International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

State estimation for linear and non-linear equality-constrained systems

Bruno O. S. Teixeira [ab]; Jaganath Chandrasekar [b]; Leonardo A.B. Tôrres [a]; Luis A. Aguirre [a]; Dennis S. Bernstein [b]
[a] Department of Electronic Engineering, Federal University of Minas Gerais, Belo Horizonte, MG, Brazil [b] Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA

First Published on: 23 March 2009

To cite this Article Teixeira, Bruno O. S., Chandrasekar, Jaganath, A.B. Tôrres, Leonardo, A. Aguirre, Luis and S. Bernstein, Dennis(2009)'State estimation for linear and non-linear equality-constrained systems',International Journal of Control,

To link to this Article: DOI: 10.1080/00207170802370033

URL: http://dx.doi.org/10.1080/00207170802370033

International Journal of Control
2009, 1–19, iFirst

Taylor & Francis
Taylor & Francis Group

State estimation for linear and non-linear equality-constrained systems

Bruno O.S. Teixeira[ab]*, Jaganath Chandrasekar[b], Leonardo A.B. Tôrres[a], Luis A. Aguirre[a] and Dennis S. Bernstein[b]

[a]*Department of Electronic Engineering, Federal University of Minas Gerais, Belo Horizonte, MG, Brazil;* [b]*Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA*

(*Received 29 June 2007; final version received 5 September 2008*)

This article addresses the state-estimation problem for linear and non-linear systems for the case in which prior knowledge is available in the form of an equality constraint. The equality-constrained Kalman filter (KF) is derived as the maximum-a-posteriori solution to the equality-constrained state-estimation problem for linear and Gaussian systems and is compared to alternative algorithms. Then, four novel algorithms for non-linear equality-constrained state estimation based on the unscented KF are presented, namely, the equality-constrained unscented KF, the projected unscented KF, the measurement-augmentation unscented KF, and the constrained unscented KF. Finally, these methods are compared on linear and non-linear examples.

Keywords: equality constraints; state estimation; Kalman filtering; unscented Kalman filter

1. Introduction

The classical Kalman filter (KF) for linear systems provides optimal state estimates under standard noise and model assumptions (Jazwinski 1970). In practice, however, additional information about the system may be available, and this information may be useful for improving state estimates. A scenario we have in mind is the case in which the dynamics and the disturbances are such that the states of the system satisfy an equality or inequality constraint (Robertson and Lee 2002; Goodwin, Seron, and de Doná 2005). For example, in a chemical reaction, the species concentrations are non-negative (Massicotte, Morawski, and Barwicz 1995; Chaves and Sontag 2002), whereas in a compartmental model with zero net inflow (Bernstein and Hyland 1993), mass is conserved. Likewise, in undamped mechanical systems, such as a system with Hamiltonian dynamics, conservation laws hold, while, in the quaternion-based attitude estimation problem, the attitude vector must have unit norm (Crassidis and Markley 2003; Choukroun, Bar-Itzhack, and Oshman 2006). Additional examples arise in optimal control (Maciejowski 2002; Goodwin et al. 2005), parameter estimation (Chia, Chow, and Chizeck 1991; Aguirre, Barroso, Saldanha, and Mendes 2004; Nepomuceno, Takahashi, Aguirre, Neto, and Mendes 2004; Walker 2006; Aguirre, Alves, and Corrêa 2007), tracking and navigation (Wen and Durrant-Whyte 1992; Alouani and Blair 1993; Dissanayake, Sukkarieh, and Nebot 2001; Shen, Honga, and Cong 2006) and aeronautics (Rotea and Lana 2005; Simon and Simon 2006). In such cases, we wish to obtain state estimates that take advantage of prior knowledge of the states and use this information to obtain better estimates than those provided by the Kalman filter in the absence of such information.

Various algorithms have been developed for equality-constrained state estimation. One of the most popular techniques is the measurement-augmentation Kalman filter (MAKF), in which a perfect 'measurement' of the constrained quantity is appended to the physical measurements (Porrill 1988; Tahk and Speyer 1990; Wen and Durrant-Whyte 1992; Alouani and Blair 1993; Chen and Chiang 1993; De Geeter, van Brussel, and De Schutter 1997; Walker 2006). In addition, estimate-projection (Simon and Chia 2002), system-projection (Ko and Bitmead 2007) and gain-projection (Gupta and Hauser 2007; Teixeira et al. 2008b) methods have been considered. A two-step projection algorithm for handling non-linear equality constraints has also been presented (Julier and LaViola 2007).

For state estimation with inequality constraints, moving horizon estimators (MHE) (Rao, Rawlings, and Lee 2001; Rao, Rawlings, and Mayne 2003), unscented filtering algorithms (Vachhani, Narasimhan, and Rengaswamy 2006; Teixeira 2008; Teixeira, Tôrres, Aguirre, and Bernstein 2008d) and probabilistic methods (Rotea and Lana 2005) have been developed. However, inequality constraints

*Corresponding author. Email: brunoot@cpdee.ufmg.br

ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207170802370033
http://www.informaworld.com

are outside the scope of the present article. Finally, a general framework for state estimation with an equality constraint on the estimator gain (aiming at enforcing special properties on the state estimates) is presented in Teixeira et al. (2008b).

The contributions of the present article are as follows. First, we investigate how a linear equality state constraint arises in a linear dynamic system and present necessary conditions on both the dynamics and process noise for the state to be equality constrained. In Ko and Bitmead (2007), this problem is stated in the opposite way, that is, given that a system satisfies an equality constraint, the goal is to characterise the process noise. In these cases, we show that an equality-constrained linear system is controllable from the process noise only in the subspace defined by the equality constraint and that additional information regarding the initial condition provided by the equality constraint is useful for improving the classical Kalman filter estimates.

Second, we derive the equality-constrained Kalman filter (ECKF) as the maximum-a-posteriori analytical solution to the equality-constrained state-estimation problem for linear and Gaussian systems. We also prove the equivalence of ECKF and MAKF and discuss connections with the estimate-projection and system-projection approaches. We compare these four algorithms by means of a compartmental system example in which the disturbances are constrained so that mass is conserved.

Next, our main contribution in this article is to develop and compare four suboptimal algorithms for equality-constrained state estimation for non-linear systems, namely, the equality-constrained unscented Kalman filter (ECUKF) the projected unscented Kalman filter (PUKF), the measurement-augmentation unscented Kalman filter (MAUKF) and the constrained unscented Kalman filter (CUKF). These methods, which extend algorithms for constrained state estimation developed for linear systems, are based on the unscented Kalman filter (UKF) (Julier and Uhlmann 2004), which is an example of sigma-point Kalman filters (SPKF) (van der Merwe, Wan, and Julier 2004). In addition, CUKF is based on MHE with unitary horizon. Recent work (Julier, Uhlmann, and Durrant-Whyte 2000; Haykin 2001; Lefebvre, Bruyninckx, and De Schutter 2002; Lefebvre, Bruyninckx, and De Schutter 2004; Romanenko and Castro 2004; van der Merue et al. 2004; Hovland et al. 2005; Choi, Yeap, and Bouchard 2005; Crassidis 2006; Chandrasekar, Ridley, and Bernstein 2007; Teixeira, Santillo, Erwin, and Bernstein 2008e) illustrates the improved performance of SPKF compared to the extended Kalman filter (EKF) (Jazwinski 1970), which is prone to numerical problems such as initialisation sensitivity, divergence, and instability for strongly non-linear systems (Reif, Günther, Yaz, and Unbehauen 1999). A quaternion-based attitude estimation problem (Crassidis and Markley 2003) is used to illustrate UKF, ECUKF PUKF, MAUKF and CUKF. Although the state of the process model satisfies the unit norm constraint, this constraint is violated by the state estimates obtained from unconstrained Kalman filtering.

Finally, we use equality-constrained Kalman filtering techniques to improve estimation when an approximate discretised model is used to represent a continuous-time process. The problem of using UKF with a discrete-time model obtained from black-box identification to perform state estimation for a continuous-time non-linear system is treated in Aguirre, Teixeira, and Tôrres (2005). According to Rao et al. (2003), constraints can also be used to correct model error. We illustrate the application of equality-constrained unscented Kalman filter techniques to this problem through an example of a discretised model of an undamped single-degree-of-freedom pendulum without external disturbances. Although energy is conserved in the original, continuous-time system, the discretised model is approximate, and the energy constraint is intended to improve estimates of the discretised states. Additionally, an application of equality-constrained Kalman filtering to magnetohydrodynamics data assimilation (Chandrasekar, Barrero, Ridley, Bernstein, and De Moor 2004; Chandrasekar, Ridley, and Bernstein 2007), in which the zero-divergence constraint is enforced on the magnetic field using finite-volume discretised models, is discussed in Teixeira, Ridley, Tôrres, Aguirre, and Bernstein (2008c). The present article is based on research in Teixeira (2008), while preliminary versions of it appear as Teixeira, Chandrasekar, Tôrres, Aguirre, and Bernstein (2007, 2008a).

2. State estimation for linear systems

For the linear stochastic discrete-time dynamic system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + G_{k-1}w_{k-1}, \qquad (1)$$

$$y_k = C_k x_k + v_k, \qquad (2)$$

where $A_{k-1} \in \mathbb{R}^{n\times n}$, $B_{k-1} \in \mathbb{R}^{n\times p}$, $G_{k-1} \in \mathbb{R}^{n\times q}$, and $C_k \in \mathbb{R}^{m\times n}$ are known matrices, the state-estimation problem can be described as follows. Assume that, for all $k \geq 1$, the known data are the measurements $y_k \in \mathbb{R}^m$, the inputs $u_{k-1} \in \mathbb{R}^p$, and the statistical properties of x_0, w_{k-1} and v_k. The initial state vector $x_0 \in \mathbb{R}^n$ is assumed to be Gaussian with mean $\hat{x}_{0|0}$ and error-covariance $P^{xx}_{0|0} \triangleq \mathrm{E}[(x_0 - \hat{x}_{0|0}) \times (x_0 - \hat{x}_{0|0})^{\mathrm{T}}]$. The process noise $w_{k-1} \in \mathbb{R}^q$, which represents unknown input disturbances, and the measurement noise $v_k \in \mathbb{R}^m$, concerning inaccuracies in the measurements, are assumed white, Gaussian, zero mean and mutually independent with known covariance matrices Q_{k-1} and R_k, respectively. Next, define the profit function

$$J(x_k) \triangleq \rho(x_k|(y_1, \ldots, y_k)), \tag{3}$$

which is the conditional probability density function of the state vector $x_k \in \mathbb{R}^n$ given the past and present measured data $y_1, \ldots, y_k$. Under the stated assumptions, the maximisation of (3) is the state-estimation problem, while the maximiser $\hat{x}_{k|k}$ of J is the optimal state estimate.

The optimal state estimate $\hat{x}_{k|k}$ is given by KF (Jazwinski 1970), whose *forecast* step is given by

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}, \tag{4}$$

$$P^{xx}_{k|k-1} = A_{k-1}P^{xx}_{k-1|k-1}A^{\mathrm{T}}_{k-1} + G_{k-1}Q_{k-1}G^{\mathrm{T}}_{k-1}, \tag{5}$$

$$\hat{y}_{k|k-1} = C_k\hat{x}_{k|k-1}, \tag{6}$$

$$P^{yy}_{k|k-1} = C_kP^{xx}_{k|k-1}C^{\mathrm{T}}_k + R_k, \tag{7}$$

$$P^{xy}_{k|k-1} = P^{xx}_{k|k-1}C^{\mathrm{T}}_k, \tag{8}$$

where $P^{xx}_{k|k-1} \triangleq \mathrm{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^{\mathrm{T}}]$, $P^{yy}_{k|k-1} \triangleq \mathrm{E}[(y_k - \hat{y}_{k|k-1})(y_k - \hat{y}_{k|k-1})^{\mathrm{T}}]$, and $P^{xy}_{k|k-1} \triangleq \mathrm{E}[(x_k - \hat{x}_{k|k-1})(y_k - \hat{y}_{k|k-1})^{\mathrm{T}}]$, and whose *data-assimilation* step is given by

$$K_k = P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}, \tag{9}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}), \tag{10}$$

$$P^{xx}_{k|k} = P^{xx}_{k|k-1} - K_kP^{yy}_{k|k-1}K^{\mathrm{T}}_k, \tag{11}$$

where $P^{xx}_{k|k} \triangleq \mathrm{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^{\mathrm{T}}]$ is the error-covariance matrix and K_k is the Kalman gain matrix. The notation $\hat{z}_{k|k-1}$ indicates an estimate of z_k at time k based on information available up to and including time $k-1$. Likewise, $\hat{z}_{k|k}$ indicates an estimate of z at time k using information available up to and including time k. Model information is used during the forecast step, while measurement data are injected into the estimates during the data-assimilation step, specifically (10).

3. State estimation for equality-constrained linear systems

In (1), assume that $\mathrm{rank}(G_{k-1}) = q < n$, and define $r \triangleq n - q$, where $1 \le r \le n$. The case $r = n$ indicates that $G_{k-1}w_{k-1}$ is absent. Therefore, there exists $E_{k-1} \in \mathbb{R}^{r\times n}$ such that $\mathrm{rank}(E_{k-1}) = r$ and

$$E_{k-1}G_{k-1} = 0_{r\times q}. \tag{12}$$

Let $T_{1,k-1} \in \mathbb{R}^{(n-r)\times n}$ be such that

$$T_{k-1} \triangleq \begin{bmatrix} T_{1,k-1} \\ E_{k-1} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is invertible. For example, we can choose $T_{1,k-1} \triangleq G^{\mathrm{T}}_{k-1}$. Multiplying (1) by T_{k-1} yields

$$\begin{bmatrix} T_{1,k-1} \\ E_{k-1} \end{bmatrix} x_k = \begin{bmatrix} T_{1,k-1}A_{k-1} \\ E_{k-1}A_{k-1} \end{bmatrix} x_{k-1} + \begin{bmatrix} T_{1,k-1}B_{k-1} \\ E_{k-1}B_{k-1} \end{bmatrix} u_{k-1} + \begin{bmatrix} T_{1,k-1}G_{k-1} \\ 0_{r\times q} \end{bmatrix} w_{k-1}.$$

Hence, for all $k \ge 1$,

$$E_{k-1}x_k = e_{k-1}, \tag{13}$$

where $e_{k-1} \triangleq E_{k-1}(A_{k-1}x_{k-1} + B_{k-1}u_{k-1})$. Note that e_{k-1} is not necessarily constant even if system (1) and (2) is time invariant. Since $\mathrm{rank}(G_{k-1}) < n$, $G_{k-1}w_{k-1}$ has singular covariance $G_{k-1}Q_{k-1}G^{\mathrm{T}}_{k-1}$ (Goodwin et al. 2005; Ko and Bitmead 2007).

