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Structured matrix norms for robust stability and performance with block-structured uncertainty

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In this paper we introduce new lower and upper robust stability bounds for structured uncertainty involving arbitrary spatial norms. Specifically, we consider a norm-bounded block-structured uncertainty characterization wherein the defining spatial norm is not necessarily the maximum singular value. This new uncertainty characterization leads to the notion of *structured matrix norms* for characterizing the allowable size of the nominal transfer function for robust stability. The lower and upper bounds are specialized to specific matrix norms including Hölder, unitarily invariant, and induced norms to provide conditions for robust stability with several different characterizations of plant uncertainty. One of the key advantages of the proposed approach over the structured singular value is the reduction is computational complexity gained by directly addressing a given uncertainty characterization without having to transform it to a spectral-norm type characterization. Finally, we introduce a performance block within the structured matrix norm framework to address robust performance in the face of structured uncertainty.

Nomenclature

$\mathbb{R}, \mathbb{C}$	real numbers, complex numbers
$\mathbb{R}^{n\times m}, \mathbb{C}^{n\times m}$	$n \times m$ real matrices, $n \times m$ complex matrices
x_i	ith entry of x
$\Vert\cdot\Vert, \vert\vert\vert\cdot\vert\vert\vert$	vector or matrix norm, vector or matrix operator norm
$\Vert x\Vert_2$	Euclidian norm of vector x $(= \sqrt{x^*x})$
A^*	Complex conjugate transpose of A
$\det A, \operatorname{tr} A$	determinant of A, trace of A
$\sigma_i(A)$	ith singular value of A
$\sigma_{\max}(A)$	maximum singular value of A
$\Vert A\Vert_{\mathrm{F}}$	Frobenius norm of A $(= (\operatorname{tr} AA^*)^{1/2})$
$\operatorname{spec}(A)$	spectrum of A
$\rho(A), \rho_{\mathrm{R}}(A)$	spectral radius of A, real spectral radius of A
$A_{(i,j)}$	(i,j)th entry of A
$\operatorname{row}_i(A), \operatorname{col}_i(A)$	ith row of A, ith column of A
E_{ij}	elementary matrix with unity in the (i,j) position and zeros elsewhere
$\Vert A\Vert_p$	$\left[\sum_{i=1}^m \sum_{j=1}^n \vert A_{(i,j)}\vert^p\right]^{1/p}, 1 \le p < \infty$

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$\lVert A \rVert_{\infty}$	$\max_{i,j} \lvert A_{(i,j)} \rvert$
$\lVert A \rVert_{\sigma p}$	$\left[\sum_{i=1}^{r} \sigma_i^p(A)\right]^{1/p}$, $1 \le p < \infty$, $r = \operatorname{rank} A$
$\lVert A \rVert_{\sigma\infty}$	$\sigma_{\max}(A)$
$A \le\le B (<< B)$	$A_{(i,j)} \le B_{(i,j)} (A_{(i,j)} < B_{(i,j)})$ for all i and j, where A and B are real matrices of the same size
$A \otimes B$	Kronecker product of A and B
H_{∞}^{n}	Hardy space of $n \times 1$ functions bounded on the imaginary axis with analytic continuation into the right half plane

1. Introduction

The ability to address block-structured uncertainty is essential for reducing conservatism in the analysis and synthesis of control systems involving robust stability and performance objectives. Accordingly, the structured singular value provides a generalization of the spectral (maximum singular value) norm to permit small-gain-type analysis of systems involving block-structured uncertainty (Fan *et al.* 1991, Packard and Doyle 1993). The role of the structured singular value in robust analysis can readily be understood by observing that bounds on the structured singular value prevent the multivariable Nyquist plot of the loop gain from encircling the critical point when the uncertainty has a norm-bounded block-diagonal structure (Bernstein *et al.* 1995, Haddad *et al.* 1996).

For block-structured uncertainty with non-spectral norm characterization, a robustness theory has been developed by Chen and Nett (1992) wherein the uncertainty is bounded by equi-induced Hölder norms and robust stability bounds are developed in terms of the Perron root of matrix majorants and interaction parameters. Robustness theory for block-structured uncertainty has also been developed in the context of operator norms. In particular, block-structured extensions of small-gain theory have been developed in ℓ_p theory (Khammash and Pearson 1993, Young and Dahleh 1995).

