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Systems & Control Letters 27 (1996) 55–66

SYSTEMS & CONTROL LETTERS

Real-μ bounds based on fixed shapes in the Nyquist plane: Parabolas, hyperbolas, cissoids, nephroids, and octomorphs [1]

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Received 17 November 1994; revised 18 July 1995

Abstract

In this paper we introduce new bounds for the real structured singular value. The approach is based on absolute stability criteria with plant-dependent multipliers that exclude the Nyquist plot from fixed plane curve shapes containing the critical point $-1 + \jmath 0$. Unlike half-plane and circle-based bounds the critical feature of the fixed curve bounds is their ability to differentiate between the real and imaginary components of the uncertainty. Since the plant-dependent multipliers have the same functional form at all frequencies, the resulting graphical interpretation of the absolute stability criteria are frequency independent in contrast to the frequency-dependent off-axis circles that arise in standard real-μ bounds.

Keywords: Stability criteria; Fixed plane curves; Plant-dependent multipliers; Real-μ bounds

1. Introduction

The quest for less conservative and easily computable bounds for the real structured singular value remains as one of the primary goals of robust control theory. The bounds given in [2], which are based upon frequency-dependent scaling matrices that exploit the block structure and phase restriction of the real uncertainty, are now known to be closely related to frequency-domain absolute stability tests such as the Popov criterion [1, 3–6]. This relationship is based upon the fact that, for two-sided uncertainty, all of these bounds involve the construction of frequency-dependent off-axis circles to enclose the Nyquist plot at each frequency.

An alternative bound for the real structured singular value that does not involve off-axis circles was given in [3]. For the case of one-sided uncertainty, the bound given in [3] is based on exclusion of the Nyquist plot from a parabolic region that encompasses the critical point, thus ensuring stability. Transforming this parabola by means of an appropriate linear fractional transformation to address two-sided uncertainty yields an octomorphic (figure-eight-shaped) region. For two-sided uncertainty, stability is guaranteed by requiring that the Nyquist plot lies entirely inside the octomorphic region, which yields a bound for the real structured singular value.

Unlike circle-based bounds, the critical feature of the octomorphic bound for the real structured singular value is its ability to differentiate between the real and imaginary components of the uncertainty. This difference

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[1] This research was supported in part by the National Science Foundation under Grant ECS-9496249 and the Air Force Office Scientific Research under Grant F49620-92-J-0127.

SSDI 0167-6911(95)00044-5

Table 1
Exclusion–inclusion pairs involving fixed regions

Exclusion region	Inclusion region
Half plane (positivity)	Circle
Parabola	Octomorph
Cissoid	Nephroid
Semi-cubical parabola	Epicycloid
—	Hyperbola

can be interpreted as being due to the judicious choice of a frequency-domain multiplier which is plant-dependent and which has the same functional form at all frequencies. This fixed functional form is reflected by the octomorphic region, which is independent of frequency in contrast to the frequency-dependent off-axis circles that arise in circle-based bounds.

In the present paper we develop new bounds for the real structured singular value by considering alternative regions in the Nyquist plane. For each choice of exclusion region in the Nyquist plane for one-sided uncertainty, we obtain a corresponding inclusion region in the Nyquist plane for the case of two-sided uncertainty. In this paper we show that the parabola–octomorph pair studied in [3] is only one of the many possibilities of exclusion–inclusion pairs that yield bounds for the real structured singular value. Specifically, in this paper we consider exclusion of the Nyquist plot from a cissoid, which gives rise to an inclusion condition involving a nephroid. In addition, we consider exclusion from a semi-cubical parabola which corresponds to inclusion by an epicycloid (figure-eight-shaped). Finally, for the case of two sided uncertainty a hyperbolic region is also considered. These choices draw upon a rich heritage of special plane curves in mathematics dating back to antiquity [7, 9]. The cases we consider are summarized in Table 1, which includes the classical case of the positivity and circle criteria.

As in [3], for simplicity of exposition our results are confined to the case of a single uncertainty block of the form δI. Extensions to multiple block uncertainty will be reported elsewhere.

2. Notation and mathematical preliminaries

In this paper, we use the following standard notation and definitions. Let $\mathbb{R}$ and $\mathbb{C}$ denote real and complex numbers, and let $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote real and complex $n \times m$ matrices. Let A^{T} and A^* denote the transpose and complex conjugate transpose of A. $M \geqslant 0$ ($M > 0$) denotes the fact that the Hermitian matrix M is nonnegative (positive) definite. The Hermitian and skew-Hermitian parts of an arbitrary complex matrix G are defined by $\mathrm{He}\, G \triangleq \frac{1}{2}(G+G^*)$ and $\mathrm{Sh}\, G \triangleq \frac{1}{2}(G-G^*)$, respectively. An *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half complex plane. Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state space realization of a transfer function $G(s)$, that is, $G(s) = C(sI-A)^{-1}B+D$. The parahermitian conjugate $G^{\sim}(s)$ of $G(s)$ has the realization

$$G^{\sim}(s) \sim \left[\begin{array}{c|c} -A^{\mathrm{T}} & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}} & D^{\mathrm{T}} \end{array}\right].$$

A square transfer function $G(s)$ is called *weakly positive real* [3] if $G(s)$ has no imaginary poles and $\mathrm{He}\, G(\jmath\omega)$ is nonnegative for all $\omega \in \mathbb{R}$. A square transfer function $G(s)$ is called *strict weakly positive real* [3] if $G(s)$ has no imaginary poles and $\mathrm{He}\, G(\jmath\omega)$ is positive definite for all $\omega \in \mathbb{R}$.