Let s be an integer satisfying $1 \le s \le r$, and partition $E_{k-1} \triangleq \left[\begin{smallmatrix} E_{1,k-1} \\ D_{k-1} \end{smallmatrix}\right]$, where

$$E_{1,k-1} \in \mathbb{R}^{(r-s)\times n} \text{ and } D_{k-1} \in \mathbb{R}^{s\times n}$$

It thus follows from (12) that

$$D_{k-1}G_{k-1} = 0_{s\times q}. \tag{14}$$

Proposition 3.1: *Assume that*

$$D_{k-1}A_{k-1} = D_{k-1} \tag{15}$$

and

$$D_{k-1}B_{k-1}u_{k-1} = 0_{s\times 1} \quad \textit{for all } k \ge 1. \tag{16}$$

Then, for all $k \ge 1$,

$$D_{k-1}x_k = d_{k-1}, \tag{17}$$

where

$$d_{k-1} \triangleq D_{k-1}x_{k-1}. \tag{18}$$

Proof: It follows from (13) that $D_{k-1}x_k = D_{k-1} \times (A_{k-1}x_{k-1} + B_{k-1}u_{k-1}) = D_{k-1}x_{k-1} = d_{k-1}$. □

Corollary 3.1: *If the system given by* (1) *and* (2) *is time invariant and* (14)–(16) *hold, then, for all* $k \ge 1$,

$$Dx_k = d, \tag{19}$$

where $D \triangleq D_{k-1}$ *and*

$$d \triangleq Dx_0. \tag{20}$$

Note that, if $s = r = n$, then $x_k = D^{-1}d$. Hence, this case is not of practical interest.

The next result shows that, if (1) is equality constrained, then it is not controllable in $\mathbb{R}^n$ from the process noise, but it is rather controllable in the subspace defined by (17).

Proposition 3.2: *If (14) and (15) hold, then (A_{k-1}, G_{k-1}) is not controllable in $\mathbb{R}^n$.*

Proof: Multiplying the controllability matrix

$$\mathcal{K}(A_{k-1}, G_{k-1}) \triangleq \begin{bmatrix} G_{k-1} & A_{k-1}G_{k-1} & \cdots & A_{k-1}^{n-1}G_{k-1} \end{bmatrix}$$

by D_{k-1} yields

$$\begin{aligned} D_{k-1}\mathcal{K}(A_{k-1}, G_{k-1}) &= \begin{bmatrix} D_{k-1}G_{k-1} & D_{k-1}A_{k-1}G_{k-1} & \cdots & D_{k-1}A_{k-1}^{n-1}G_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} 0_{s\times q} & D_{k-1}G_{k-1} & \cdots & D_{k-1}A_{k-1}^{n-2}G \end{bmatrix} \\ &= 0_{s\times nq}, \end{aligned}$$

implying that the columns of $\mathcal{K}$ lie on the null space of D_{k-1} such that $\operatorname{rank}(\mathcal{K}) < n$. □

Assuming that, for all $k \geq 1$, $D_{k-1} \in \mathbb{R}^{s\times n}$ satisfying (14)–(16) and d_{k-1} defined by (18) are known, the objective of the *equality-constrained state-estimation* problem is to maximise (3) subject to (17).

4. Equality-constrained Kalman filter

In this section, we solve the equality-constrained state-estimation problem to obtain ECKF. Let $\hat{x}^{\mathrm{p}}_{k|k}$ denote the solution of the equality-constrained state-estimation problem.

Lemma 4.1: *Assume that $P^{xx}_{k|k-1}$ given by (5) and R_k are positive definite. $\hat{x}^{\mathrm{p}}_{k|k}$ maximises J given by (3) subject to (17) if and only if $\hat{x}^{\mathrm{p}}_{k|k}$ minimises*

$$\mathcal{J}(x_k) \triangleq \Big[(x_k - \hat{x}_{k|k-1})^{\mathrm{T}}(P^{xx}_{k|k-1})^{-1}(x_k - \hat{x}_{k|k-1}) + (y_k - C_k x_k)^{\mathrm{T}} R_k^{-1}(y_k - C_k x_k)\Big] \quad (21)$$

subject to (17), where $\hat{x}_{k|k-1}$ is given by (4).

Proof: The proof is similar to the unconstrained counterpart investigated in Maybeck (1979, pp. 234–235) or Jazwinski (1970, pp. 207–208) and is omitted for brevity. For completeness, the proof to the constrained case is presented in Teixeira (2008, Lemma 5.2.1). □

Define the projected error covariance $P^{xx\mathrm{p}}_{k|k} \triangleq \mathrm{E}[(x_k - \hat{x}^{\mathrm{p}}_{k|k})(x_k - \hat{x}^{\mathrm{p}}_{k|k})^{\mathrm{T}}] + \delta I_{n\times n}$, where $0 < \delta \ll 1$ is a very small number compared to the entries of $P^{xx}_{k|k}$ that guarantees that $P^{xx\mathrm{p}}_{k|k}$ is positive definite. From our experience, we set $10^{-15} \leq \delta \leq 10^{-9}$ for computational implementation.

Proposition 4.1: *Let $\hat{x}_{k|k-1}$ and $P^{xx}_{k|k-1}$ be given by*

$$\hat{x}_{k|k-1} \triangleq A_{k-1}\hat{x}^{\mathrm{p}}_{k-1|k-1} + B_{k-1}u_{k-1}, \quad (22)$$

$$P^{xx}_{k|k-1} \triangleq A_{k-1}P^{xx\mathrm{p}}_{k-1|k-1}A^{\mathrm{T}}_{k-1} + G_{k-1}Q_{k-1}G^{\mathrm{T}}_{k-1}. \quad (23)$$

Define

$$\hat{d}_{k-1|k} \triangleq D_{k-1}\hat{x}_{k|k}, \quad (24)$$

$$P^{dd}_{k|k} \triangleq D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}, \quad (25)$$

$$P^{xd}_{k|k} \triangleq P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}, \quad (26)$$

$$K^{\mathrm{p}}_k \triangleq P^{xd}_{k|k}(P^{dd}_{k|k})^{-1}, \quad (27)$$

where $\hat{x}_{k|k}$ is given by (10) and $P^{xx}_{k|k}$ is given by (11). Then $\hat{x}^{\mathrm{p}}_{k|k}$ and $P^{xx\mathrm{p}}_{k|k}$ are given by

$$\hat{x}^{\mathrm{p}}_{k|k} = \hat{x}_{k|k} + K^{\mathrm{p}}_k(d_{k-1} - \hat{d}_{k-1|k}), \quad (28)$$

$$P^{xx\mathrm{p}}_{k|k} = P^{xx}_{k|k} - K^{\mathrm{p}}_k P^{dd}_{k|k} K^{\mathrm{p}\,\mathrm{T}}_k + \delta I_{n\times n}. \quad (29)$$

Proof: Using Lemma 4.1, let $\lambda \in \mathbb{R}^s$ and define the Lagrangian

$$L \triangleq \mathcal{J}(x_k) + 2\lambda^{\mathrm{T}}(D_{k-1}x_k - d_{k-1}).$$

The necessary conditions for a minimiser $\hat{x}^{\mathrm{p}}_{k|k}$ are given by

$$\frac{\partial L}{\partial x_k} = (P^{xx}_{k|k-1})^{-1}(\hat{x}^{\mathrm{p}}_{k|k} - \hat{x}_{k|k-1}) - C^{\mathrm{T}}_k R^{-1}_k(y_k - C_k\hat{x}^{\mathrm{p}}_{k|k}) + D^{\mathrm{T}}_{k-1}\lambda = 0_{n\times 1}, \quad (30)$$

$$\frac{\partial L}{\partial \lambda} = D_{k-1}\hat{x}^{\mathrm{p}}_{k|k} - d_{k-1} = 0_{s\times 1}. \quad (31)$$

It follows from (30) that

$$\begin{aligned} &((P^{xx}_{k|k-1})^{-1} + C^{\mathrm{T}}_k R^{-1}_k C_k)(\hat{x}^{\mathrm{p}}_{k|k} - \hat{x}_{k|k-1}) \\ &\quad = C^{\mathrm{T}}_k R^{-1}_k(y_k - C_k\hat{x}_{k|k-1}) - D^{\mathrm{T}}_{k-1}\lambda. \end{aligned} \quad (32)$$

From (11), using (7)–(9) and the matrix inversion lemma (Bernstein 2005), we have

$$\begin{aligned} P^{xx}_{k|k} &= P^{xx}_{k|k-1} - K_k P^{yy}_{k|k-1}K^{\mathrm{T}}_k \\ &= P^{xx}_{k|k-1} - P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}P^{yy}_{k|k-1}(P^{yy}_{k|k-1})^{-\mathrm{T}}(P^{xy}_{k|k-1})^{\mathrm{T}} \\ &= P^{xx}_{k|k-1} - P^{xx}_{k|k-1}C^{\mathrm{T}}_k(C_k P^{xx}_{k-1|k-1}C^{\mathrm{T}}_k + R_k)^{-1}C_k P^{xx}_{k|k-1} \\ &= ((P^{xx}_{k|k-1})^{-1} + C^{\mathrm{T}}_k R^{-1}_k C_k))^{-1}. \end{aligned} \quad (33)$$

Furthermore, from (9), using (7) and (8), we have

$$\begin{aligned}
K_k &= P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1} \\
&= P^{xx}_{k|k-1}C_k^{\mathrm{T}}(C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + R_k)^{-1} \\
&= P^{xx}_{k|k}(P^{xx}_{k|k})^{-1}P^{xx}_{k|k-1}C_k^{\mathrm{T}}(C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + R_k)^{-1} \\
&= P^{xx}_{k|k}(C_k^{\mathrm{T}}R_k^{-1}C_k + (P^{xx}_{k|k-1})^{-1})P^{xx}_{k|k-1}C_k^{\mathrm{T}} \\
&\quad \times (C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + R_k)^{-1} \\
&= P^{xx}_{k|k}(C_k^{\mathrm{T}}R_k^{-1}C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + C_k^{\mathrm{T}}R_k^{-1}R_k) \\
&\quad \times (C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + R_k)^{-1} \\
&= P^{xx}_{k|k}C_k^{\mathrm{T}}R_k^{-1}(C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + R_k)(C_k P^{xx}_{k|k-1}C_k^{\mathrm{T}} + R_k)^{-1} \\
&= P^{xx}_{k|k}C_k^{\mathrm{T}}R_k^{-1}. \qquad (34)
\end{aligned}$$

Substituting (33) and (34) into (32) and multiplying by $P^{xx}_{k|k}$ yields

$$\hat{x}^{\mathrm{p}}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1}) - P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}\lambda. \qquad (35)$$

Substituting (35) into (31) yields

$$\begin{aligned}
d_{k-1} &= D_{k-1}\hat{x}_{k|k-1} + D_{k-1}P^{xx}_{k|k}C_k^{\mathrm{T}}R_k^{-1}(y_k - C_k\hat{x}_{k|k-1}) \\
&\quad - D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}\lambda,
\end{aligned}$$

which implies

$$\begin{aligned}
\lambda &= (D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1}(D_{k-1}\hat{x}_{k|k-1} - d_{k-1}) \\
&\quad + (D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1}D_{k-1}K_k(y_k - C_k\hat{x}_{k|k-1}). \qquad (36)
\end{aligned}$$

Likewise, substituting (36) into (35) yields

$$\begin{aligned}
\hat{x}^{\mathrm{p}}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1}) \\
&\quad - P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}(D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1}(D_{k-1}\hat{x}_{k|k-1} - d_{k-1}) \\
&\quad - P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}(D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1}D_{k-1}K_k(y_k - C_k\hat{x}_{k|k-1}) \\
&= \hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1}) \\
&\quad - P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}(D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1}\big(D_{k-1}\hat{x}_{k|k-1} - d_{k-1} \\
&\quad + D_{k-1}K_k y_k - D_{k-1}K_k C_k\hat{x}_{k|k-1}\big) \\
&= \hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1}) \\
&\quad + P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}(D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1} \\
&\quad \times \big[d_{k-1} - D_{k-1}(\hat{x}_{k|k-1} + K_k(y_k - C_k\hat{x}_{k|k-1}))\big].
\end{aligned}$$

Now using (24)–(27), (9)–(11), we obtain

$$\begin{aligned}
\hat{x}^{\mathrm{p}}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}) \\
&\quad + P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}(D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1} \\
&\quad \times \big[d_{k-1} - D_{k-1}(\hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}))\big] \\
&= \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}) + K^{\mathrm{p}}_k(d_{k-1} - \hat{d}_{k-1|k}) \\
&= \hat{x}_{k|k} + K^{\mathrm{p}}_k(d_{k-1} - \hat{d}_{k-1|k}),
\end{aligned}$$

which proves (28).

Given the symmetry between (28) and (10) and recalling that $P^{xx}_{k|k}$ is given by (11), it follows that $P^{xx\mathrm{p}}_{k|k}$ is given by (29). □

Remark 4.1: Note that ECKF is expressed in three steps, namely, the *forecast* step (22) and (23), (6)–(8), the *data-assimilation* step (9)–(11) and the *projection* step (24)–(29), where the updated estimates are projected onto the hyperplane defined by the equality constraint (17).

Lemma 4.2: *Let $\mathcal{N}(D_{k-1})$ denote the null space of D_{k-1}, let $W \in \mathbb{R}^{n\times n}$ be positive definite, and define*

$$\mathcal{P}_{\mathcal{N}(D_{k-1})} \triangleq I_{n\times n} - WD^{\mathrm{T}}_{k-1}(D_{k-1}WD^{\mathrm{T}}_{k-1})^{-1}D_{k-1}. \qquad (37)$$

Then $\mathcal{P}_{\mathcal{N}(D_{k-1})}$ is an oblique projector whose range is $\mathcal{N}(D_{k-1})$, that is, $\mathcal{P}^2_{\mathcal{N}(D_{k-1})} = \mathcal{P}_{\mathcal{N}(D_{k-1})}$ but it is not necessarily symmetric.