In the present paper we derive structured-singular-value-type robustness conditions involving arbitrary spatial norms. In particular, we consider norm-bounded, block-structured uncertainty for which the defining norm is not necessarily the maximum singular value. This generalization of the structured singular value leads to the notion of *structured matrix norms* for characterizing the size of the nominal transfer function. Structured matrix norms thus include the structured singular value as a special case.

The usefulness of structured matrix norms lies in their ability to characterize plant uncertainty that is not consistent with the geometry of singular value bounds. For example, an uncertain matrix block whose entries have independently bounded magnitudes or whose entries satisfy a bound on the sum of their absolute values can be directly characterized by an appropriate matrix norm, but can only be conservatively bounded by the spectral norm. For block-structured uncertainty with non-spectral norm characterization, structured matrix norms thus provide a useful extension of the structured singular value.

Since all norms defined on finite-dimensional spaces are equivalent (Horn and Johnson 1985, Stewart and Sun 1990), it follows that robust stability guarantees obtained via the standard small-μ theorem for system uncertainties bounded by the spectral norm can be used to obtain robust stability guarantees for system uncertainties bounded by arbitrary norms. However, these bounds are conservative for the given uncertainty characterization, and thus the resulting robust stability guarantees are sufficient but not necessary. Hence, this framework is useful for reducing the conservatism of robust stability predictions for systems involving uncertainties bounded by arbitrary norms.

A practical advantage of structured matrix norms over the structured singular value is the reduction in computational complexity gained by directly addressing a given uncertainty characterization without having to transform it to a spectral-norm type characterization. In particular, to apply structured singular value analysis to systems with multiple sources of uncertainty, system uncertainty is typically recast into a block-diagonal structure and robust stability guarantees are computed with respect to this uncertainty structure. However, as pointed out by Chen *et al.* (1996 a,b) the conversion of a given uncertainty characterization into a block-diagonal characterization amenable to structured singular value analysis can increase the problem size and thus the computational complexity. Alternatively, structured matrix norms do not require that system uncertainty be recast into a block-diagonal structure. Specifically, structured matrix norms permit nonzero off-diagonal subblocks with each subblock perturbed independently. This feature is demonstrated in Example 1.

The paper begins with several definitions concerning submultiplicative matrix norms, induced norms, and equi-induced norms. A useful reference on this topic is Stewart and Sun (1990), which provides numerous relevant results. Next, we turn our attention in §3 to the definition of the structured matrix norm. This definition is then used in Theorem 1 where necessary and sufficient conditions for robust stability are given in terms of the structured matrix norm. This result is then followed by the development of lower and upper bounds for the structured matrix norm. In §5 these bounds are specialized to specific matrix norms to provide conditions for robust stability in terms of several different characterizations of plant uncertainty. In §6 we extend the results of the previous sections to block-norm uncertainty where the size of the uncertain blocks is characterized by different spatial norms. In §7, as in the structured singular value framework, we introduce a performance block within the structured matrix norm framework to address robust performance in the face of structured uncertainty. Section 8 considers several numerical examples and compares the proposed bounds to the structured singular value bounds. Finally, we draw some conclusions in §9.

2. Mathematical preliminaries

In this section we give several definitions and lemmas concerning matrix norms (Stewart and Sun 1990). Let $\|\cdot\|'', \|\cdot\|', \|\cdot\|$ denote matrix norms on $\mathbb{C}^{l\times n}$, $\mathbb{C}^{l\times m}$, and $\mathbb{C}^{m\times n}$, respectively. We say $(\|\cdot\|'', \|\cdot\|', \|\cdot\|)$ is a *submultiplicative triple* of matrix norms if $\|AB\|'' \le \|A\|'\|B\|$, for all $A \in \mathbb{C}^{l\times m}$ and $B \in \mathbb{C}^{m\times n}$. Furthermore, if $l = m = n$ and $(\|\cdot\|, \|\cdot\|, \|\cdot\|)$ is a submultiplicative triple of matrix norms, then $\|\cdot\|$ is said to be *submultiplicative*. A matrix norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is *unitarily invariant* if $\|UAV\| = \|A\|$ for all $A \in \mathbb{C}^{m\times n}$ and for all unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$. Furthermore, a unitarily invariant matrix norm $\|\cdot\|$ on $\mathbb{C}^{m\times n}$ is *normalized* if $\|A\| = \sigma_{\max}(A)$ for all rank-one matrices $A \in \mathbb{C}^{m\times n}$.