3. Absolute stability criteria with plant-dependent multipliers

In this section we consider a robust stability problem involving an uncertain matrix of the form FI_m, where F is a real scalar, in a negative feedback interconnection with the $m \times m$ asymptotically stable transfer function

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

The uncertain scalar F is assumed to satisfy $0 \leqslant F \leqslant M$, where $M \in \mathbb{R}$ is nonnegative.

First we present the following lemmas which are needed for the main result.

Lemma 3.1. *Let $G(s)$ be asymptotically stable, let $H = H^* \in \mathbb{C}^{m\times m}$, and let $N \in \mathbb{R}$. Furthermore, define*

$$Z(s) \triangleq (I_m - G(s)H)(I_m - HG^{\sim}(s)) - N(G(s) - G^{\sim}(s)).$$

If $Z(s)(I_m + MG(s))$ is strict weakly positive real, then $\det(I_m + FG(\jmath\omega)) \neq 0$ *for all $0 \leqslant F \leqslant M$ and $\omega \in \mathbb{R}$.*

Proof. The result is immediate if $M = 0$. Hence, assume $M > 0$ and suppose that there exist $\omega \in \mathbb{R}$ and $F \in [0, M]$ such that $\det(I_m + FG(\jmath\omega)) = 0$. Then there exists nonzero $x \in \mathbb{C}^m$ such that $FG(\jmath\omega)x = -x$. Now, since $F \in [0, M]$ it follows that $F^2 - FM \leqslant 0$ and

$$x^* G^*(\jmath\omega)(I_m - G(\jmath\omega)H)(F^2 - FM)(I_m - HG^*(\jmath\omega))G(\jmath\omega)x \leqslant 0,$$

which simplifies to

$$x^* \left[(I_m - G(\jmath\omega)H)(I_m - HG^*(\jmath\omega)) + \mathrm{He}[M(I_m - G(\jmath\omega)H)(I_m - HG^*(\jmath\omega))G(\jmath\omega)]\right] x \leqslant 0. \quad (1)$$

Alternatively, since $Z(s)(I_m + MG(s))$ is strict weakly positive real it follows that

$$\begin{aligned} & x^* \left[(I_m - G(\jmath\omega)H)(I_m - HG^*(\jmath\omega)) + \mathrm{He}[M(I_m - G(\jmath\omega)H)(I_m - HG^*(\jmath\omega))G(\jmath\omega)]\right] x \\ & > \frac{MN}{2} x^* [(G(\jmath\omega) - G^*(\jmath\omega))G(\jmath\omega) + G^*(\jmath\omega)(G^*(\jmath\omega) - G(\jmath\omega))] x \\ & = -\frac{MNF}{2} x^* G^*(\jmath\omega)[G(\jmath\omega) - G^*(\jmath\omega) + G^*(\jmath\omega) - G(\jmath\omega)]G(\jmath\omega)x = 0, \end{aligned}$$

which contradicts (1). Consequently, $\det(I_m + FG(\jmath\omega)) \neq 0$ for all $F \in [0, M]$ and for all $\omega \in \mathbb{R}$. □

Lemma 3.2. *Let $k_1, k_2 \in \mathbb{R}$, $k_1 \leqslant k_2$. Assume that the negative feedback interconnection of*

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

and $k_1 I_m$ is asymptotically stable and $\det(I_m + G(\jmath\omega)F) \neq 0$ *for all $\omega \in \mathbb{R}$ and $F \in [k_1, k_2]$. Then the negative feedback interconnection of $G(s)$ and FI_m is asymptotically stable for all $F \in [k_1, k_2]$.*

Proof. The result is trivial if $k_1 = k_2$. Now, suppose $k_1 < k_2$ and there exists $\hat{F} \in (k_1, k_2]$ such that the negative feedback interconnection of $G(s)$ and $\hat{F} I_m$ is not asymptotically stable or, equivalently, $A - \hat{F}BC$ is not Hurwitz. Since, the negative feedback interconnection of $G(s)$ and $k_1 I_m$ is asymptotically stable, it follows that $A - k_1 BC$ is Hurwitz and there exists $F \in (k_1, \hat{F}]$ such that $A - FBC$ has an eigenvalue $\jmath\hat{\omega}$ on the imaginary axis.

Next, note that

$$\begin{aligned}\det(I_m + FG(\jmath\hat{\omega})) &= \det(I_m + k_1 G(\jmath\hat{\omega}) + (F - k_1)G(\jmath\hat{\omega}))\\ &= \det(I_m + k_1 G(\jmath\hat{\omega}))\det(I_m + (F-k_1)(I + k_1 G(\jmath\hat{\omega}))^{-1}G(\jmath\hat{\omega}))\\ &= \det(I_m + k_1 G(\jmath\hat{\omega}))\det(I_m + (F-k_1)C(\jmath\hat{\omega}I - A + k_1 BC)^{-1}B)\\ &= \det(I_m + k_1 G(\jmath\hat{\omega}))\det(\jmath\hat{\omega}I - A + k_1 BC)^{-1}\det(\jmath\hat{\omega}I - (A - FBC)).\end{aligned}$$

Since $\det(\jmath\hat{\omega}I - (A - FBC)) = 0$, it follows that $\det(I_m + FG(\jmath\hat{\omega})) = 0$, which is a contradiction. □

Next we present our main result.