For the following two results, let $\hat{x}_{k|k}$ given by (10) and $P^{xx}_{k|k}$ given by (11) denote the updated estimate and updated error covariance provided by ECKF. Also, let $\hat{x}^{\mathrm{p}}_{k|k}$ given by (28) and $P^{xx\mathrm{p}}_{k|k}$ given by (29) denote the projected estimate and projected error covariance of ECKF.

Proposition 4.2: *Set $W = P^{xx}_{k|k}$ in (37). Then, the projection step (24)–(29) is equivalent to*

$$\hat{x}^{\mathrm{p}}_{k|k} = \mathcal{P}_{\mathcal{N}(D_{k-1})}\hat{x}_{k|k} + \bar{d}_{k-1}, \qquad (38)$$

$$P^{xx\mathrm{p}}_{k|k} = \mathcal{P}_{\mathcal{N}(D_{k-1})}P^{xx}_{k|k} + \delta I_{n\times n}, \qquad (39)$$

where $\bar{d}_{k-1} \triangleq P^{xx}_{k|k}D^{\mathrm{T}}_{k-1}(D_{k-1}P^{xx}_{k|k}D^{\mathrm{T}}_{k-1})^{-1}d_{k-1}$.

Proof: Using Lemma 4.2 and substituting (24)–(27) into (28) and (29) yields (38) and (39). □

Proposition 4.3: *Assume that (1) and (2) is time invariant. Also, assume that D in (19) satisfies (14)–(16). Furthermore, assume that, for a given $k-1$, $D\hat{x}^{\mathrm{p}}_{k-1|k-1} = d$ and $DP^{xx\mathrm{p}}_{k-1|k-1} = 0_{s\times n}$ and set $\delta = 0$. Then $D\hat{x}_{k|k} = d$, $DP^{xx}_{k|k} = 0_{s\times n}$, $\hat{x}^{\mathrm{p}}_{k|k} = \hat{x}_{k|k}$, and $P^{xx\mathrm{p}}_{k|k} = P^{xx}_{k|k}$.*

Proof: Multiplying (22) and (23) by D yields

$$\begin{aligned}
D\hat{x}_{k|k-1} &= DA\hat{x}^{\mathrm{p}}_{k-1|k-1} + DBu_{k-1} \\
&= D\hat{x}^{\mathrm{p}}_{k-1|k-1} + 0_{s\times 1} = d, \qquad (40)
\end{aligned}$$

$$\begin{aligned}
DP^{xx}_{k|k-1} &= DAP^{xx\mathrm{p}}_{k-1|k-1}A^{\mathrm{T}} + DGQ_{k-1}G^{\mathrm{T}} \\
&= DP^{xx\mathrm{p}}_{k-1|k-1}A^{\mathrm{T}} + 0_{s\times q}Q_{k-1}G^{\mathrm{T}} \\
&= 0_{s\times n}A^{\mathrm{T}} = 0_{s\times n}. \qquad (41)
\end{aligned}$$

With (8) and (41), multiplying (9) by D yields

$$DK_k = DP^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1} = DP^{xx}_{k|k-1}C^{\mathrm{T}}(P^{yy}_{k|k-1})^{-1}$$
$$= 0_{s\times n}C^{\mathrm{T}}(P^{yy}_{k|k-1})^{-1} = 0_{s\times m}. \tag{42}$$

With (40) and (42), multiplying (10) and (11) by D yields

$$D\hat{x}_{k|k} = D\hat{x}_{k|k-1} + DK_k(y_k - \hat{y}_{k|k-1})$$
$$= d + 0_{s\times m}(y_k - \hat{y}_{k|k-1}) = d, \tag{43}$$

$$DP^{xx}_{k|k} = DP^{xx}_{k|k-1} - DK_kP^{yy}_{k|k-1}K^{\mathrm{T}}_k$$
$$= 0_{s\times n} + 0_{s\times m}P^{yy}_{k|k-1}K^{\mathrm{T}}_k = 0_{s\times n}. \tag{44}$$

Given (43) and (44) and $\delta = 0$, from (38) and (39), we have $\hat{x}^{\mathrm{p}}_{k|k} = \hat{x}_{k|k}$ and $P^{xx\mathrm{p}}_{k|k} = P^{xx}_{k|k}$. □

Corollary 4.1: *Assume that $D\hat{x}^{\mathrm{p}}_{1|1} = d$ and $DP^{xx\mathrm{p}}_{1|1} = 0_{s\times n}$. Then, for all $k \geq 2$, $D\hat{x}_{k|k} = d$ and $DP^{xx}_{k|k} = 0_{s\times n}$.*

Remark 4.2: Therefore, for time-invariant systems, whenever (14)–(16) hold, the projection step of ECKF given by (24)–(29) is required only at $k = 1$, so that, for all $k \geq 2$, the updated estimate $\hat{x}_{k|k}$ given by (10) satisfies $D\hat{x}_{k|k} = d$.

5. Connections between ECKF and alternative approaches

We now compare ECKF to three Kalman filtering algorithms that yield state estimates satisfying an equality constraint.

First we consider *MAKF* (Porrill 1988; Tahk and Speyer 1990; Wen and Durrant-Whyte 1992), which treats (17) as perfect measurements. In Appendix I, we present the MAKF equations and prove that the MAKF and ECKF estimates are equal.

In Appendix II, we show that the *projected Kalman filter by system projection* (PKF-SP) (Ko and Bitmead 2007), which, assuming that (14)–(16) hold, incorporates the information provided by (17) only in filter initialisation, that is, $k = 0$, is a special case of ECKF for time-invariant systems.

In Appendix III, we briefly review the *projected Kalman filter by estimate projection* (PKF-EP) (Simon and Chia 2002; Simon 2006), which projects $\hat{x}_{k|k}$ onto the hyperplane (17) for all $k \geq 1$. Unlike ECKF, the projected estimate of PKF-EP is not recursively fed back in the next iteration.

Figure 1 illustrates how the forecast, data-assimilation and projection steps are connected for ECKF, PKF-SP, PKF-EP and MAKF.

Also, for the special case of unitary horizon, ECKF and MHE (Rao et al. 2001) minimise the same cost function, specifically (21). However, ECKF provides the analytical solution to the equality-constrained optimisation problem. Moreover, unlike MHE, ECKF enforces the constraint information on the error covariance in addition to the state estimate.

We also remark that, if $m = 0$, then the problem solved by ECKF in Proposition 4.1 is similar to the case in which Kalman filter is used as an iterative solver for systems of linear algebraic equations such as $Dx = d$, where $D \in \mathbb{R}^{s\times n}$, $\mathrm{d} \in \mathbb{R}^s$, and $\mathrm{rank}(D) = s$ (Pinto and Rios Neto 1990, Theorem 2).

Note that, if d_{k-1} is uncertain, then (17) can be replaced by the noisy equality constraint (De Geeter et al. 1997; Walker 2006; Ko and Bitmead 2007)

$$d_{k-1} = D_{k-1}x_k + v^{\mathrm{d}}_k, \tag{45}$$

where $v^{\mathrm{d}}_k \in \mathbb{R}^s$ is a white, Gaussian, zero-mean noise with covariance R^{d}_k, and it is treated as an extra noisy measurement by MAKF (Appendix I) but with

$$\tilde{R}^{\mathrm{d}}_k \triangleq \begin{bmatrix} R_k & 0_{m\times s} \\ 0_{s\times m} & R^{\mathrm{d}}_k \end{bmatrix}$$

replacing $\tilde{R}_k$.

6. State estimation for equality-constrained non-linear systems

For the non-linear stochastic discrete-time dynamic system

$$x_k = f(x_{k-1}, u_{k-1}, w_{k-1}, k-1), \tag{46}$$

$$y_k = h(x_k, k) + v_k, \tag{47}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{N} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ are, respectively, the process and observation models, and whose state vector x_k is known to satisfy the equality constraint

$$g(x_k, k-1) = d_{k-1}, \tag{48}$$

where $g : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^s$ and d_{k-1} is known, the objective of the equality-constrained state-estimation problem is, for all $k \geq 1$, to maximise (3) subject to (48). However, the solution to this problem is complicated (Daum 2005) by the fact that, for non-linear systems, $\rho(x_k|(y_1, \ldots, y_k))$ is not completely characterised by its first-order and second-order moments. We thus use an approximation based on the classical Kalman filter to provide a suboptimal solution to the non-linear case.

6.1 *Unscented Kalman filter*

First, to address the unconstrained case, we consider UKF (Julier and Uhlmann 2004) to provide

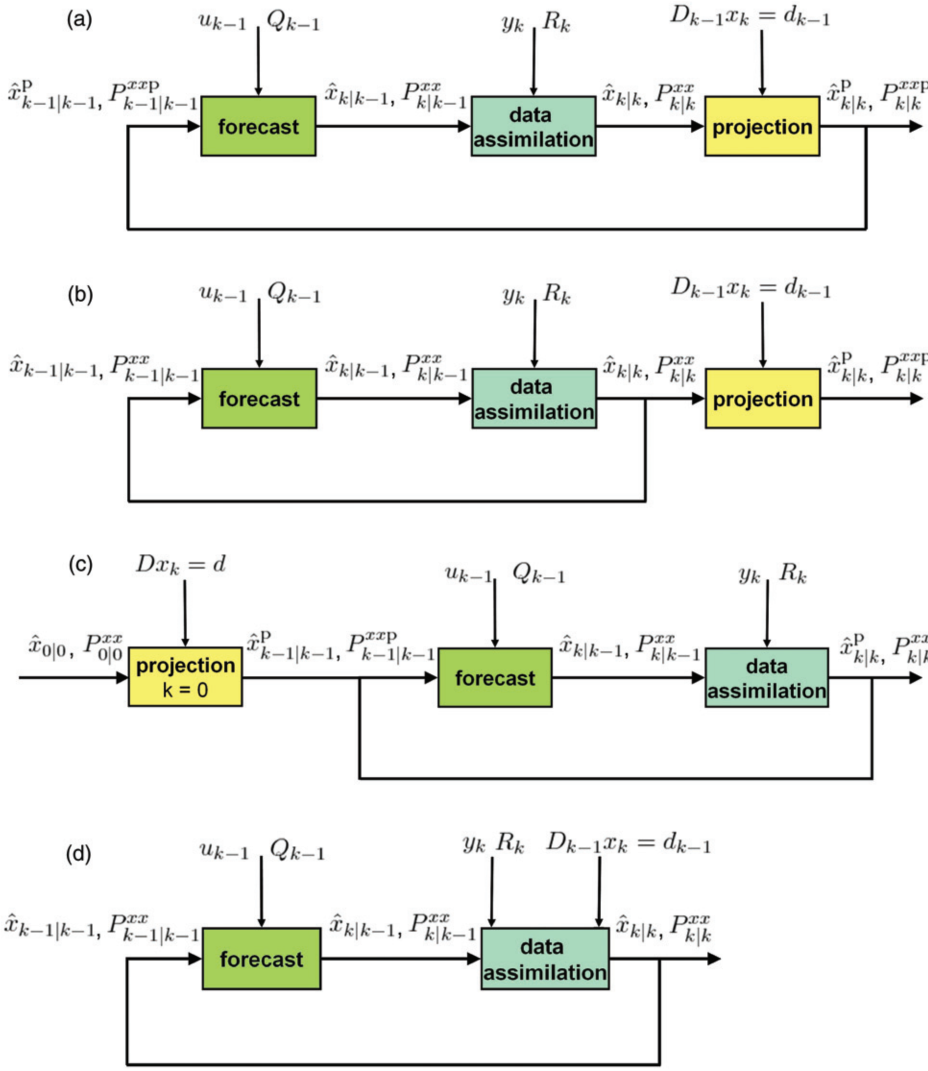

Figure 1. Comparative diagram of (a) ECKF, (b) PKF-EP, (c) PKF-SP and (d) MAKF. Unlike PKF-EP, the projection step of ECKF is connected by feedback recursion. In PKF-SP, in the context of time-invariant dynamics, the initial state estimate and the associated error covariance carry the information provided by the equality constraint. Note that, since (14) holds such that $GQ_{k-1}G^{\mathrm{T}}$ is a 'constrained' covariance, Q_{k-1} need not to be modified in the PKF-SP implementation. Otherwise, we can replace $GQ_{k-1}G^{\mathrm{T}}$ by $\mathcal{P}_{\mathcal{N}(D)}GQ_{k-1}G^{\mathrm{T}}$ in (5), where $\mathcal{P}_{\mathcal{N}(D)}$ is given by (A.25). In MAKF, the equality constraint is enforced at the data-assimilation step.

a suboptimal solution to the state-estimation problem. Instead of analytically or numerically linearising (46) and (47) and using (4)–(11), UKF employs the unscented transform (UT) (Julier et al. 2000), which approximates the posterior mean $\hat{y} \in \mathbb{R}^m$ and covariance $P^{yy} \in \mathbb{R}^{m\times m}$ of a random vector y obtained from the non-linear transformation $y = h(x)$, where x is a prior random vector whose mean $\hat{x} \in \mathbb{R}^n$ and covariance $P^{xx} \in \mathbb{R}^{n\times n}$ are assumed to be known. UT yields the true mean $\hat{y}$ and the true covariance P^{yy} if $h = h_1 + h_2$, where h_1 is linear and h_2 is quadratic (Julier et al. 2000). Otherwise, $\hat{y}_k$ is a *pseudo mean* and P^{yy} is a *pseudo covariance*.