Next, let $\|\cdot\|'$ and $\|\cdot\|''$ denote vector norms of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Then $\|\cdot\| : \mathbb{C}^{n\times m} \to \mathbb{R}$ defined by $\|A\| \triangleq \max_{\|x\|'=1} \|Ax\|''$ is the *matrix norm induced by* $\|\cdot\|'$ and $\|\cdot\|''$. If $m = n$ and $\|\cdot\|' = \|\cdot\|''$, then $\|\cdot\|$ is an *equi-induced norm.* If $\|\cdot\|' = \|\cdot\|_p$ and $\|\cdot\|'' = \|\cdot\|_q$, where $p, q \in [1, \infty]$, then the resulting induced matrix norm on $\mathbb{C}^{n\times m}$ is denoted by $\|\cdot\|_{q,p}$.

Finally, consider a partitioned matrix $A \in \mathbb{C}^{m\times m}$ such that $A = [A_{ij}], i,j = 1,\ldots,r$, where $A_{ij} \in \mathbb{C}^{m_i\times m_j}$ and $\sum_{i=1}^{r} m_i = m$. Then the *block-norm matrix* of A is the matrix $[\|A_{ij}\|_{(i,j)}] \in \mathbb{R}^{r\times r}$ whose (i,j)th entry is $\|A_{ij}\|_{(i,j)}$, where $\|\cdot\|_{(i,j)}$ is a specified matrix norm on $\mathbb{C}^{m_i\times m_j}$. For convenience we write $\|\cdot\|_{(i)}$ for $\|\cdot\|_{(i,j)}$. Furthermore, if $\|\cdot\|_{(i,j)} = \|\cdot\|'$, $i,j = 1,\ldots,r$, where $\|\cdot\|'$ is a matrix norm of $\mathbb{C}^{m_i\times m_j}$, then $[\|A_{ij}\|_{(i,j)}]$ is denoted by $[\|A_{ij}\|']$.

The following lemmas are needed for developing robustness bounds in later sections.

Lemma 1: *Let $1 \le p \le \infty$ and $1 \le q \le \infty$ be such that $1/p + 1/q = 1$ and let $A \in \mathbb{C}^{m\times n}$. Then*

$$\|A\|_{p,q} \le \|A\|_p \tag{1}$$

Proof: First, suppose $p = \infty$ and $q = 1$. Then for all $x \in \mathbb{C}^n$ it follows that

$$\|Ax\|_\infty = \max_{i=1,\ldots,m} \left| \sum_{j=1}^{n} A_{(i,j)} x_j \right| \le \max_{i=1,\ldots,m} \sum_{j=1}^{n} |A_{(i,j)}| |x_j| \le \|A\|_\infty \|x\|_1$$

Next, suppose $1 \le p < \infty$. Then for all $x \in \mathbb{C}^n$ it follows from Hölder's inequality that

$$\|Ax\|_p = \left[\sum_{i=1}^{m} |\mathrm{row}_i(A)x|^p \right]^{1/p} \le \left[\sum_{i=1}^{m} \|\mathrm{row}_i(A)\|_p^p \|x\|_q^p \right]^{1/p}$$

$$= \left[\sum_{i=1}^{m} \sum_{j=1}^{n} |A_{(i,j)}|^p \right]^{1/p} \|x\|_q = \|A\|_p \|x\|_q$$

which implies that $\|A\|_{p,q} \le \|A\|_p$. □

Remark 1: In several special cases (1) is an equality. For example, if $p = \infty$ and $q = 1$ then $\|A\|_{\infty,1} = \|A\|_\infty$ for all $A \in \mathbb{C}^{m\times n}$ (Kahan 1966). Alternatively, if $p = 1$, $q = \infty$, and A has all nonnegative entries, then $\|A\|_1 = \|Ax\|_1 / \|x\|_\infty$, where $x = [1, 1, \ldots, 1]^{\mathrm{T}}$, and hence $\|A\|_{1,\infty} = \|A\|_1$.