Theorem 3.1. *Let*

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

be asymptotically stable, let $H = H^* \in \mathbb{C}^{m\times m}$, and let $N \in \mathbb{R}$. If $Z(s)(I_m + MG(s))$ is strict weakly positive real, then the negative feedback interconnection of $G(s)$ and FI_m is asymptotically stable for all $0 \leqslant F \leqslant M$.

Proof. The result is a direct consequence of Lemmas 3.1 and 3.2. □

Next, we extend Theorem 3.1 to the case of upper and lower uncertainty bounds. To do this let $M_1, M_2 \in \mathbb{R}$ be such that $M_1 \leqslant M_2$, let $M \triangleq M_2 - M_1$, and define the shifted uncertainty $F_s \in [M_1, M_2]$ along with the shifted transfer function

$$G_s(s) \triangleq (I_m + M_1 G(s))^{-1}G(s) \sim \left[\begin{array}{c|c} A - M_1 BC & B \\ \hline C & 0 \end{array}\right],$$

and shifted multiplier

$$Z_s(s) \triangleq (I_m - G_s(s)H)(I_m - HG_s^{\sim}(s)) - N(G_s(s) - G_s^{\sim}(s)).$$

For upper and lower uncertainty bounds we have the following corollary to Theorem 3.1.

Corollary 3.1. *Let*

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

assume $G_s(s)$ is asymptotically stable, let $H = H^* \in \mathbb{C}^{m\times m}$, and let $N \in \mathbb{R}$. If $Z_s(s)(I_m + MG_s(s))$ is strict weakly positive real then the negative feedback interconnection of $G(s)$ and $F_s I_m$ is asymptotically stable for all $F_s \in [M_1, M_2]$.

Proof. Since $Z_s(s)(I_m + MG_s(s))$ is strict weakly positive real it follows from Lemma 3.1 that $\det(I_m + FG_s(\jmath\omega)) \neq 0$ for all $F \in [0, M]$ and $\omega \in \mathbb{R}$ or, equivalently, $\det(I_m + (F_s - M_1)G_s(\jmath\omega)) \neq 0$ for all $\omega \in \mathbb{R}$ and for all $F_s \in [M_1, M_2]$. Now,

$$\begin{aligned}\det(I_m + (F_s - M_1)G_s(\jmath\omega)) &= \det(I_m + (F_s - M_1)(I_m + M_1 G(\jmath\omega))^{-1}G(\jmath\omega))\\ &= \det(I + M_1 G(\jmath\omega))^{-1}\det(I + F_s G(\jmath\omega)),\end{aligned}$$

and hence, $\det(I + G(\jmath\omega)F_s) \neq 0$ for all $\omega \in \mathbb{R}$ and $F_s \in [M_1, M_2]$. The result now follows from Lemma 3.2. □

Remark 3.1. It is important to note that even though Theorem 3.1 and Corollary 3.1 consider a single uncertainty block of the form FI_m the results also apply to multiple-block uncertainty when $G(\jmath\omega)F = FG(\jmath\omega)$. This can be easily seen by tracing through the proof of Lemma 3.1. This is not surprising since as in mixed-μ theory the multipliers, which in this case are plant-dependent, are required to commute with the block uncertainty structure. Alternatively, choosing specific multiplier structures, for example,

$$Z(s) = \operatorname{tr}[(I_m - G(s)H)(I_m - HG^{\sim}(s)) - N(G(s) - G^{\sim}(s))]I_m,$$

where "tr" denotes the trace operator, the commuting assumption between the multiplier and the plant is no longer required for addressing multiple-block uncertainty.

The form of the absolute stability criteria in Theorem 3.1 and Corollary 3.1 is similar to classical absolute stability criteria which involve a feedback system containing a linear time-invariant plant and a memoryless (possibly time-varying) sector bounded nonlinearity [8]. As discussed in [3] the transfer function $Z(s)$ is a stability multiplier that distinguishes the class of allowable nonlinearities. Furthermore, as in [3], the multipliers $Z(s)$ and $Z_s(s)$ depend upon the plant itself. As will be shown in the next sections, in the SISO case these plant-dependent multipliers provide stability tests involving fixed functional forms that differentiate between the real and imaginary components of the uncertainty while eliminating a large class of feedback nonlinearities and hence providing tight bounds for real parameter uncertainty.

Remark 3.2. Since the form of the absolute stability conditions of Theorem 3.1 and Corollary 3.1 are similar to classical absolute stability criteria, a natural question is for what class of memoryless feedback nonlinearities do these criteria predict stability? This question could be addressed by constructing specialized Lyapunov functions predicated on Kalman–Yakubovitch–Popov equations corresponding to the weak positive real conditions of Theorem 3.1 and Corollary 3.1. For the case $H = 0$, Kalman–Yakubovitch–Popov conditions are given in [3].