UT is based on a set of deterministically chosen vectors known as sigma points. To capture the mean $\hat{x}^{\mathrm{a}}_{k-1|k-1}$ of the augmented prior state vector

$$x^{\mathrm{a}}_{k-1} \triangleq \begin{bmatrix} x_{k-1} \\ w_{k-1} \end{bmatrix}, \tag{49}$$

where $x^{\mathrm{a}}_{k-1} \in \mathbb{R}^{n_{\mathrm{a}}}$ and $n_{\mathrm{a}} \triangleq n + q$, as well as the augmented prior error covariance

$$P^{xx\mathrm{a}}_{k-1|k-1} \triangleq \begin{bmatrix} P^{xx}_{k-1|k-1} & 0_{n\times q} \\ 0_{q\times n} & Q_{k-1} \end{bmatrix}, \tag{50}$$

the sigma-point matrix $\mathcal{X}_{k-1} \in \mathbb{R}^{n_a \times (2n_a+1)}$ is chosen as

$$\mathcal{X}_{k-1|k-1} = \hat{x}^{a}_{k-1|k-1} 1_{1\times(2n_a+1)} + \sqrt{(n_a+\lambda)} \times \left[0_{n_a\times 1} \quad \left(P^{xxa}_{k-1|k-1}\right)^{1/2} \quad -\left(P^{xxa}_{k-1|k-1}\right)^{1/2} \right], \tag{51}$$

with weights

$$\begin{cases} \gamma_0^{(m)} \triangleq \dfrac{\lambda}{n_a+\lambda}, \\ \gamma_0^{(c)} \triangleq \dfrac{\lambda}{n_a+\lambda} + 1 - \alpha^2 + \beta, \\ \gamma_i^{(m)} \triangleq \gamma_i^{(c)} \triangleq \gamma_{i+n_a}^{(m)} \triangleq \gamma_{i+n_a}^{(c)} \triangleq \dfrac{1}{2(n_a+\lambda)}, \\ \qquad i = 1, \ldots, n_a, \end{cases} \tag{52}$$

where $(\cdot)^{1/2}$ is the Cholesky square root, $0 < \alpha \le 1$, $\beta \ge 0$, $\kappa \ge 0$, and $\lambda \triangleq \alpha^2(\kappa+n_a) - n_a > -n_a$. We set $\alpha = 1$ and $\kappa = 0$ (Haykin 2001) such that $\lambda = 0$ (Julier and Uhlmann 2004) and set $\beta = 2$ (Haykin 2001). Alternative schemes for choosing sigma points are given in Julier and Uhlmann (2004) and references therein.

The UKF *forecast* equations are given by (51) and (52) together with

$$\mathrm{col}_i(\mathcal{X}^x_{k|k-1}) = f(\mathrm{col}_i(\mathcal{X}^x_{k-1|k-1}),\ u_{k-1},\ \mathrm{col}_i(\mathcal{X}^w_{k-1|k-1}), k-1),\quad i = 0, \ldots, 2n_a, \tag{53}$$

$$\hat{x}_{k|k-1} = \sum_{i=0}^{2n_a} \gamma_i^{(m)} \mathrm{col}_i(\mathcal{X}^x_{k|k-1}), \tag{54}$$

$$P^{xx}_{k|k-1} = \sum_{i=0}^{2n_a} \gamma_i^{(c)} [\mathrm{col}_i(\mathcal{X}^x_{k|k-1}) - \hat{x}_{k|k-1}] \times [\mathrm{col}_i(\mathcal{X}^x_{k|k-1}) - \hat{x}_{k|k-1}]^{\mathrm{T}}, \tag{55}$$

$$\mathrm{col}_i(\mathcal{Y}_{k|k-1}) = h(\mathrm{col}_i(\mathcal{X}^x_{k|k-1}), k),\quad i = 0, \ldots, 2n_a, \tag{56}$$

$$\hat{y}_{k|k-1} = \sum_{i=0}^{2n_a} \gamma_i^{(m)} \mathrm{col}_i(\mathcal{Y}_{k|k-1}), \tag{57}$$

$$P^{yy}_{k|k-1} = \sum_{i=0}^{2n_a} \gamma_i^{(c)} [\mathrm{col}_i(\mathcal{Y}_{k|k-1}) - \hat{y}_{k|k-1}] \times [\mathrm{col}_i(\mathcal{Y}_{k|k-1}) - \hat{y}_{k|k-1}]^{\mathrm{T}} + R_k, \tag{58}$$

$$P^{xy}_{k|k-1} = \sum_{i=0}^{2n_a} \gamma_i^{(c)} [\mathrm{col}_i(\mathcal{X}^x_{k|k-1}) - \hat{x}_{k|k-1}] \times [\mathrm{col}_i(\mathcal{Y}_{k|k-1}) - \hat{y}_{k|k-1}]^{\mathrm{T}}, \tag{59}$$

where

$$\begin{bmatrix} \mathcal{X}^x_{k-1|k-1} \\ \mathcal{X}^w_{k-1|k-1} \end{bmatrix} \triangleq \mathcal{X}_{k-1|k-1},$$

$$\mathcal{X}^x_{k-1|k-1} \in \mathbb{R}^{n\times(2n_a+1)}$$

and

$$\mathcal{X}^w_{k-1|k-1} \in \mathbb{R}^{q\times(2n_a+1)}$$

The UKF *data-assimilation* equations are given by (9)–(11).

7. Equality-constrained unscented Kalman filters

In this section, by using UT, we extend the algorithms PKF-EP, ECKF and MAKF to the non-linear case. These unscented-based approaches provide suboptimal solutions to the equality-constrained state-estimation problem for non-linear systems. Unlike the linear case (§ 4), these approaches do not guarantee that the non-linear equality constraint (48) is exactly satisfied, but they provide approximate solutions.

Furthermore, to obtain state estimates satisfying (48) at a given tolerance by solving an optimisation problem online, we also present an unscented extension of the constrained Kalman filter (CKF) for linear systems, which is a special case of MHE with unitary horizon (Rao et al. 2001).

7.1 *Projected unscented Kalman filter*

The projection step of ECKF given by (24)–(29) is now extended to the non-linear case by means of UT. Choosing sigma points and associated weights as indicated in (51) and (52), we have

$$\mathcal{X}^x_{k|k} = \hat{x}_{k|k} 1_{1\times(2n+1)} + \sqrt{(n+\lambda)} \times \left[0_{n\times 1} \quad \left(P^{xx}_{k|k}\right)^{1/2} \quad -\left(P^{xx}_{k|k}\right)^{1/2} \right], \tag{60}$$

where $\hat{x}_{k|k}$ and $P^{xx}_{k|k}$ are given by (10) and (11), respectively. Then the sigma points $\mathrm{col}_i(\mathcal{X}^x_{k|k}) \in \mathbb{R}^n$, $i = 0, \ldots, 2n$, are propagated through (48), yielding

$$\mathrm{col}_i(\mathcal{D}_{k|k}) = g(\mathrm{col}_i(\mathcal{X}^x_{k|k}), k-1),\quad i = 0, \ldots, 2n, \tag{61}$$

such that $\hat{d}_{k-1|k}$, $P^{dd}_{k|k}$, and $P^{xd}_{k|k}$ are given by

$$\hat{d}_{k-1|k} = \sum_{i=0}^{2n} \gamma_i^{(m)} \mathrm{col}_i(\mathcal{D}_{k|k}), \tag{62}$$

$$P^{dd}_{k|k} = \sum_{i=0}^{2n} \gamma_i^{(c)} [\mathrm{col}_i(\mathcal{D}_{k|k}) - \hat{d}_{k-1|k}][\mathrm{col}_i(\mathcal{D}_{k|k}) - \hat{d}_{k-1|k}]^{\mathrm{T}}, \tag{63}$$

$$P^{xd}_{k|k} = \sum_{i=0}^{2n} \gamma_i^{(c)} [\mathrm{col}_i(\mathcal{X}^x_{k|k}) - \hat{x}_{k|k}][\mathrm{col}_i(\mathcal{D}_{k|k}) - \hat{d}_{k-1|k}]^{\mathrm{T}}, \tag{64}$$

and K^{p}_k, $\hat{x}^{\mathrm{p}}_{k|k}$, and $P^{xx\mathrm{p}}_{k|k}$ are given by (27), (28) and (29), respectively.

Appending the projection step (60)–(64), (27)–(29) to UKF (51)–(59), (9)–(11) without feedback recursion

(Figure 1(b)) yields *PUKF*, which extends PKF-EP to non-linear systems.

7.2 *Equality-constrained unscented Kalman filter*

Define

$$x^{\mathrm{ap}}_{k-1} \triangleq \begin{bmatrix} x^{\mathrm{p}}_{k-1} \\ w_{k-1} \end{bmatrix},$$

and

$$P^{xx\mathrm{ap}}_{k-1|k-1} \triangleq \begin{bmatrix} P^{xx\mathrm{p}}_{k-1|k-1} & 0_{n\times q} \\ 0_{q\times n} & Q_{k-1} \end{bmatrix},$$

such that the sigma points

$$\mathcal{X}_{k-1|k-1} = \hat{x}^{\mathrm{ap}}_{k-1|k-1} 1_{1\times(2n_{\mathrm{a}}+1)} + \sqrt{(n_{\mathrm{a}}+\lambda)} \times \left[0_{n_{\mathrm{a}}\times 1} \quad \left(P^{xx\mathrm{ap}}_{k-1|k-1}\right)^{1/2} \quad -\left(P^{xx\mathrm{ap}}_{k-1|k-1}\right)^{1/2} \right], \tag{65}$$

are chosen based on $\hat{x}^{\mathrm{p}}_{k-1|k-1}$ given by (28). Then, by appending the projection step (60)–(64), (27)–(29) to Equations (65), (52)–(59), (9)–(11), we obtain *ECUKF*. Note that, unlike PUKF, ECUKF connects the projection step to UKF by feedback recursion; see Figure 1(a).

7.3 *Measurement-augmentation unscented Kalman filter*

To extend the MAKF algorithm to the non-linear case, we replace (47) by the augmented observation

$$\tilde{y}_k \triangleq \tilde{h}(x_k, k) + \tilde{v}_k, \tag{66}$$

where

$$\tilde{y}_k \triangleq \begin{bmatrix} y_k \\ d_{k-1} \end{bmatrix}, \tilde{h}(x_k, k) \triangleq \begin{bmatrix} h(x_k, k) \\ g(x_k, k-1) \end{bmatrix}$$

and

$$\tilde{v}_k \triangleq \left[\begin{smallmatrix} v_k \\ 0_{s\times 1} \end{smallmatrix}\right]$$

With (66), *MAUKF* combines (51)–(55) with the augmented forecast equations

$$\mathrm{col}_i(\tilde{\mathcal{Y}}_{k|k-1}) = \tilde{h}(\mathrm{col}_i(\mathcal{X}^x_{k|k-1}), k), \quad i = 0, \ldots, 2n_{\mathrm{a}}, \tag{67}$$

$$\hat{\tilde{y}}_{k|k-1} = \sum_{i=0}^{2n_{\mathrm{a}}} \gamma_i^{(m)} \mathrm{col}_i(\tilde{\mathcal{Y}}_{k|k-1}), \tag{68}$$

$$\tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1} = \sum_{i=0}^{2n_{\mathrm{a}}} \gamma_i^{(c)} [\mathrm{col}_i(\tilde{\mathcal{Y}}_{k|k-1}) - \hat{\tilde{y}}_{k|k-1}] \times [\mathrm{col}_i(\tilde{\mathcal{Y}}_{k|k-1}) - \hat{\tilde{y}}_{k|k-1}]^{\mathrm{T}} + \tilde{R}_k, \tag{69}$$

$$\tilde{P}^{x\tilde{y}}_{k|k-1} = \sum_{i=0}^{2n_{\mathrm{a}}} \gamma_i^{(c)} [\mathrm{col}_i(\mathcal{X}^x_{k|k-1}) - \hat{x}_{k|k-1}] \times [\mathrm{col}_i(\tilde{\mathcal{Y}}_{k|k-1}) - \hat{\tilde{y}}_{k|k-1}]^{\mathrm{T}}, \tag{70}$$

where

$$\tilde{R}_k \triangleq \begin{bmatrix} R_k & 0_{m\times s} \\ 0_{s\times m} & 0_{s\times s} \end{bmatrix}, \tag{71}$$

and the KF data-assimilation equations

$$\tilde{K}_k = \tilde{P}^{x\tilde{y}}_{k|k-1} (\tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1})^{-1}, \tag{72}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \tilde{K}_k(\tilde{y}_k - \hat{\tilde{y}}_{k|k-1}), \tag{73}$$

$$P^{xx}_{k|k} = P^{xx}_{k|k-1} - \tilde{K}_k \tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1} \tilde{K}^{\mathrm{T}}_k. \tag{74}$$

Recall that, unlike the linear case, the ECUKF and MAUKF estimates are not necessarily equivalent.

In practice, to circumvent ill-conditioning problems in the inversion of $\tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1}$ in (69), we replace $\tilde{R}_k$ by $\tilde{R}^{\mathrm{d}}_k$, where we set $R^{\mathrm{d}}_k = \delta I_{s\times s}$, $10^{-15} \le \delta \le 10^{-9}$.

7.4 *Constrained unscented Kalman filter*

We now extend CKF to the non-linear case by combining the forecast step of UKF with the data-assimilation step of CKF. Then we obtain *CUKF*, whose forecast equations are given by (51)–(59) and whose data-assimilation equations are given by

$$\hat{x}_{k|k} = \underset{\{x_k:\, g(x_k, k-1) = d_{k-1}\}}{\arg\min} \mathcal{J}_1(x_k), \tag{75}$$

together with (9) and (11), where $\mathcal{J}_1(x_k)$ is given by

$$\mathcal{J}_1(x_k) \triangleq \Big[(x_k - \hat{x}_{k|k-1})^{\mathrm{T}} (P^{xx}_{k|k-1})^{-1} (x_k - \hat{x}_{k|k-1}) + \big(y_k - h(x_k,k)\big)^{\mathrm{T}} R_k^{-1} (y_k - h(x_k,k))\Big], \tag{76}$$

where $\hat{x}_{k|k-1}$ is given by (54) and $P^{xx}_{k|k-1}$ is given by (55). Various optimisation methods can be used to solve online the constrained optimisation problem (75) (Fletcher 2000). Note that (76) is the non-linear counterpart of (21).

Note that, unlike ECUKF and MAUKF, the equality-constraint information is not assimilated by the error-covariance $P^{xx}_{k|k}$ in (11). Also, CUKF allows the enforcement of inequality constraints in addition to (48) in (75).

8. Numerical examples

In this section, a linear example is investigated using KF, ECKF, MAKF, PKF-EP and PKF-SP, and two non-linear examples are discussed using UKF, ECUKF, MAUKF, PUKF and CUKF.

To compare the performance of these algorithms, we use two metrics over a c-run Monte Carlo simulation. First, the *accuracy* of the state estimate $\hat{x}_{i,k|k,j}$ given by (10), $i = 1, \ldots, n$ and $j = 1, \ldots, c$, from time $k = k_0$ to k_f is quantified by the root mean square error (RMSE) index given by

$$\mathrm{RMSE}_i = \frac{1}{c}\sum_{j=1}^{c}\sqrt{\frac{1}{k_f - k_0 + 1}\sum_{k=k_0}^{k_f}(x_{i,k} - \hat{x}_{i,k|k,j})^2},\quad i = 1, \ldots, n \tag{77}$$

where $x_{i,k}$ is the true value. For ECKF and PKF-EP, as well as for ECUKF and PUKF, $\hat{x}_{i,k|k,j}$ is replaced by $\hat{x}^{\mathrm{p}}_{i,k|k,j}$ given by (28) and (A.27), respectively. Note that, to calculate RMSE_i, $x_{i,k}$ must be known, and thus this index is restricted to simulation.