Lemma 2: *Let $\|\cdot\|$ be a unitarily invariant matrix norm on $\mathbb{C}^{m\times n}$ and let $A \in \mathbb{C}^{m\times n}$. Then $\sigma_{\max}(A)\|E_{11}\| \le \|A\|$.*

Proof: If $A = 0$ the result is immediate. Next, let $A \ne 0$ and note that E_{11} is a rank-one matrix. Hence $\sigma_{\max}(\sigma_{\max}(A)E_{11}) = \sigma_{\max}(A)$ is the only non-zero singular value of $\sigma_{\max}(A)E_{11}$. The result is now a direct consequence of Theorem II.3.7 of Stewart and Sun (1990). □

Corollary 1 (Stewart and Sun 1990, p. 80): *Let $\|\cdot\|$ be a normalized untarily invariant matrix norm on $\mathbb{C}^{m\times n}$ and let $A \in \mathbb{C}^{m\times n}$. Then $\sigma_{\max}(A) \le \|A\|$.*

Lemma 3 (Stewart and Sun 1990, p. 68): *Let $\|\cdot\|, \|\cdot\|'$, and $\|\cdot\|''$ denote vector norms on $\mathbb{C}^m$. Let $\|\cdot\|''''$ be induced by $\|\cdot\|$ and $\|\cdot\|''$, let $\|\cdot\|'''''$ be induced by $\|\cdot\|'$ and $\|\cdot\|''$, and let $\|\cdot\|'''$ be induced by $\|\cdot\|$ and $\|\cdot\|'$. Then $(\|\cdot\|'''', \|\cdot\|''''', \|\cdot\|''')$ is a submultiplicative triple.*

Corollary 2: *Every equi-induced matrix norm is submultiplicative.*

Remark 2: If we set $\|\cdot\| = \|\cdot\|'' = \|\cdot\|_q$ and $\|\cdot\|' = \|\cdot\|_p$, where $1 \le p \le \infty$ and $1 \le q \le \infty$ in Lemma 3 then it follows from Lemma 3 that $(\|\cdot\|_{q,q}, \|\cdot\|_{q,p}, \|\cdot\|_{p,q})$ is a submultiplicative triple. Furthermore, if $(1/p) + (1/q) = 1$ it follows from Lemma 1 that

$$\|AB\|_{q,q} \le \|A\|_{q,p}\|B\|_{p,q} \le \|A\|_q\|B\|_p$$

which implies that $(\|\cdot\|_{q,q}, \|\cdot\|_q, \|\cdot\|_p)$ is a submultiplicative triple.

Lemma 4: *Let $\|\cdot\|$ be a matrix norm on $\mathbb{C}^{m\times m}$ and define the matrix norm $\|\cdot\|'$ on $\mathbb{C}^{m\times m}$ by $\|\cdot\|' \triangleq k\|\cdot\|$, where $k \ge \max\{\|AB\| : A, B \in \mathbb{C}^{m\times m}, \ \|A\| \le 1, \|B\| \le 1\}$. Then $\|\cdot\|'$ is submultiplicative on $\mathbb{C}^{m\times m}$.*

Proof: Since

$$\begin{aligned} k &\ge \max\{\|AB\| : A, B \in \mathbb{C}^{m\times m}, \|A\| \le 1, \|B\| \le 1\} \\ &= \max\left\{\frac{\|AB\|}{\|A\|\|B\|} : A, B \in \mathbb{C}^{m\times m}, A, B \ne 0\right\}, \end{aligned}$$

it follows that if $A, B \in \mathbb{C}^{m\times m}$, then $\|AB\| \le k\|A\|\|B\|$, and hence $k\|AB\| \le k^2\|A\|\|B\|$, which shows that $k\|\cdot\|$ is submultiplicative on $\mathbb{C}^{m\times m}$. □

Remark 3: Note that if $\|\cdot\|$ is submultiplicative on $\mathbb{C}^{m\times m}$ then $\max\{\|AB\| : A, B \in \mathbb{C}^{m\times m}, \ \|A\| \le 1, \ \|B\| \le 1\} \le 1$. Hence, $k\|\cdot\|$ is submultiplicative for all $k \ge 1$. Furthermore, it can be shown that $\max\{\|AB\|_\infty : A, B \in \mathbb{C}^{m\times m}, \ \|A\|_\infty \le 1, \ \|A\|_\infty \le 1, \ \|B\|_\infty \le 1\} = m$ and hence $m\|\cdot\|_\infty$ is submultiplicative on $\mathbb{C}^{m\times m}$.