4. Geometric interpretation of stability criteria

In this section, we give geometric interpretations of the criteria developed in Theorem 3.1 and Corollary 3.1 in the scalar uncertainty case ($m = 1$). To do this, let $G(\jmath\omega) = x(\jmath\omega) + \jmath y(\jmath\omega)$. For notational convenience we write x for $x(\jmath\omega)$ and y for $y(\jmath\omega)$. In this case, $\mathrm{He}[Z(\jmath\omega)(I_m + MG(\jmath\omega))] > 0$ yields

$$(1 + Mx)[(1 - Hx)^2 + H^2y^2] + 2MNy^2 > 0. \tag{2}$$

Similarly, let $G_s(\jmath\omega) = x_s + \jmath y_s$. In this case, $\mathrm{He}[Z_s(\jmath\omega)(I_m + (M_2 - M_1)G_s(\jmath\omega)] > 0$ yields

$$[1 + (M_2 - M_1)x_s][(1 - Hx_s)^2 + H^2y_s^2] + 2(M_2 - M_1)Ny_s^2 > 0. \tag{3}$$

Next, since $G_s(\jmath\omega) = (1 + M_1G(\jmath\omega))^{-1}G(\jmath\omega)$ it follows that x_s and y_s are given by

$$x_s = \frac{x + M_1(x^2 + y^2)}{(1 + M_1x)^2 + M_1^2y^2}, \qquad y_s = \frac{y}{(1 + M_1x)^2 + M_1^2y^2}$$

and hence (3) can be written as

$$[(1 + M_1x)(1 + M_2x) + M_1M_2y^2](\alpha^2 + H^2y^2) + 2(M_2 - M_1)N[(1 + M_1x)^2 + M_1^2y^2]y^2 > 0, \tag{4}$$

where $\alpha \triangleq 1 + (2M_1 - H)x + M_1(M_1 - H)(x^2 + y^2)$. Note that (2) and (4) correspond to the strict weakly positive real conditions in Theorem 3.1 and Corollary 3.1, respectively. Conditions (2) and (4) have a graphical interpretation in the Nyquist plane in terms of fixed shapes which are symmetric about the real axes.

Next we consider several special cases of (2) and (4) by choosing specific values for H and N.

4.1. Positivity and circle criteria

First, choosing $N = 0$ and $H = 0$ it follows from Theorem 3.1 and Corollary 3.1 that robust stability is guaranteed if

$$\mathrm{He}(I_m + MG(\jmath\omega)) > 0, \quad \omega \in \mathbb{R}, \tag{5}$$

and

$$\mathrm{He}(I_m + (M_2 - M_1)G_s(\jmath\omega)) > 0, \quad \omega \in \mathbb{R}. \tag{6}$$

In this case, (2) and (4) simplify to

$$1 + Mx > 0 \tag{7}$$

and

$$(1 + M_1 x)(1 + M_2 x) + M_1 M_2 y^2 > 0, \tag{8}$$

which correspond to the classical positivity and circle criteria with a graphical interpretation in the Nyquist plane in terms of a half plane exclusion and a circle inclusion, respectively.

4.2. Parabolic and octomorphic criteria

Next, we specialize Theorem 3.1 and Corollary 3.1 to the parabolic and octomorphic criteria given in [3]. Specifically, choosing $H = 0$ it follows from Theorem 3.1 and Corollary 3.1 that robust stability is guaranteed if

$$\mathrm{He}[I_m + M(I_m - N(G(\jmath\omega) - G^*(\jmath\omega)))G(\jmath\omega)] > 0, \quad \omega \in \mathbb{R}, \tag{9}$$

and

$$\mathrm{He}[I_m + (M_2 - M_1)(I_m - N(G_s(\jmath\omega) - G_s^*(\jmath\omega)))G_s(\jmath\omega)] > 0, \quad \omega \in \mathbb{R}. \tag{10}$$

In this case, (2) and (4) specialize to

$$1 + Mx + 2MNy^2 > 0 \tag{11}$$

and

$$[(1 + M_1 x)(1 + M_2 x) + M_1 M_2 y^2][(1 + M_1 x)^2 + M_1^2 y^2] + 2(M_2 - M_1)Ny^2 > 0. \tag{12}$$

Condition (11) has a graphical interpretation in the Nyquist plane in terms of a parabola which is symmetric about the real axis with vertex $(-1/M, 0)$ and parameter N governing the curvature (Fig. 1). For the two-sided uncertainty case, condition (12) requires that the Nyquist plot of $G(\jmath\omega)$ lies inside the octomorphic region in the Nyquist plane with real-axis intercepts $-1/M_2$ and $-1/M_1$ shown in Fig. 2.

4.3. Cissoid and nephroid criteria

Next with $H = -M$ it follows from Theorem 3.1 and Corollary 3.1 that robust stability is guaranteed if

$$\mathrm{He}[((I_m + MG(\jmath\omega))(I_m + MG^*(\jmath\omega)) - N(G(\jmath\omega) - G^*(\jmath\omega)))(I_m + MG(\jmath\omega))] > 0, \quad \omega \in \mathbb{R}, \tag{13}$$

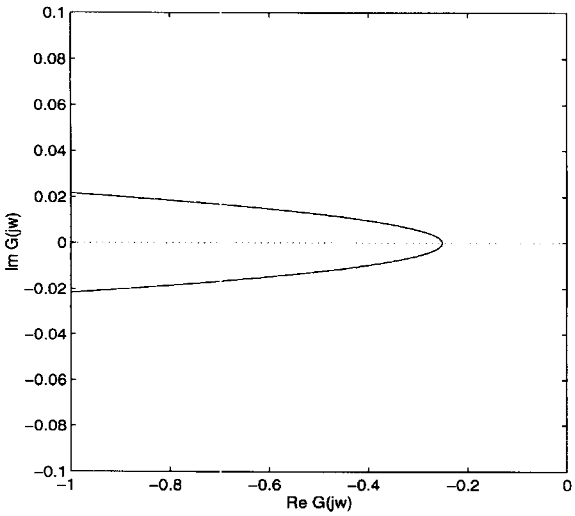

Fig. 1. Parabolic region with x-axis intercept $-1/M$.