Next, we assess how *informative* (Lefebvre et al. 2004) the state estimate $\hat{x}_{k|k}$ is by evaluating the mean trace (MT) of $P^{xx}_{k|k}$ given by (11) from time $k = k_0$ to k_f, that is,

$$\mathrm{MT} = \frac{1}{c}\sum_{j=1}^{c}\left[\frac{1}{k_f - k_0 + 1}\sum_{k=k_0}^{k_f}\mathrm{tr}(P^{xx}_{k|k,j})\right]. \tag{78}$$

MT measures the uncertainty in the estimate $\hat{x}_{k|k}$. For ECKF and PKF-EP, as well as for their non-linear counterparts, $P^{xx}_{k|k,j}$ is replaced by $P^{xxp}_{k|k,j}$ given by (29) and (39), respectively.

8.1 *Compartmental system*

Consider the linear discrete-time compartmental model (1) and (2) (Bernstein and Hyland 1993) whose matrices are given by

$$A = \begin{bmatrix} 0.94 & 0.028 & 0.019 \\ 0.038 & 0.95 & 0.001 \\ 0.022 & 0.022 & 0.98 \end{bmatrix}, \quad B = 0_{3\times 1},$$
$$G = \begin{bmatrix} 0.05 & -0.03 \\ -0.02 & 0.01 \\ -0.03 & 0.02 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{79}$$

with initial condition $x_0 = [1 \quad 1 \quad 1]^{\mathrm{T}}$ and process noise and observation noise covariance matrices $Q_{k-1} = \sigma_w^2 I_{3\times 3}$ and $R_k = \sigma_v^2 I_{2\times 2}$. The data-free simulation of this system is shown in Figure 2 for $\sigma_w = 1.0$. Note that (14)–(16) hold for (79) such that the trajectory of $x_k \in \mathbb{R}^3$ lies on the plane (19), whose parameters are assumed to be known and are given by

$$D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad d = 3, \tag{80}$$

that is, conservation of mass is verified.

For state estimation, the KF algorithm is initialised with

$$\hat{x}_{0|0} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^{\mathrm{T}}, \quad P^{xx}_{0|0} = I_{3\times 3}. \tag{81}$$

Figure 3 shows that the KF estimates do not lie on the plane (19). Even if $\hat{x}_{0|0} = x_0$ or $\sigma_w = 0$, KF does not produce estimates satisfying (19).

Next, we implement the ECKF algorithm. From a 100-run Monte Carlo simulation for each one of these process noise levels, namely, $\sigma_w = 0$, 0.1, 0.5, 1.0, and $\sigma_v = 0.01$, Table 1 shows that the ECKF estimates satisfy the equality constraint. In addition, these estimates are both more accurate and more informative than the KF estimates.

For MAKF, PKF-EP and PKF-SP, initialisation is defined as in (81), except for PKF-SP (see (A.23) in Appendix II). Table 1 summarises the results. ECKF, MAKF, PKF-SP and PKF-EP guarantee that (19) is satisfied at machine precision and yield improved estimates compared to KF. All equality-constrained methods produce similar results concerning RMSE and MT for this time-invariant system, which is in accordance with Ko and Bitmead (2007, Theorem 2) regarding PKF-SP and PKF-EP. Recall that the numerical differences in Table 1 regarding the RMS constraint error for the equality-constrained algorithms are neglegible. However, though not shown in Table 1, it is relevant to mention that PKF-EP produces less accurate and less informative forecast estimates $\hat{x}_{k|k-1}$ compared to the other constrained algorithms. This is expected because PKF-EP do not use $\hat{x}^{\mathrm{p}}_{k-1|k-1}$ to calculate $\hat{x}_{k|k-1}$; see Figure 1(b).

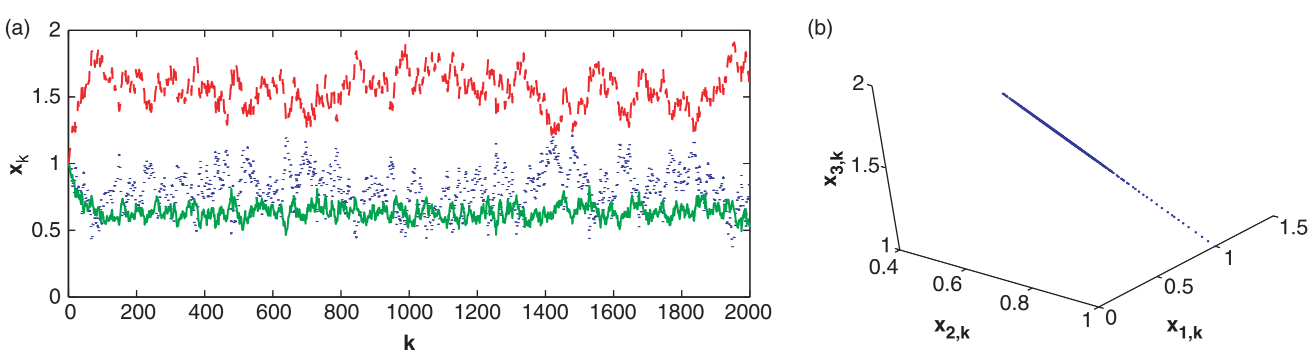

Figure 2. Data-free simulation of the compartmental model. In (a), the state components are shown evolving with time and, in (b), in state space.

In addition, a land-based vehicle linear example with kinematic constraints (Simon and Chia 2002; Ko and Bitmead 2007) is investigated using KF, ECKF, MAKF, PKF-SP and PKF-EP in Teixeira (2008, Chapter 5).

8.2 *Attitude estimation*

Consider an attitude estimation problem (Crassidis and Markley 2003; Choukroun et al. 2006), whose kinematics are modelled as

$$\dot{e}(t) = \frac{1}{2}\Omega(t)e(t), \tag{82}$$

where the state vector is the quaternion vector $e(t) = [e_0(t) \quad e_1(t) \quad e_2(t) \quad e_3(t)]^{\mathrm{T}}$, the matrix $\Omega(t)$ is given by

$$\Omega(t) = \begin{bmatrix} 0 & r(t) & -q(t) & p(t) \\ -r(t) & 0 & p(t) & q(t) \\ q(t) & -p(t) & 0 & r(t) \\ -p(t) & -q(t) & -r(t) & 0 \end{bmatrix}, \tag{83}$$

and the angular velocity vector $u(t) = [p(t) \quad q(t) \quad r(t)]^{\mathrm{T}}$ is a known input. Since $\|e(0)\|_2 = 1$ and $\Omega(t)$ is skew symmetric, it follows that, for all $t > 0$,

$$\|e(t)\|_2 = 1. \tag{84}$$

We set $e(0) = [0.9603 \quad 0.1387 \quad 0.1981 \quad 0.1387]^{\mathrm{T}}$ and

$$u(t) = \left[0.03\sin\left(\frac{2\pi}{600}t\right) \quad 0.03\sin\left(\frac{2\pi}{600}t - 300\right) \quad 0.03\sin\left(\frac{2\pi}{600}t - 600\right) \right]^{\mathrm{T}}. \tag{85}$$

To perform attitude estimation, we assume that

$$\breve{u}_{k-1} = u((k-1)T) + \beta_{k-1} + w^u_{k-1} \tag{86}$$

is measured by rate gyros, where T is the discretisation step, $w^u_{k-1} \in \mathbb{R}^3$ is zero-mean Gaussian noise and $\beta_{k-1} \in \mathbb{R}^3$ is drift error. The discrete-time equivalent of (82) augmented by the gyro drift random-walk model (Crassidis and Markley 2003) is given by

$$\begin{bmatrix} e_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} A_{k-1} & 0_{4\times 3} \\ 0_{3\times 4} & I_{3\times 3} \end{bmatrix} \begin{bmatrix} e_{k-1} \\ \beta_{k-1} \end{bmatrix} + \begin{bmatrix} 0_{4\times 1} \\ w^{\beta}_{k-1} \end{bmatrix}, \tag{87}$$

where $e_k \triangleq e(kT)$, $w^{\beta}_{k-1} \in \mathbb{R}^3$ is process noise associated with the drift-error model,

$$x_k \triangleq \begin{bmatrix} e_k \\ \beta_k \end{bmatrix} \in \mathbb{R}^7$$

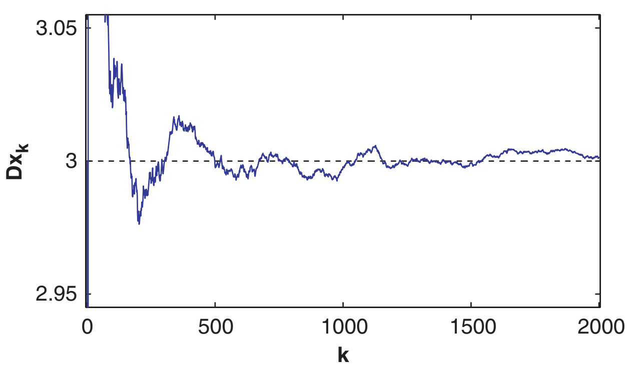

Figure 3. Estimate of the total mass (constraint) Dx_k in the conservative compartmental system (79) using KF (——) in comparison with the true value (– –).

Table 1. Percent RMS constraint error, RMSE (77), and MT (78), from $k = 1500$ to 2000, for 100-run Monte Carlo simulation for compartmental system, concerning process noise levels $\sigma_w = 0$, 0.1, 0.5 and 1.0, and KF, ECKF, MAKF, PKF-EP and PKF-SP algorithms.

σ_w	KF	ECKF	MAKF	PKF-EP	PKF-SP
Percent RMS constraint error					
0	0.12	4.52×10^{-15}	4.24×10^{-11}	4.53×10^{-15}	8.19×10^{-12}
0.1	0.22	4.52×10^{-15}	2.01×10^{-11}	4.52×10^{-15}	4.05×10^{-12}
0.5	0.40	4.50×10^{-15}	0.88×10^{-11}	4.51×10^{-15}	3.92×10^{-12}
1.0	0.62	4.53×10^{-15}	0.50×10^{-11}	4.51×10^{-15}	3.98×10^{-12}
RMSE$_i$, $i = 1, 2, 3$ ($\times10^{-3}$)					
0	0.57, 0.36, 2.93		0.10, 0.16, 0.21		
0.1	6.26, 2.60, 7.34		6.25, 2.54, 4.19		
0.5	9.01, 4.58, 13.2		9.01, 4.55, 6.75		
1.0	9.35, 5.58, 19.7		9.35, 5.56, 8.07		
MT ($\times10^{-4}$)					
0	0.0996		0.0012		
0.1	1.0515		0.6352		
0.5	2.8057		1.4722		
1.0	5.4646		1.8387		

Table 2. Percent RMS constraint error, RMSE (77), and MT (78), from $k = 5000$ to 10 000, for 100-run Monte Carlo simulation for attitude estimation using UKF, MAUKF, PUKF, ECUKF and CUKF.

UKF	MAUKF	PUKF	ECUKF	CUKF
Percent RMS constraint error ($\times 10^{-4}$)				
254.1	6.50	8.31	6.49	0.58
RMSE$_i$ for e_0, e_1, e_2, e_3, β_1, β_2, β_3 ($\times 10^{-3}$)				
1.331	1.398	1.334	1.398	1.332
1.366	1.434	1.365	1.433	1.365
1.320	1.377	1.320	1.376	1.318
1.319	1.374	1.315	1.374	1.313
0.124	0.130	0.124	0.130	0.124
0.119	0.125	0.119	0.125	0.119
0.120	0.126	0.120	0.125	0.120
MT for attitude ($\times 10^{-6}$)				
8.18	5.42	8.16	5.42	8.18
MT for drift ($\times 10^{-7}$)				
1.17	1.09	1.17	1.09	1.17

is the state vector,

$$w_{k-1} \triangleq \begin{bmatrix} w^u_{k-1} \\ w^\beta_{k-1} \end{bmatrix} \in \mathbb{R}^6$$

is the process noise, and

$$A_{k-1} \triangleq \cos(s_{k-1}) I_{4\times 4} - \frac{1}{2}\frac{T\sin(s_{k-1})}{s_{k-1}}\Omega_{k-1}, \qquad (88)$$

$$\Omega_{k-1} \triangleq \Omega((k-1)T), \qquad (89)$$

$$s_{k-1} \triangleq \frac{T}{2}\left\| \breve{u}_{k-1} - \beta_{k-1} - w^u_{k-1} \right\|_2. \qquad (90)$$

The constraint (84) also holds for (87), that is,

$$x^2_{1,k} + x^2_{2,k} + x^2_{3,k} + x^2_{4,k} = 1. \qquad (91)$$

We set $T = 0.1$ s, $\beta_{k-1} = [0.001 \;\; -0.001 \;\; 0.0005]^{\mathrm{T}}$ rad s^{-1}, and $Q_{k-1} = \mathrm{diag}(10^{-5} I_{3\times 3}, 10^{-10} I_{3\times 3})$. Attitude observations $y^{[i]}_k \in \mathbb{R}^3$ for a direction sensor are given by (Crassidis and Markley 2003)

$$y^{[i]}_k = C_k r^{[i]} + v^{[i]}_k, \qquad (92)$$

where $r^{[i]} \in \mathbb{R}^3$ is a reference direction vector to a known point, and C_k is the rotation matrix from the reference to the body-fixed frame

$$C_k = \begin{bmatrix} x^2_{1,k} - x^2_{2,k} - x^2_{3,k} + x^2_{4,k} & 2(x_{1,k}x_{2,k} + x_{3,k}x_{4,k}) & 2(x_{1,k}x_{3,k} - x_{2,k}x_{4,k}) \\ 2(x_{1,k}x_{2,k} - x_{3,k}x_{4,k}) & -x^2_{1,k} + x^2_{2,k} - x^2_{3,k} + x^2_{4,k} & 2(x_{1,k}x_{4,k} + x_{2,k}x_{3,k}) \\ 2(x_{1,k}x_{3,k} + x_{2,k}x_{4,k}) & 2(-x_{1,k}x_{4,k} + x_{2,k}x_{3,k}) & -x^2_{1,k} - x^2_{2,k} + x^2_{3,k} + x^2_{4,k} \end{bmatrix}. \qquad (93)$$

We assume that two directions are available (Crassidis and Markley 2003; Lee, Leok, McClamroch, and Sanyal 2007), which can be obtained using either a star tracker or a combined three-axis accelerometer/three-axis magnetometer. We set $r^{[1]} = [1 \;\; 0 \;\; 0]^{\mathrm{T}}$, $r^{[2]} = [0 \;\; 1 \;\; 0]^{\mathrm{T}}$, and $R_k = 10^{-4} I_{6\times 6}$. These direction measurements are assumed to be provided at a lower rate, specifically, at 1 Hz, which corresponds to a sample interval of $10T$ s.

