $\mathrm{erf}(\infty) = 1$, and hence $y(\infty) = 1/2$. By a variable substitution, we can write $y(t)$ as

$$y(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{e^{-4\tau}}{\sqrt{\tau}} d\tau,$$

which shows that the integrand is dominated by the exponential function e^{-4t}. The Laplace transform of $y(t)$ can be found as

$$\hat{y}(s) = \frac{1}{s\sqrt{s+4}}.$$

Application of the standard final value theorem to $y(\cdot)$ yields

$$y(\infty) = \lim_{s \to 0} s\hat{y}(s) = \lim_{s \to 0} \frac{1}{\sqrt{s+4}} = \frac{1}{2},$$

as expected.

To allow infinite limits, we now state a generalization of the extended final value theorem that applies to irrational Laplace transforms. This result is adapted from [5, p. 91].

Generalized Final Value Theorem

Let $y(t)$ be Laplace transformable, let $\lambda > -1$, and assume that $\lim_{t \to \infty} y(t)/t^\lambda$ and $\lim_{s \downarrow 0} s^{\lambda+1}\hat{y}(s)$ exist. Then

$$\lim_{t \to \infty} \frac{y(t)}{t^\lambda} = \frac{1}{\Gamma(\lambda+1)} \lim_{s \downarrow 0} s^{\lambda+1}\hat{y}(s). \tag{3}$$

In the above result, $\Gamma(x)$ denotes the *gamma function* defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

If n is a positive integer, then $\Gamma(n+1) = n!$, $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(n + (1/2)) = [1 \cdot 3 \cdots (2n-1)]\sqrt{\pi}/2^n$.

If $\lambda = 0$, then the generalized final value theorem reduces to the standard final value theorem. Furthermore, whenever the limit on the right hand side of (3) is finite, $y(t)$ approaches infinity in the order of

$$y(t) \sim \left(\frac{1}{\Gamma(\lambda+1)} \lim_{s \downarrow 0} s^{\lambda+1} \hat{y}(s) \right) t^{\lambda}.$$

Since $\Gamma(x)$ is positive, whether $y(t)$ approaches ∞ or $-\infty$ is determined by $\lim_{s\downarrow 0} s^{\lambda+1} \hat{y}(s)$.

The generalized final value theorem generalizes the extended final value theorem since $\hat{y}(s)$ need not be a rational function. To illustrate, consider $y(t) = \sqrt{t}$, which has the Laplace transform

$$\hat{y}(s) = \frac{1}{2s}\sqrt{\frac{\pi}{s}}.$$

The function $y(t)t^{-1/2}$ approaches 1 as $t \to \infty$, which is consistent with the calculation

$$\lim_{t\to\infty} \frac{y(t)}{\sqrt{t}} = \frac{1}{\Gamma(3/2)} \lim_{s\downarrow 0} s^{3/2} \hat{y}(s) = \frac{1}{\Gamma(3/2)} \frac{\sqrt{\pi}}{2} = 1.$$

CONCLUSIONS

For rational Laplace transforms with poles in the OLHP or at the origin, the extended final value theorem provides the correct infinite limit. For irrational Laplace transforms, the generalized final value theorem provides the analogous result.

Finally, we point to a detailed analysis of the final value theorem for piecewise continuous functions given in [6, chap. 12].

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» EDUCATION *(continued from page 94)*

that explains the importance and historical context of the paper. For instance, the U.S. Patent Office took nine years to approve Black's patent for the feedback amplifier because they did not believe it would work. The papers in this volume illustrate the difficulties and routes to fundamental results in control. *Kirsten Morris, University of Waterloo, Canada*

THOSE WITH NO FAVORITE BOOK

Many people who were surveyed said they could not think of a favorite book or were too busy to write up their thoughts. A few people explained why they have no favorite book:

» I have no favorite classical control book. I feel that we need a fresh approach to teaching classical control that, for either the institution or the instructor, provides a broader view of the field, reaches out to other disciplines such as biology and physics, incorporates more key ideas developed in the last 15–20 years on robustness, nonlinear control, and discrete event or hybrid systems, and includes a broader range of applications to demonstrate the richness and broad applicability of the concepts. *Kevin Passino, Ohio State University*

» To satisfy my teenage daughter's curiosity, I showed her my favorite textbook on classical control. Afterwards, she told her friends that I am a mathematician. To help me out, my physician wife explained to her the fine difference: a mathematician solves equations, but your dad creates his own equations. So, I need another favorite control book. It must be short and can be read during breakfast and on the train, by teenagers and physicians. It tells people that control is interesting and important, and we create systems, in addition to equations. It is a daunting task but achievable. *Leyi Wang, Wayne State University, Michigan*

» I'm sorry, but I didn't reply because I don't have a favorite. In fact, I don't have any that I particularly like. *Pablo Iglesias, John Hopkins University, Baltimore, Maryland*

THE FINAL WORD

We close with one of our favorite responses to the survey:

» I don't want to be quoted for this, but I have never read a book on classical control. The first book I ever read was *Linear Multivariable Control* by Murray Wonham, which was for years a bible for me. You can, however, not be further from the truth than claiming that this book is a book on classical control. Later, I realized the need to know about classical control, which I handled by sitting in on a lecture series. *An anonymous control engineer*