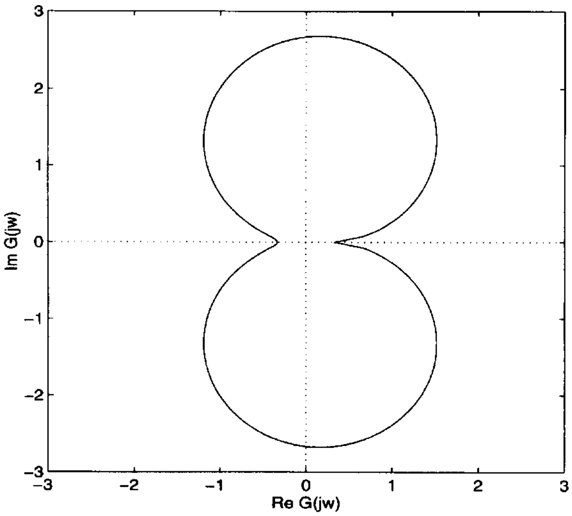

Fig. 2. Octomorphic region with x-axis intercepts $-1/M_2$ and $-1/M_1$.

and

$$\mathrm{He}[((I_m + MG_s(\jmath\omega))(I_m + MG_s^*(\jmath\omega)) - N(G_s(\jmath\omega) - G_s^*(\jmath\omega)))(I_m + MG_s(\jmath\omega))] > 0, \quad \omega \in \mathbb{R}. \tag{14}$$

In this case (2) and (4) specialize to

$$(1 + Mx)^3 + My^2[M(1 + Mx) + 2N] > 0 \tag{15}$$

and

$$\beta^3 + (M_2 - M_1)^2\beta y^2 + 2(M_2 - M_1)N[(1 + M_1x)^2 + M_1^2y^2]y^2 > 0, \tag{16}$$

where $\beta \triangleq (1 + M_1x)(1 + M_2x) + M_1M_2y^2$. Condition (15) has a graphical interpretation in the Nyquist plane in terms of a cissoid which is symmetric about the real-axis with vertex $(-1/M, 0)$, asymptote $x = -(2N+M)/M^2$, and parameter N governing the curvature (Fig. 3). In the two-sided uncertainty case, condition (16) requires that the Nyquist plot of $G(\jmath\omega)$ lie inside a nephroid in the Nyquist plane with real-axis intercepts $-1/M_2$ and $-1/M_1$ shown in Fig. 4.

4.4. Hyperbolic criterion

Finally, for two-sided uncertainty, choosing $H = M_1$ it follows from Corollary 3.1 that robust stability is guaranteed if

$$\mathrm{He}[(I_m + M_1G(\jmath\omega))^{-1}(I_m - N(G(\jmath\omega) - G^*(\jmath\omega)))(I_m + M_1G^*(\jmath\omega))^{-1}(I_m + MG_s(\jmath\omega))] > 0, \tag{17}$$

for all $\omega \in \mathbb{R}$. In this case (4) specializes to

$$(1 + M_1x)(1 + M_2x) + [M_1M_2 + 2(M_2 - M_1)N]y^2 > 0, \tag{18}$$

which has a graphical interpretation in the Nyquist plane in terms of a hyperbola when N satisfies $N < M_1M_2/(2(M_1 - M_2))$ and an ellipse when N satisfies $N > M_1M_2/(2(M_1 - M_2))$. In both cases the real-axis intercepts are $-1/M_2$ and $-1/M_1$, with center $(-(M_1 + M_2)/2M_1M_2, 0)$, and principal axes $x = -(M_1 + M_2)/2M_1M_2$ and $y = 0$.

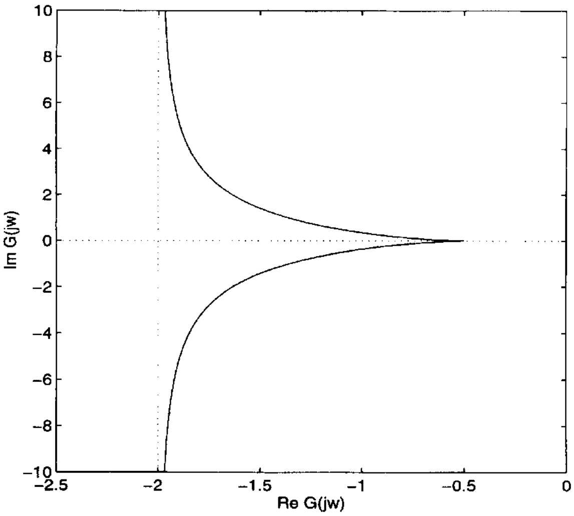

Fig. 3. Cissoidal region with x-axis intercept $-1/M$.