We implement Kalman filtering using UKF, ECUKF, PUKF, MAUKF and CUKF with (87), (92) and constraint (91). We initialise these algorithms with $\hat{x}_{0|0} = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$ and $P^{xx}_{0|0} = \mathrm{diag}(0.5 I_{4\times 4} \;\; 0.01 I_{3\times 3})$. We use the `fmincon` algorithm of Matlab in the CUKF implementation.

Table 2 shows the results obtained from a 100-run Monte Carlo simulation. Note that, for the non-linear case, MT given by (78) is obtained from the pseudo-error covariance, and thus, smaller values of MT do not guarantee a smaller spread about the mean. Nevertheless, with the usage of prior knowledge by ECUKF and MAUKF, the values of MT show that more informative estimates are produced compared to the unconstrained estimates given by UKF. However, a slight increase in the RMS error is observed for algorithms ECUKF and MAUKF implying loss of accuracy around 5% compared to UKF. On the other hand, PUKF and CUKF yield estimates as accurate as UKF does. The equality constraint is more closely tracked whenever a constrained filter is employed; see Figure 4. RMS errors around 0.0007% are obtained for PUKF, MAUKF and ECUKF, which exhibits the smallest constraint error among these three algorithms. Compared to PUKF, MAUKF and ECUKF, note that our implementation of CUKF using the `fmincon`

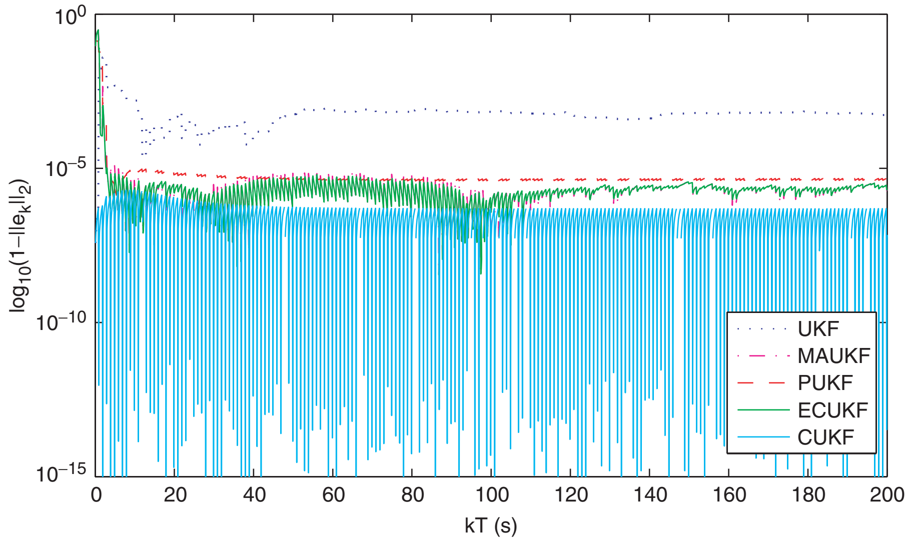

Figure 4. Estimation error of the quaternion vector norm using UKF (····), ECUKF (thick ——), PUKF (– – –), MAUKF (– · –) and CUKF (thin ——) algorithms. The ECUKF and MAUKF estimates almost coincide, while the constraint error for PUKF is slightly larger than the constraint error for ECUKF and MAUKF. CUKF estimates satisfy the equality constraint at machine precision at most times.

algorithm provides a 11 times smaller constraint error for the quaternion norm, but at a two times larger processing time.

Also, for comparison, we implement the EKF-counterparts of ECUKF, PUKF and MAUKF, namely, ECEKF, PEKF (Simon and Chia 2002) and MAEKF (Alouani and Blair 1993). The results (not shown) indicate that the unscented approaches yield improved results.

8.3 *Simple pendulum*

We consider the continuous-time undamped and unforced pendulum

$$\ddot{\theta}(t) + \frac{g}{L}\sin\theta(t) = 0, \tag{94}$$

where $\theta(t)$ is the angular position such that $\theta = 0$ rad corresponds to the stable equilibrium position, g is the gravity acceleration and L is the length. Given noisy measurements of the angular velocity of the pendulum, we want to obtain states that satisfy the energy conservation property.

Using Euler discretisation with time step T, such that $t = kT$, and defining $x_{1,k} = \theta(kT)$ and $x_{2,k} = \dot{\theta}(kT)$, we obtain the approximate discretised model

$$\begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} = \begin{bmatrix} x_{1,k-1} + Tx_{2,k-1} \\ x_{2,k-1} - T\frac{g}{L}\sin\left(x_{1,k-1}\right) \end{bmatrix} + \begin{bmatrix} w_{1,k} \\ w_{2,k} \end{bmatrix}, \tag{95}$$

with noisy measurements of the true (continuous-time model) values of angular velocity given by

$$y_k = x_2(kT) + v_k. \tag{96}$$

By conservation of energy in (94) we have

$$-mgL\cos(x_1(t)) + \frac{mL^2}{2}x_2^2(t) = E(0), \tag{97}$$

where $E(0)$ is the total mechanical energy and m is the pendulum mass. Next, we define the approximate energy E_k of the discrete-time model by

$$E_k \triangleq -mgL\cos(x_{1,k}) + \frac{mL^2}{2}x_{2,k}^2. \tag{98}$$

We use the fourth-order Runge–Kutta integration scheme to obtain $x(kT)$ for $L = 1$ m, $T = 10$ ms, and initial conditions $\theta(0) = 3\pi/4$, $\dot{\theta}(0) = \pi/50$. We assume that $E(0)$ is known and we implement equality-constrained state estimation by constraining $E_k = E(0)$ for all $k \geq 1$. The state estimation is initialised with $Q_{k-1} = \sigma_w^2 I_{2\times 2}$, $R_k = \sigma_v^2$, $\hat{x}_{0|0} = [1\ 1]^{\mathrm{T}}$, and $P_{0|0}^{xx} = I_{2\times 2}$, where three values of observation noise are tested, namely, $\sigma_v = 0.1$, 0.25 and 0.5, and process noise with $\sigma_w = 0.007$ is set to help convergence of estimates (Xiong, Zhang, and Chan 2007). A 100-run Monte Carlo simulation is performed for each σ_v.

Table 3 shows the percent RMS errors related to the equality constraint (98). Figure 5 compares the accuracy of the algorithms with relation to (97). It can be noticed, in this example, that the data-free simulation of the discretised model results in an unrealistic increasing energy E_k. Note that UKF is not able to closely track the constraint. For higher observation noise levels, that is, $\sigma_v = 0.5$, RMS constraint errors between 4% and 7% are observed. Nonetheless, whenever PUKF, MAUKF and ECUKF are employed, these indices are reduced

Table 3. Percent RMS constraint error, RMSE (77), and MT (78), from $k = 3000$ to 4000, for 100-run Monte Carlo simulation for the discretised pendulum (95), concerning different levels of observation noise $\sigma_v = 0.1$, 0.25 and 0.5, and algorithms, namely, UKF, ECUKF, PUKF, MAUKF and CUKF. We show in italics results for the case in which the true continuous-time model (94) is used with $\sigma_w = 0.0003$ to help convergence.

σ_v	UKF	MAUKF	PUKF	ECUKF	CUKF
Percent RMS constraint error ($\times 10^{-2}$)					
0.1	356.30	1.95	5.65	1.95	0.000066
	21.81	*0.06*	*0.27*	*0.06*	*0.000151*
0.25	459.40	3.50	9.11	3.51	0.000050
	25.68	*0.14*	*1.09*	*0.14*	*0.000134*
0.5	594.61	5.98	15.93	5.97	0.000040
	29.80	*0.32*	*3.16*	*0.31*	*0.000119*
RMSE_i, $i = 1, 2$ ($\times 10^{-2}$)					
0.1	2.95, 2.88	0.91, 1.92	1.15, 2.12	0.91, 1.92	1.20, 2.07
	0.59, *1.16*	*0.17*, *0.36*	*0.39*, *0.82*	*0.17*, *0.36*	*0.32*, *0.67*
0.25	3.93, 5.59	1.32, 3.05	1.76, 3.84	1.32, 3.04	1.84, 3.75
	0.98, *2.15*	*0.30*, *0.68*	*0.70*, *1.55*	*0.29*, *0.68*	*0.59*, *1.32*
0.5	5.56, 9.61	1.80, 4.00	2.76, 5.93	1.80, 3.99	2.86, 6.03
	1.60, *3.55*	*0.62*, *1.35*	*1.15*, *2.54*	*0.60*, *1.30*	*0.98*, *2.14*
MT ($\times 10^{-4}$)					
0.1	26.79	8.09	9.08	8.09	26.89
	2.18	*0.37*	*1.09*	*0.37*	*0.37*
0.25	61.66	20.67	26.63	20.66	61.54
	8.31	*0.94*	*4.18*	*0.94*	*0.94*
0.5	139.94	42.13	66.86	42.11	138.60
	23.17	*2.10*	*11.73*	*2.10*	*2.10*

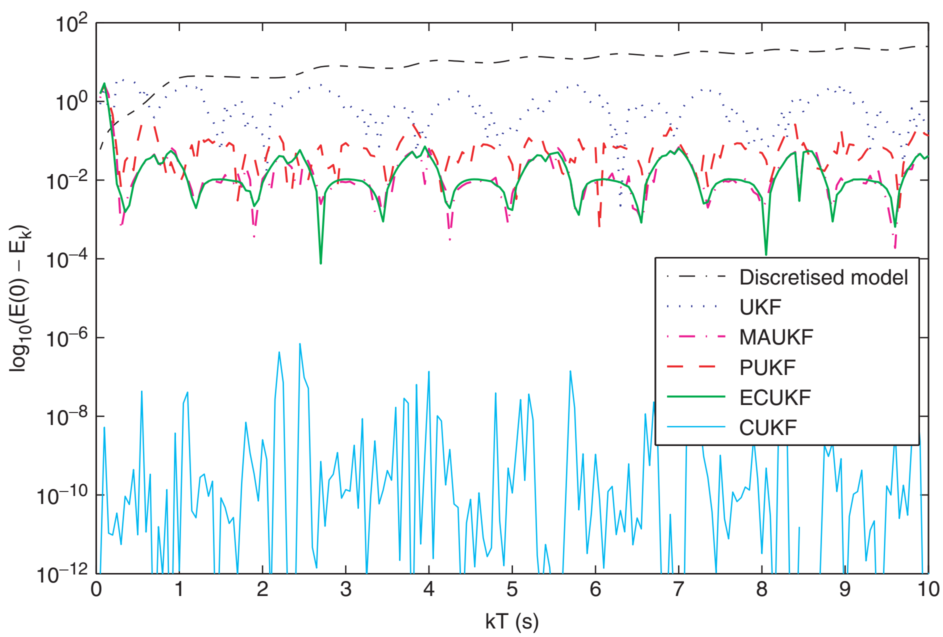

Figure 5. Estimation error of the total energy $E(0)$ for the discretised pendulum (95)–(96) for UKF ($\cdots\cdot$), MAUKF (thick $-\cdot-$), PUKF ($---$), ECUKF (thick —) and CUKF (thin —) with $\sigma_v = 0.25$. For comparison, the thin dot-dashed line, which is above the remaining lines, refers to the energy E_k calculated from data-free simulation of the discretised model (95). ECUKF and MAUKF estimates almost coincide, while CUKF estimates satisfy the equality constraint at machine precision at most times.

by two orders of magnitude, while, by using CUKF, we observe a reduction of seven orders of magnitude. In addition to the improvement in the accuracy of constraint satisfaction, the usage of prior knowledge also results in more accurate and more informative estimates.

According to the indices in italics in Table 3, the same comparative analysis is applicable when the true continuous-time model (94) is used replacing (95). The use of a fourth Runge–Kutta integration with UKF yields 30% smaller RMSE indices and two times smaller MT index compared to Euler discretisation with ECUKF, but with approximately seven times larger RMS constraint error and with approximately four times larger processing time. Similar to § 8.2, note that the use of a process model whose state vector satisfies an equality constraint does not guarantee that the state estimates satisfy this constraint because such information is not taken into account during data assimilation.

For this numerical example, ECUKF and MAUKF yield more accurate and more informative estimates than PUKF and CUKF. Moreover, the performance of ECUKF and MAUKF almost coincide for this non-linear example. When it comes to constraint satisfaction, CUKF yields the most accurate results.

In addition, we implement the EKF-counterparts of ECUKF, PUKF and MAUKF. The results (not shown) indicate that the unscented approaches yield competitive results compared to the extended approaches.

9. Concluding remarks

We have shown that the problem of equality-constrained state estimation for linear and Gaussian systems arises from the definition of both process noise and dynamic equations with special properties, specifically, (14)–(16), such that the system is not controllable in $\mathbb{R}^n$ from the process noise. In this case, the classical Kalman filter does not guarantee that its estimates satisfy the equality constraint.

Then we have solved the equality-constrained state-estimation problem for linear and Gaussian systems using the maximum-a-posteriori approach, yielding ECKF. Moreover, we have proved the equivalence of ECKF to MAKF and have presented its connections to PKF-SP and PKF-EP. We have compared these four methods by means of a compartmental system example with mass conservation.

For the non-linear case, where it is the main contribution of the present article, four suboptimal algorithms based on UKF were presented, namely, ECUKF, PUKF, MAUKF, CUKF. CUKF, which is an optimisation-based approach, allows the enforcement of both equality and inequality constraints at a given tolerance. These methods were compared on two examples, including a quaternion-based attitude estimation problem, as well as a mechanical system with conserved energy.