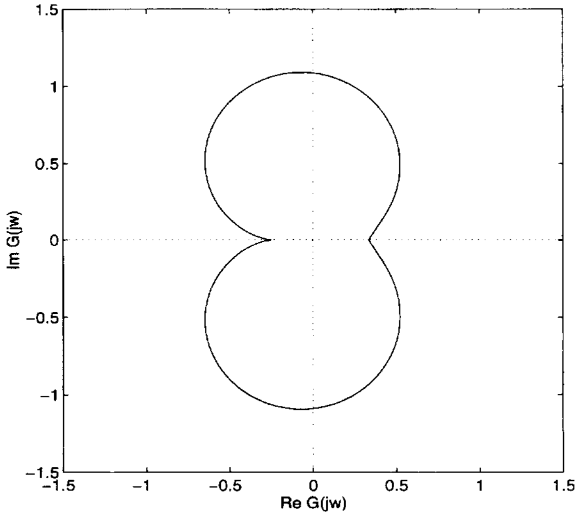

Fig. 4. Nephroidal region with x-axis intercepts $-1/M_2$ and $-1/M_1$.

Remark 4.1. Note that in the single sector case, $M_1 = 0$, (18) yields (11) which gives an absolute stability test with a graphical interpretation in terms of a parabola.

Remark 4.2. It is important to note that unlike many of the classical criteria the above absolute stability criteria allow the Nyquist plot of $G(\jmath\omega)$ to enter all four quadrants of the complex plane while avoiding encirclements and crossings of the critical point $-1/M + \jmath 0$.

Remark 4.3. Note that if we replace N by $-Ns^2$ in $Z(s)$ and $Z_s(s)$ the stability criteria in the above subsections yield *frequency-dependent* shapes in the Nyquist plane involving parabolas, octomorphs, cissoids, nephroids, and hyperbolas, respectively.

5. Alternative fixed shapes in the Nyquist plane

In this section we present a different plant-dependent multiplier that specializes to frequency domain stability criteria with alternative fixed shapes in the Nyquist plane in the case of scalar uncertainty. The proofs of the following results are similar to those of Theorem 3.1 and Corollary 3.1 and hence are omitted.

Theorem 5.1. *Let*

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

be asymptotically stable and let $N \in \mathbb{R}$. Furthermore, define

$$\hat{Z}(s) \triangleq \left[I_m + \tfrac{1}{2}M(G(s) + G^{\sim}(s))\right]^2 - N(G(s) - G^{\sim}(s)).$$

If $\hat{Z}(s)(I_m + MG(s))$ is strict weakly positive real then the negative feedback interconnection of $G(s)$ and FI_m is asymptotically stable for all $0 \leqslant F \leqslant M$.

Corollary 5.1. *Let*

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

assume $G_s(s)$ is asymptotically stable, and let $N \in \mathbb{R}$. Furthermore, define

$$\hat{Z}_s(s) \triangleq \left[I_m + \tfrac{1}{2}M(G_s(s) + G_s^{\sim}(s))\right]^2 - N(G_s(s) - G_s^{\sim}(s)).$$

If $\hat{Z}_s(s)(I_m + MG_s(s))$ is strict weakly positive real then the negative feedback interconnection of $G(s)$ and $F_s I_m$ is asymptotically stable for all $F_s \in [M_1, M_2]$.

Now we specialize the above results to the case of scalar uncertainty. Specifically, letting $G(\jmath\omega) = x + \jmath y$ it follows that the strict weakly positive real conditions of Theorem 5.1 and Corollary 5.1 can be written as

$$(1 + Mx)^3 + 2MNy^2 > 0, \tag{19}$$

and

$$[(1 + M_1 x)(1 + M_2 x) + M_1 M_2 y^2]^3 + 2(M_2 - M_1)N[(1 + M_1 x)^2 + M_1^2 y^2]y^2 > 0. \tag{20}$$

Condition (19) has a graphical interpretation in the Nyquist plot in terms of a semi-cubical parabola which is symmetric about the real axis with vertex $(-1/M, 0)$ and parameter N governing the curvature. The geometric shape for a semi-cubical parabola is similar to a cissoid but, unlike a cissoid, does not involve a y-axis asymptote [7, 9]. In the two-sided uncertainty case the stability criterion of Corollary 5.1 requires the Nyquist plot of $G(\jmath\omega)$ lie inside the plane curve characterized by (20) which corresponds to an epicycloid (nephroid-like shape) with real-axis intercepts $-1/M_2$ and $-1/M_1$.

6. Application to real-μ upper bounds

In this section, we use Corollary 3.1 to obtain real-μ upper bounds which do not involve off-axis circles and frequency-dependent scales. Note that similar results could be obtained using the absolute stability criterion given by Corollary 5.1. To make connections with real-μ theory we set $-M_1 = M_2 = \gamma^{-1}$, where $\gamma > 0$, and consider the set of uncertain matrices

$$\mathbf{\Delta} \triangleq \{\Delta \in \mathbb{R}^{m\times m} : \Delta = \delta I_m, \delta \in \mathbb{R}\}, \tag{21}$$

and

$$\mathbf{\Delta}_\gamma = \{\Delta \in \mathbf{\Delta} : -\gamma^{-1} \leqslant \delta \leqslant \gamma^{-1}\}.$$

Next, recall that for real uncertainty $\Delta \in \mathbf{\Delta}$, the structured singular value $\mu(G(\jmath\omega))$ is defined by

$$\mu(G(\jmath\omega)) \triangleq \left(\min_{\Delta\in\mathbf{\Delta}}\{\sigma_{\max}(\Delta) : \det(I + G(\jmath\omega)\Delta) = 0\}\right)^{-1}, \tag{22}$$

while $\mu(G(\jmath\omega)) \triangleq 0$ if there does not exist $\Delta \in \mathbf{\Delta}$ such that $\det(I + G(\jmath\omega)\Delta) = 0$. The following result provides an upper bound for $\mu(G(\jmath\omega))$ defined by (22).