Numerical results suggest that, in addition to exactly, that is, at machine precision, (in the linear case and for CUKF in the non-linear case) or very closely (for ECUKF, MAUKF and PUKF in the non-linear case) satisfying a known equality constraint of the system, the proposed methods can yield more accurate and more informative estimates than KF (in the linear case) or UKF (in non-linear case). For the linear scenario, ECKF, MAKF, PKF-SP and PKF-EP have produced similar results. However, for the non-linear case, considering the examples investigated, we recommend the user to test ECUKF, MAUKF and PUKF in this order for a given equality-constrained state-estimation application. If the constraint satisfaction accuracy has priority over processing time and ease of implementation, we suggest CUKF. Recall that, since these non-linear methods are approximate, their performance depends on the application. Moreover, except for CUKF, all equality-constrained approaches have required similar processing time, which was competitive to KF (for linear algorithms) and UKF (for non-linear cases) processing time. In this case, CUKF has a larger processing time because it solves online a constrained optimisation problem. The performance of CUKF depends on the optimisation algorithm and problem. For the two non-linear examples investigated, the CUKF processing time was 2–15 times larger than the UKF processing time.

Finally, we have also addressed the case where an approximate discretised model is used to represent a continuous-time process in state estimation. Improved estimates were obtained when equality-constrained Kalman filtering algorithms were employed to enforce a conserved quantity of the original continuous-time model, but without the higher computational burden required by more accurate integration schemes.

We believe that comparisons of MHE for non-linear systems (Rao et al. 2003) against ECUKF and MAUKF would be valuable in a deeper analysis of the algorithms and that this should be pursued in the near future.

Acknowledgements

We wish to thank Prof. Harris McClamroch for assistance in the attitude estimation example.

This research was supported by the National Council for Scientific and Technological Development (CNPq), Brazil, and by the National Science Foundation Information Technology Research Initiative, through Grants ATM-0325332 and CNS-0539053 to the University of Michigan, Ann Arbor, USA.

References

Aguirre, L.A., Alves, G.B., and Corrêa, M.V. (2007), 'Steady-state Performance Constraints for Dynamical Models Based on RBF Networks', *Engineering Applications of Artificial Intelligence*, 20, 924–935.

Aguirre, L.A., Barroso, M.F.S., Saldanha, R.R., and Mendes, E.M.A.M. (2004), 'Imposing Steady-state Performance on Identified Nonlinear Polynomial Models by Means of Constrained Parameter Estimation', *IEE Proceedings in Control Theory and Applications*, 151, 174–179.

Aguirre, L.A., Teixeira, B.O.S., and Tôrres, L.A.B. (2005), 'Using Data-driven Discrete-time Models and the Unscented Kalman Filter to Estimate Unobserved Variables of Nonlinear Systems', *Physical Review E*, 72, 026226.

Alouani, A.T., and Blair, W.D. (1993), 'Use of a Kinematic Constraint in Target Tracking Constant Speed, Maneuvering Targets', *IEEE Transactions on Automatic Control*, 38, 1107–1111.

Bernstein, D.S. (2005), *Matrix Mathematics*, Princeton, USA: Princeton University Press.

Bernstein, D.S., and Hyland, D.C. (1993), 'Compartmental Modelling and Second-moment Analysis of State Space Systems', *SIAM Journal on Matrix Analysis and Applications*, 14, 880–901.

Chandrasekar, J., Barrero, O., Ridley, A.J., Bernstein, D.S., and De Moor, D. (2004), 'State Estimation for Linearised MHD Flow', in *Proceedings of the 43th IEEE Conference on Decision and Control*, Paradise Island: The Bahamas, pp. 2584–2589.

Chandrasekar, J., Ridley, A.J., and Bernstein, D.S. (2007), 'A Comparison of the Extended and Unscented Kalman Filters for Discrete-time Systems with Nondifferentiable Dynamics', in *Proceedings of the 2007 American Control Conference*, New York City, USA, pp. 4431–4436.

Chaves, M., and Sontag, E.D. (2002), 'State-estimators for Chemical Reaction Networks of Feinberg-Horn-Jackson Zero Deficiency Type', *European Journal of Control*, 8, 343–359.

Chen, Y.H., and Chiang, C.T. (1993), 'Adaptive Beamforming using the Constrained Kalman filter', *IEEE Transactions on Antennas and Propagation*, 41, 1576–1580.

Chia, T.L., Chow, P., and Chizeck, H.J. (1991), 'Recursive Parameter Identification of Constrained Systems: An Application to Electrically Stimulated Muscle', *IEEE Transactions on Biomedical Engineering*, 38, 429–441.

Choi, J., Yeap, T.H., and Bouchard, M. (2005), 'Online State-Space Modeling using Recurrent Multilayer Perceptrons with Unscented Kalman Filter', *Neural Processing Letters*, 22, 69–84.

Choukroun, D., Bar-Itzhack, I.Y., and Oshman, Y. (2006), 'Novel Quaternion Kalman Filter', *IEEE Transactions on Aerospace and Electronic Systems*, 42, 174–190.

Crassidis, J.L. (2006), 'Sigma-point Kalman Filtering for Integrated GPS and Inertial Navigation', *IEEE Transactions on Aerospace and Electronic Systems*, 42, 750–756.

Crassidis, J.L., and Markley, F.L. (2003), 'Unscented Filtering for Spacecraft Attitude Estimation', *AIAA Journal of Guidance, Control, and Dynamics*, 26, 536–542.

Daum, F.E. (2005), 'Nonlinear Filters: Beyond the Kalman Filter', *IEEE Aerospace and Electronics Systems Magazine*, 20, 57–69.

De Geeter, J., van Brussel, H., and De Schutter, J. (1997), 'A Smoothly Constrained Kalman Filter', *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 19, 1171–1177.

Dissanayake, G., Sukkarieh, S., and Nebot, E. (2001), 'The Aiding of a Low-cost Strapdown Inertial Measurement Unit using Vehicle Model Constraints for Land Vehicle Applications', *IEEE Transactions on Robotics and Automation*, 17, 731–747.

Fletcher, R. (2000), *Practical Methods of Optimization*, Chichester, UK: John Wiley & Sons.

Goodwin, G.C., Seron, M.M., and de Doná, J.A. (2005), *Constrained Control and Estimation: An Optimisation Approach*, London: Springer.

Gupta, N., and Hauser, R. (2007), 'Kalman Filtering with Equality and Inequality State Constraints', Technical Report 07/18, Numerical Analysis Group, Oxford University Computing Laboratory, University of Oxford. http://arxiv.org/abs/0709.2791

Haykin, S. (Ed.) (2001), *Kalman Filtering and Neural Networks*, New York City, USA: Wiley Publishing.

Hovland, G.E., von Hoff, T.P., Gallestey, E.A., Antoine, M., Farruggio, D., and Paice, A.D.B. (2005), 'Nonlinear Estimation Methods For Parameter Tracking in Power Plants', *Control Engineering Practice*, 13, 1341–1355.

Jazwinski, A.H. (1970), *Stochastic Processes and Filtering Theory*, New York City, USA: Academic Press, Inc.

Julier, S.J., and LaViola Jr, J.J. (2007), 'On Kalman Filtering with Nonlinear Equality Constraints', *IEEE Transactions on Signal Processing*, 55, 2774–2784.

Julier, S.J., and Uhlmann, J.K. (2004), 'Unscented Filtering and Nonlinear Estimation', *Proceedings of the IEEE*, 92, 401–422.

Julier, S.J., Uhlmann, J.K., and Durrant-Whyte, H.F. (2000), 'A New Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators', *IEEE Transactions on Automatic Control*, 45, 477–482.

Ko, S., and Bitmead, R.R. (2007), 'State Estimation for Linear Systems with State Equality Constraints', *Automatica*, 43, 1363–1368.

Lee, T., Leok, M., McClamroch, N.H., and Sanyal, A. (2007), 'Global Attitude Estimation using Single Direction Measurements', in *Proceedings of the 2007 American Control Conference*, New York City, USA, pp. 3659–3664.

Lefebvre, T., Bruyninckx, H., and De Schutter, J. (2002), 'Comment on 'A New Method for the Nonlinear Transformation of Means and Covariances in Filters and

Estimators', *IEEE Transactions on Automatic Control*, 47, 1406–1408.

——— (2004), 'Kalman Filters for Nonlinear Systems: A Comparison of Performance', *International Journal of Control*, 77, 639–653.

Maciejowski, J.M. (2002), *Predictive Control with Constraints*, Essex, UK: Pearson Education Ltd.

Massicotte, D., Morawski, R.Z., and Barwicz, A. (1995), 'Incorporation of a Positivity Constraint into a Kalman-Filter-Based Algorithm for Correction of Spectrometric Data', *IEEE Transactions on Instrumentation and Measurement*, 44, 2–7.

Maybeck, P. (1979), *Stochastic Models, Estimation and Control*, Vol. 1, San Diego, USA: Academic Press.

Nepomuceno, E.G., Takahashi, R.H.C., Aguirre, L.A., Neto, O.M., and Mendes, E.M.A.M. (2004), 'Multiobjective Nonlinear System Identification: A Case Study with Thyristor Series Capacitor (TCSC)', *International Journal of Systems Science*, 35, 537–546.

Nikoukhah, R., Campbell, S.L., and Delebecque, F. (1999), 'Kalman Filtering for General Discrete-Time Linear Systems', *IEEE Transactions on Automatic Control*, 44, 1829–1839.

Pinto, R.L.F., and Rios Neto, A. (1990), 'An Optimal Linear Estimation Approach to Solve Systems of Linear Algebraic Equations', *Journal of Computational and Applied Mathematics*, 33, 261–268.

Porrill, J. (1988), 'Optimal Combination and Constraints for Geometrical Sensor Data', *International Journal of Robotics Research*, 7, 66–77.

Rao, C.V., Rawlings, J.B., and Lee, J.H. (2001), 'Constrained Linear State Estimation – A Moving Horizon Approach', *Automatica*, 37, 1619–1628.

Rao, C.V., Rawlings, J.B., and Mayne, D.Q. (2003), 'Constrained State Estimation for Nonlinear Discrete-time Systems: Stability and Moving Horizon Approximations', *IEEE Transactions on Automatic Control*, 48, 246–258.

Reif, K., Günther, S., Yaz, E., and Unbehauen, R. (1999), 'Stochastic Stability of the Discrete-time Extended Kalman Filter', *IEEE Transactions on Automatic Control*, 44, 714–728.

Robertson, D.G., and Lee, J.H. (2002), 'On the Use of Constraints in Least Squares Estimation and Control', *Automatica*, 38, 1113–1123.

Romanenko, A., and Castro, J.A.A.M. (2004), 'The Unscented Filter as an Alternative to the EKF for Non-linear State Estimation: A Simulation Case Study', *Computers and Chemical Engineering*, 28, 347–355.

Rotea, M., and Lana, C. (2008), 'State Estimation with Probability Constraints', *International Journal of Control*, 81, 920–930.

Shen, S., Honga, L., and Cong, S. (2006), 'Reliable Road Vehicle Collision Prediction with Constrained Filtering', *Signal Processing*, 86, 3339–3356.

Simon, D. (2006), *Optimal State Estimation: Kalman, H_∞ and Nonlinear Approaches*, Hoboken, NJ: Wiley-Interscience.

Simon, D., and Chia, T.L. (2002), 'Kalman Filtering with State Equality Constraints', *IEEE Transactions on Aerospace and Electronic Systems*, 38, 128–136.

Simon, D., and Simon, D.L. (2006), 'Kalman Filtering with Inequality Constraints for Turbofan Engine Health Estimation', *IEE Proceedings – Control Theory and Applications*, 153, 371–378.

Tahk, M., and Speyer, J.L. (1990), 'Target Tracking Problems Subject to Kinematic Constraints', *IEEE Transactions on Automatic Control*, 35, 324–326.

Teixeira, B.O.S. (2008), 'Constrained State Estimation for Linear and Nonlinear Dynamic Systems', Ph.D. thesis, Belo Horizonte – MG, Brazil: Graduate Program in Electrical Engineering, Federal University of Minas Gerais.

Teixeira, B.O.S., Chandrasekar, J., Tôrres, L.A.B., Aguirre, L.A., and Bernstein, D.S. (2007), 'State Estimation for Equality-constrained Linear Systems', in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, USA, pp. 6220–6225.

——— (2008a), 'Unscented Filtering for Equality-constrained Nonlinear Systems', in *Proceedings of the 2008 American Control Conference*, Seattle, USA, pp. 39–44.

Teixeira, B.O.S., Chandrasekar, J., Palanthandalam-Madapusi, H.J., Tôrres, L.A.B., Aguirre, L.A., and Bernstein, D.S. (2008b), 'Gain-constrained Kalman Filtering for Linear and Nonlinear Systems', *IEEE Transactions on Signal Processing* 56, 4113–4123.

Teixeira, B.O.S., Ridley, A., Tôrres, L.A.B., Aguirre, L.A., and Bernstein, D.S. (2008c), 'Data Assimilation for Magnetohydrodynamics with a Zero-divergence Constraint on the Magnetic Field', in *Proceedings of the 2008 American Control Conference*, Seattle, USA, pp. 2534–2539.

Teixeira, B.O.S., Tôrres, L.A.B., Aguirre, L.A., and Bernstein, D.S. (2008d), 'Unscented Filtering for Interval-constrained Nonlinear Systems', in *Proceedings of the 47th IEEE Conference on Decision and Control*, Cancun, Mexico, pp. 5116–5121.

Teixeira, B.O.S., Santillo, M.A, Erwin, R.S. and Bemstein, D.S. (2008e), 'Spacecraft Tracking using sampled-data Kalman Filters', *IEEE Control Systems Magazine*, 28(4), 78–94.

Vachhani, P., Narasimhan, S., and Rengaswamy, R. (2006), 'Robust and Reliable Estimation via Unscented Recursive Nonlinear Dynamic Data Reconciliation', *Journal of Process Control*, 16, 1075–1086.

van der Merwe, R., Wan, E.A., and Julier, S.J. (2004), 'Sigma-point Kalman Filters for Nonlinear Estimation and Sensor-fusion – Applications to Integrated Navigation', in *Proceedings of the AIAA Guidance, Navigation & Control Conference,*, Providence, USA. No. AIAA2004-5120.

Walker, D.M. (2006), 'Parameter Estimation using Kalman Filters with Constraints', *International Journal of Bifurcation and Chaos*, 4, 1067–1078.

Wen, W., and Durrant-Whyte, H.F. (1992), 'Model-based Multi-sensor Data Fusion', in *Proceedings of the 1992 IEEE International Conference on Robotics and Automation*, Nice, France, pp. 1720–1726.

Xiong, K., Zhang, H., and Chan, C. (2007), 'Author's Reply to 'Comments on 'Performance Evaluation of UKF-based Nonlinear Filtering' ', *Automatica*, 43, 569–570.