Theorem 6.1. *Let $\omega \in \mathbb{R}$ and $G(\jmath\omega) \in \mathbb{C}^{m\times m}$. Then*

$$\mu(G(\jmath\omega)) \leqslant \bar{\mu}(G(\jmath\omega)), \tag{23}$$

where

$$\bar{\mu}(G(\jmath\omega)) \triangleq \inf\left\{\gamma > 0 : \textit{there exist } H = H^* \in \mathbb{C}^{m\times m}, \textit{ and } N \in \mathbb{R} \right.$$

$$\left. \textit{such that } \mathrm{He}\left[Z_s(\jmath\omega)\left(\frac{\gamma}{2}I_m + G_s(\jmath\omega)\right)\right] > 0\right\}, \tag{24}$$

where

$$G_s(\jmath\omega) \triangleq (I - \gamma^{-1}G(\jmath\omega))^{-1}G(\jmath\omega).$$

Proof. First note that with $-M_1 = M_2 = \gamma^{-1}$, the inequality in (24) is the strict weakly positive real condition given in Corollary 3.1. Hence, it follows from the proof of Corollary 3.1 that $\det(I + G(\jmath\omega)\Delta) \neq 0$, $\Delta \in \Delta_\gamma$. Now, since $\bar{\mu}(G(\jmath\omega))$ is defined as the infimum over all γ such that the inequality in (24) is satisfied, it follows that $\det(I + G(\jmath\omega)\Delta) \neq 0$, $\Delta \in \Delta_\gamma$, for each such γ. Now, it follows from the definition of $\mu(G(\jmath\omega))$ that $\mu(G(\jmath\omega)) \leqslant \bar{\mu}(G(\jmath\omega))$. □

Remark 6.1. It is important to note that since $\bar{\mu}(G(\jmath\omega))$ given by (24) involves a strict weakly positive real condition which in turn can be captured by Kalman–Yakubovitch–Popov type equations the resulting μ bound can be computed using standard BMI techniques.

Remark 6.2. In the SISO case $\bar{\mu}(G(\jmath\omega))$ can be written as

$$\bar{\mu}(G(\jmath\omega)) = \inf\{\gamma > 0 : \text{there exist } H, N \in \mathbb{R} \text{ such that (4) is satisfied with } -M_1 = M_2 = \gamma^{-1}\}.$$

Note that if $y = 0$ then (4) is satisfied if and only if $\gamma > |x|$ which yields $\bar{\mu}(G(\jmath\omega)) = |x|$. Alternatively, if $y \neq 0$ then there exists $N \in \mathbb{R}$ for all $\gamma > 0$ such that (4) is satisfied which yields $\bar{\mu}(G(\jmath\omega)) = 0$. Hence the allowable real parameter uncertainty predictions using Theorem 6.1 correspond to the real axis intercepts of the Nyquist plot of the plant $G(\jmath\omega)$ which shows that $\bar{\mu}(G(\jmath\omega))$ given by (24) is totally nonconservative for SISO systems.

7. Illustrative numerical example

To illustrate Theorem 3.1 and Corollary 3.1 consider the asymptotically stable plant

$$G(s) = \frac{-0.25s + 1}{3s^2 + s + 3}.$$

It follows from the Nyquist plot of $G(s)$ (Fig. 5) that the negative feedback interconnection of $G(s)$ and F is asymptotically stable for $-3 \leqslant F \leqslant 4$. Setting $M = 4$ the strict weakly positive real condition in Theorem 3.1 is satisfied for $H = 0$, $N = 200$ and for $H = -M$, $N = 5$. The parabola and the cissoid corresponding to these stability conditions are shown in Figs. 5 and 6, respectively.

To illustrate Corollary 3.1, consider two-sided uncertainty $M_1 \leqslant F \leqslant M_2$. Letting $M_1 = -3$ and $M_2 = 3$, it follows that the Nyquist plot is contained in the octomorphic region with intercepts $-1/M_2 = -0.33$ and $-1/M_1 = 0.33$. This inclusion is shown in Fig. 7 where $N = 8$. For the full uncertainty range, this inclusion is obtained with $N = 4200$. Next, letting $M_1 = -3$ and $M_2 = 4$ it follows that the Nyquist plot is contained in the nephroid region with $N = 12$ and $H = -7$ (Fig. 8). Finally, setting $M_1 = -2.5$ and $M_2 = 2.5$ it follows that the Nyquist plot is contained within the hyperbolic branches with $N = 3$ and $H = -2.5$ (Fig. 9). For the full range of uncertainty this inclusion is obtained with $N = 1000$ and $H = -3$. For this example the positive real criterion and the circle criterion yield the conservative bounds $0 \leqslant F \leqslant 1.71$ and $-0.95 \leqslant F \leqslant 0.97$, respectively.