Appendix I: Equivalence between ECKF and MAKF

Define the augmented observation

$$\tilde{y}_k \triangleq \begin{bmatrix} y_k \\ d_{k-1} \end{bmatrix} = \tilde{C}_k x_k + \begin{bmatrix} v_k \\ 0_{s\times 1} \end{bmatrix}, \tag{A.1}$$

where

$$\tilde{C}_k \triangleq \begin{bmatrix} C_k \\ D_{k-1} \end{bmatrix}. \tag{A.2}$$

With (A.1), MAKF uses (4) and (5) together with the augmented forecast equations

$$\hat{\tilde{y}}_{k|k-1} = \tilde{C}_k \hat{x}_{k|k-1}, \tag{A.3}$$

$$\tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1} = \tilde{C}_k P^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k + \tilde{R}_k, \tag{A.4}$$

$$\tilde{P}^{x\tilde{y}}_{k|k-1} = P^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k, \tag{A.5}$$

where $\tilde{R}_k$ is given by (71), and the augmented data-assimilation equations given by (72)–(74).

For convenience, let $\tilde{x}_{k|k-1} \triangleq \hat{x}_{k|k-1}$ (4) denote the forecast estimate provided by MAKF. Furthermore, let $\tilde{P}^{xx}_{k|k-1} \triangleq P^{xx}_{k|k-1}$ (5) be the associated forecast error covariance of MAKF. Also let $\hat{x}_{k|k-1}$ (22) and $P^{xx}_{k|k-1}$ (23) denote the forecast estimate and the associated error covariance of ECKF. Assume that δ in (29) is sufficiently small and can be neglected.

Proposition 9.1: *Assume that $\tilde{x}_{k|k-1} = \hat{x}_{k|k-1}$ and $\tilde{P}^{xx}_{k|k-1} = P^{xx}_{k|k-1}$. Then $\tilde{x}_{k+1|k} = \hat{x}_{k+1|k}$ and $\tilde{P}^{xx}_{k+1|k} = P^{xx}_{k+1|k}$.*

Proof: $\tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1}$ (A.4) is equivalent to

$$\tilde{P}^{\tilde{y}\tilde{y}}_{k|k-1} = \begin{bmatrix} P^{yy}_{k|k-1} & C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} \\ D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k & P^{dd}_{k|k-1} \end{bmatrix}.$$

It follows from Bernstein (2005) that $\tilde{P}^{\tilde{y}\tilde{y}-1}_{k|k-1}$ has entries

$$\tilde{P}^{\tilde{y}\tilde{y}-1}_{k|k-1} = \begin{bmatrix} (\tilde{P}^{-1}_{k|k-1})_1 & (\tilde{P}^{-1}_{k|k-1})_{12} \\ (\tilde{P}^{-1}_{k|k-1})^{\mathrm{T}}_{12} & (\tilde{P}^{-1}_{k|k-1})_2 \end{bmatrix},$$

where

$$\begin{aligned}
(\tilde{P}^{-1}_{k|k-1})_1 &\triangleq \Big(P^{yy}_{k|k-1} - C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} (P^{dd}_{k|k-1})^{-1} D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k\Big)^{-1}, \\
(\tilde{P}^{-1}_{k|k-1})_{12} &\triangleq -\Big(P^{yy}_{k|k-1} - C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} (P^{dd}_{k|k-1})^{-1} D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k\Big)^{-1} \\
&\quad \times C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} (P^{dd}_{k|k-1})^{-1}, \\
(\tilde{P}^{-1}_{k|k-1})_2 &\triangleq \Big(P^{dd}_{k|k-1} - D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1} C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1}\Big)^{-1}.
\end{aligned} \tag{A.6}$$

Furthermore, it can be shown that

$$\begin{aligned}
(\tilde{P}^{-1}_{k|k-1})_1 &= (P^{yy}_{k|k-1})^{-1} + (P^{yy}_{k|k-1})^{-1} C_k P^{xx}_{k|k-1} \\
&\quad \times D^{\mathrm{T}}_{k-1} (\tilde{P}^{-1}_{k|k-1})_2 D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1},
\end{aligned} \tag{A.7}$$

$$(\tilde{P}^{-1}_{k|k-1})_{12} = -(P^{yy}_{k|k-1})^{-1} C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} (\tilde{P}^{-1}_{k|k-1})_2. \tag{A.8}$$

It follows from (9) that

$$K_k = P^{xx}_{k|k-1} C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1}, \tag{A.9}$$

Furthermore, substituting (A.9) into (11) yields

$$P^{xx}_{k|k} = P^{xx}_{k|k-1} - P^{xx}_{k|k-1} C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1} C_k P^{xx}_{k|k-1}.$$

Hence,

$$\begin{aligned}
(D_{k-1} P^{xx}_{k|k} D^{\mathrm{T}}_{k-1})^{-1} &= \Big(D_{k-1} P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} - D_{k-1} P^{xx}_{k|k-1} \\
&\quad \times C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1} C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1}\Big)^{-1} \\
&= (\tilde{P}^{-1}_{k|k-1})_2.
\end{aligned} \tag{A.10}$$

Substituting (A.9) into (10) yields (28). Substituting (24) into (28) yields

$$\hat{x}^{\mathrm{p}}_{k|k} = \hat{x}_{k|k-1} + \big[\, K_k - K^{\mathrm{p}}_k D_{k-1} K_k \quad K^{\mathrm{p}}_k \,\big]\big(\tilde{y}_k - \tilde{C}_k \hat{x}_{k|k-1}\big). \tag{A.11}$$

It follows from (25), (A.6), (A.8) and (A.10) that

$$K^{\mathrm{p}}_k = P^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k \begin{bmatrix} (\tilde{P}^{-1}_{k|k-1})_{12} \\ (\tilde{P}^{-1}_{k|k-1})_2 \end{bmatrix}. \tag{A.12}$$

Substituting (A.8) into (A.12) and substituting the resulting expression into $K_k - K^{\mathrm{p}}_k D_{k-1} K_k$ yields

$$\begin{aligned}
&K_k - K^{\mathrm{p}}_k D_{k-1} K_k \\
&\quad = P^{xx}_{k|k-1} C^{\mathrm{T}}_k \Big[(P^{yy}_{k|k-1})^{-1} + (P^{yy}_{k|k-1})^{-1} C_k P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} \\
&\qquad \times (\tilde{P}^{-1}_{k|k-1})_2 D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1}\Big] \\
&\qquad - P^{xx}_{k|k-1} D^{\mathrm{T}}_{k-1} (\tilde{P}^{-1}_{k|k-1})_2 D_{k-1} P^{xx}_{k|k-1} C^{\mathrm{T}}_k (P^{yy}_{k|k-1})^{-1}.
\end{aligned}$$

Hence, (A.7) and (A.8) imply that

$$K_k - K^{\mathrm{p}}_k D_{k-1} K_k = P^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k \begin{bmatrix} (\tilde{P}^{-1}_{k|k-1})_1 \\ (\tilde{P}^{-1}_{k|k-1})^{\mathrm{T}}_{12} \end{bmatrix}. \tag{A.13}$$

Therefore, it follows from (A.12) and (A.13) that

$$\big[\, K_k - K^{\mathrm{p}}_k D_{k-1} K_k \quad K^{\mathrm{p}}_k \,\big] = P^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k \tilde{P}^{\tilde{y}\tilde{y}-1}_{k|k-1}. \tag{A.14}$$

Since the estimate $\tilde{x}_{k|k}$ of MAKF is given by

$$\tilde{x}_{k|k} = \tilde{x}_{k|k-1} + \tilde{K}_k\Big(\tilde{y}_k - \tilde{C}_k \tilde{x}_{k|k-1}\Big), \tag{A.15}$$

where

$$\tilde{K}_k = \tilde{P}^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k \Big(\tilde{C}_k \tilde{P}^{xx}_{k|k-1} \tilde{C}^{\mathrm{T}}_k + \tilde{R}_k\Big)^{-1},$$

it follows from (A.14) that

$$\tilde{K}_k = \big[\, K_k - K^{\mathrm{p}}_k D_{k-1} K_k \quad K^{\mathrm{p}}_k \,\big]. \tag{A.16}$$

Therefore, (A.11) and (A.15) imply that $\tilde{x}_{k|k} = \hat{x}^{\mathrm{p}}_{k|k}$ and (4) and (22) imply that $\tilde{x}_{k+1|k} = \hat{x}^{\mathrm{p}}_{k+1|k}$.

Note that (11) and (29) can be expressed as

$$P^{xx}_{k|k} = (I_{n\times n} - K_k C_k) P^{xx}_{k|k-1} (I_{n\times n} - K_k C_k)^{\mathrm{T}} + K_k R_k K^{\mathrm{T}}_k, \tag{A.17}$$

$$P^{xx\mathrm{p}}_{k|k} = (I_{n\times n} - K^{\mathrm{p}}_k D_{k-1}) P^{xx}_{k|k} (I_{n\times n} - K^{\mathrm{p}}_k D_{k-1})^{\mathrm{T}}. \tag{A.18}$$

Substituting (A.17) into (A.18) yields

$$P^{xxp}_{k|k} = (I_{n\times n} - K^{p}_{k} D_{k-1})(I_{n\times n} - K_k C_k) P^{xx}_{k|k-1} (I_{n\times n} - K_k C_k)^{\mathrm{T}} \times (I_{n\times n} - K^{p}_{k} D_{k-1})^{\mathrm{T}} + (K_k - K^{p}_{k} D_{k-1} K_k) R_k \times (K_k - K^{p}_{k} D_{k-1} K_k)^{\mathrm{T}}. \quad (A.19)$$

Substituting (A.2) and (A.16) into (A.19) yields

$$P^{xxp}_{k|k} = (I_{n\times n} - \tilde{K}\tilde{C}_k) P^{xx}_{k|k-1} (I_{n\times n} - \tilde{K}\tilde{C}_k)^{\mathrm{T}} + \tilde{K}\tilde{R}_k \tilde{K}^{\mathrm{T}}. \quad (A.20)$$

Since, (11) implies that

$$\tilde{P}^{xx}_{k|k} = (I_{n\times n} - \tilde{K}\tilde{C}_k) \tilde{P}^{xx}_{k|k-1} (I_{n\times n} - \tilde{K}\tilde{C}_k)^{\mathrm{T}} + \tilde{K}\tilde{R}_k \tilde{K}^{\mathrm{T}}, \quad (A.21)$$

it follows from (A.20) and (A.21) that $\tilde{P}^{xx}_{k|k} = P^{xxp}_{k|k}$. Hence, (4) and (23) imply that $\tilde{P}^{xx}_{k+1|k} = P^{xx}_{k+1|k}$. □

Appendix II: Connection between ECKF and PKF-SP

Assume that system given by (1), (2) and (19) is time invariant. Also, assume that (14)–(16) hold for D in (19). In Ko and Bitmead (2007), using a descriptor system representation (Nikoukhah, Campbell, and Delebecque 1999), the system given by (1) and (2), (19) is written in a projected representation. Then, consider PKF-SP which uses KF equations (4)–(11), but initialised with $\hat{x}^{p}_{0|0}$ satisfying

$$D\hat{x}^{p}_{0|0} = d, \quad (A.22)$$

and the singular initial error covariance

$$P^{xxp}_{0|0} = \mathcal{P}_{\mathcal{N}(D)} P^{xx}_{0|0}, \quad (A.23)$$

where the projector $\mathcal{P}_{\mathcal{N}(D)} \in \mathbb{R}^{n\times n}$ is obtained by the singular value decomposition

$$D^{\mathrm{T}} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_s \\ 0_{(n-s)\times s} \end{bmatrix} \begin{bmatrix} V_1^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{bmatrix}, \quad (A.24)$$

where $U_2 \in \mathbb{R}^{n\times(n-s)}$ such that

$$\mathcal{P}_{\mathcal{N}(D)} = U_2 U_2^{\mathrm{T}}. \quad (A.25)$$

Also, note that, since (14) holds, Gw_{k-1} is constrained in $\mathcal{P}_{\mathcal{N}(D)}$ and $GQ_{k-1}G^{\mathrm{T}}$ is a 'constrained' covariance as used in Ko and Bitmead (2007).

With Corollary 4.1 and comparing (A.22) and (A.23) to (38) and (39), we see that, similar to ECKF, which performs projection only at $k=1$ to guarantee constraint satisfaction for all $k \geq 1$, PKF-SP performs projection in initialisation, that is, only at $k=0$, providing that (14)–(16) hold; see Figure 1(c).

Appendix III: Connection between ECKF and PKF-EP

PKF-EP projects the updated estimate $\hat{x}_{k|k}$ (10) onto the hyperplane defined by (17) by minimising the cost function

$$\mathrm{J}(x_k) \triangleq \left(x_k - \hat{x}_{k|k}\right)^{\mathrm{T}} W^{-1} \left(x_k - \hat{x}_{k|k}\right) \quad (A.26)$$

subject to (17), where $W \in \mathbb{R}^{n\times n}$ is positive definite. The solution $\hat{x}^{p}_{k|k}$ to (A.26) is given by

$$\hat{x}^{p}_{k|k} = \hat{x}_{k|k} + K^{p}_{k}(d_{k-1} - D_{k-1}\hat{x}_{k|k}), \quad (A.27)$$

where

$$K^{p}_{k} \triangleq WD^{\mathrm{T}}_{k-1}(D_{k-1}WD^{\mathrm{T}}_{k-1})^{-1}. \quad (A.28)$$

The projected error covariance $P^{xxp}_{k|k}$ associated with $\hat{x}^{p}_{k|k}$ is given by (39) with $\Delta = 0$.

PKF-EP is formed by forecast ((4)–(8)), data-assimilation ((9)–(11)) and projection ((A.27)–(A.28), (37), (39)) steps.

We set $W = P^{xx}_{k|k}$ in (A.28), where $P^{xx}_{k|k}$ is given by (11), such that $\hat{x}^{p}_{k|k}$ (A.27) is optimal according to the maximum-a-posteriori and minimum-variance criteria (Simon and Chia 2002). In this case, note that the projection Equations (A.27), (A.28), (37), (39) of PKF-EP are equal to the projection Equations (24)–(29) of ECKF. However, unlike ECKF, PKF-EP does not recursively feed the projected estimate $\hat{x}^{p}_{k|k}$ (A.27) and the error covariance $P^{xxp}_{k|k}$ given by (39) back in forecasts (4) and (5); see Figure 1(b). Therefore, the PKF-EP forecast estimate $\hat{x}_{k|k-1}$ (4) is different from its ECKF counterpart (22).