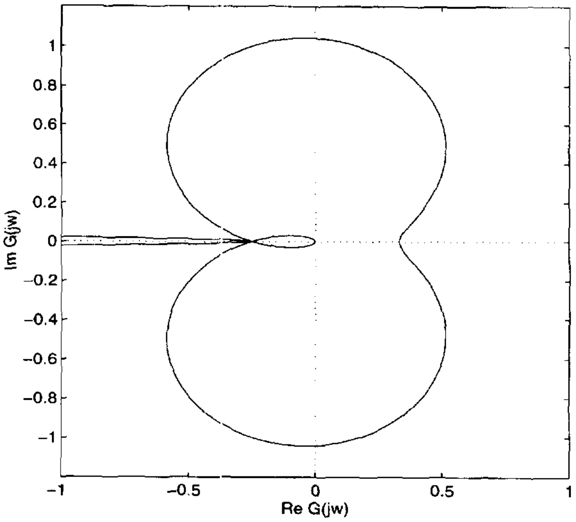

Fig. 5. Nyquist plot excluded from parabola ($M = 4$, $N = 200$).

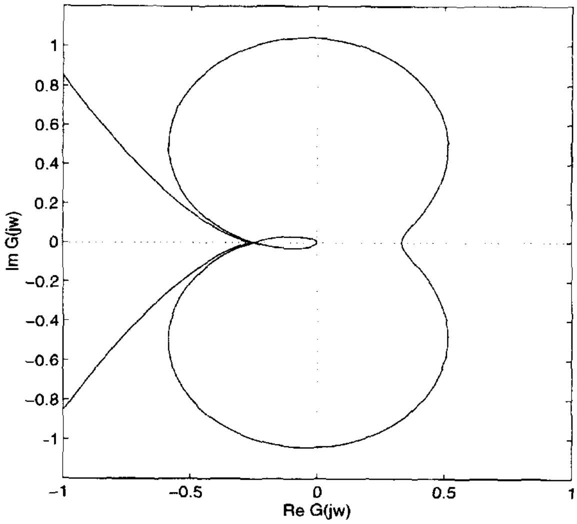

Fig. 6. Nyquist plot excluded from cissoid ($M = 4$, $N = 5$).

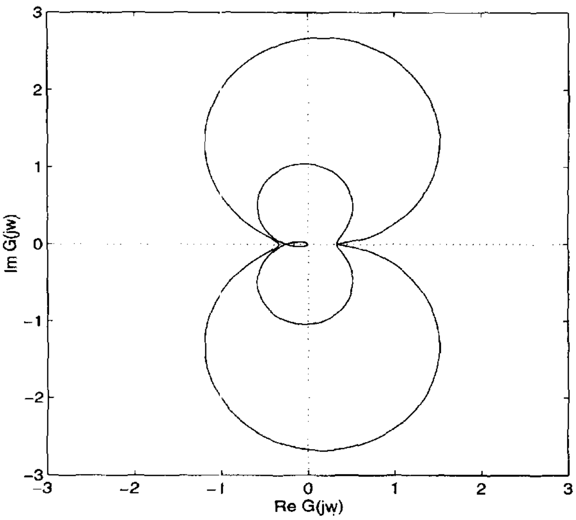

Fig. 7. Nyquist plot enclosed by octomorph ($M_1 = -3$, $M_2 = 3$, $N = 8$).

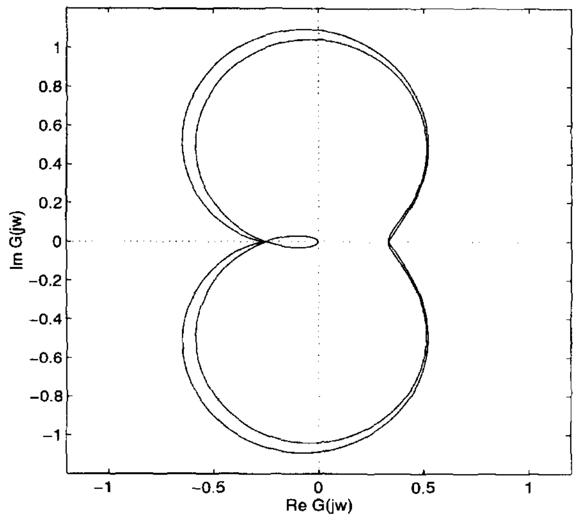

Fig. 8. Nyquist plot enclosed by nephroid ($M_1 = -3$, $M_2 = 4$, $N = 12$).

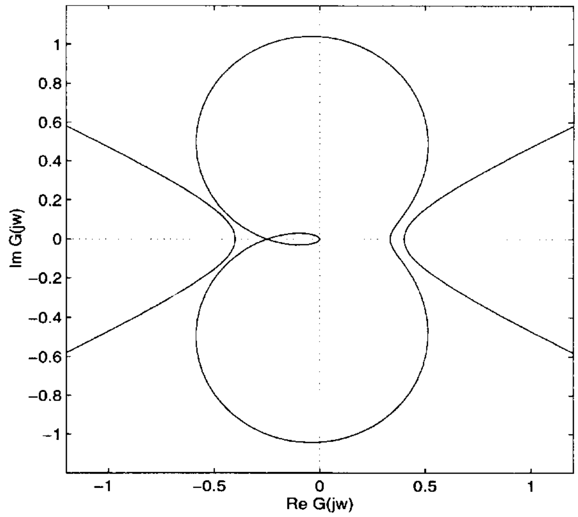

Fig. 9. Nyquist plot excluded from hyperbola ($M_1 = -2.5$, $M_2 = 2.5$, $N = 3$).

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