Lemma 5 (Horn and Johnson 1985, p. 491): *Let $A, B \in \mathbb{R}^{m\times m}$. If $0 \le\le A \le\le B$, then $\rho(A) \le \rho(B)$.*

Lemma 6 (Ostrowski 1975): *Let $A \in \mathbb{C}^{m\times m}$ be partitioned such that $A = [A_{ij}], i, j = 1, \ldots, r$, where $A_{ij} \in \mathbb{C}^{m_i\times m_j}$ and $\sum_{i=1}^r m_i = m$, and let $\|\cdot\|$ be a submultiplicative matrix norm on $\mathbb{C}^{m_i\times m_j}$. Then $\rho(A) \le \rho([\|A_{ij}\|])$.*

The following well-known specialization of Lemma 2.6 is given by Stewart and Sun (1990).

Corollary 3: *Let $\|\cdot\|$ be a submultiplicative matrix norm on $\mathbb{C}^{m\times m}$. If $A \in \mathbb{C}^{m\times m}$ then $\rho(A) \le \|A\|$.*

3. Necessary and sufficient condition for robust stability

In this section, we give a generalization of the structured singular value (Packard and Doyle 1993) and provide necessary and sufficient conditions for robust stability. First, we consider a nominal square transfer function $G(s) \in \mathbb{C}^{m\times m}$ in a negative

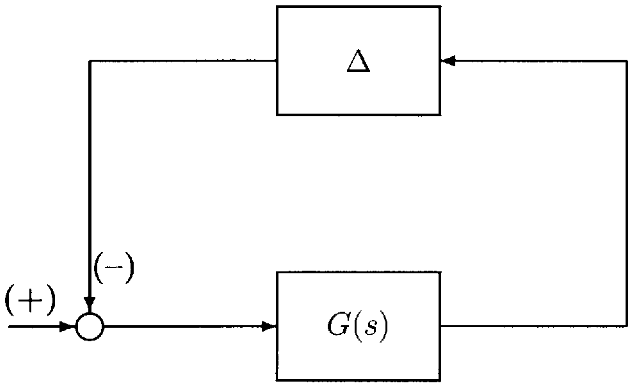

Figure 1. Interconnection of transfer function $G(s)$ with uncertain matrix Δ.

feedback interconnection with an uncertain complex square matrix $\Delta \in \mathbb{C}^{m\times m}$ as shown in figure 1. The matrix Δ belongs to the set $\boldsymbol{\Delta} \subseteq \mathbb{C}^{m\times m}$ of block-diagonal matrices defined by

$$\boldsymbol{\Delta} \triangleq \{\Delta \in \mathbb{C}^{m\times m} : \Delta = \text{block-diag}(I_{l_1} \otimes \Delta_1, I_{l_2} \otimes \Delta_2, \ldots, I_{l_{r+c}} \otimes \Delta_{r+c}),$$
$$\Delta_i \in \mathbb{R}^{m_i\times m_i}, i = 1, \ldots, r; \Delta_i \in \mathbb{C}^{m_i\times m_i}, i = r+1, \ldots, r+c\}$$

where the dimension m_i and the number of repetitions l_i of each block are given, and $r + c \geq 1$. We refer to the case $r = 0$ as *complex*, $r, c > 0$ as *mixed*, and $r = 0$, $c = 1$, $l_1 = 1$ as *single full complex block*.

Now, let $\|\cdot\|$ denote a matrix norm on $\mathbb{C}^{m\times m}$ and for $G \in \mathbb{C}^{m\times m}$ define the *structured matrix norm* $\upsilon(G)$ by

$$\upsilon(G) \triangleq \left(\min_{\Delta\in\boldsymbol{\Delta}} \{\|\Delta\| : \det[I + G\Delta] = 0\} \right)^{-1} \tag{2}$$

and if $\det[I + G\Delta] \neq 0$ for all $\Delta \in \boldsymbol{\Delta}$, then $\upsilon(G) \triangleq 0$. To show that 'min' in (2) is attained let $\beta > 0$ and define the closed set $\mathcal{X}_\beta \triangleq \{\Delta \in \boldsymbol{\Delta} : \|\Delta\| \leq \beta, \det[I + G\Delta] = 0\}$. Note that if, for all $\beta > 0$, $\mathcal{X}_\beta$ is empty then, by definition, $\upsilon(G) = 0$. Alternatively, if $\mathcal{X}_\beta$ is non-empty then if follows that $\mathcal{X}_\beta$ is compact. Hence it follows from the continuity of $\|\cdot\|$ that the $\min\{\|\Delta\| : \Delta \in \mathcal{X}_\beta\}$ exists which implies that $\upsilon(G)$ is well defined. Furthermore, for $\gamma > 0$, define the set of norm-bounded, block-diagonal uncertain matrices $\boldsymbol{\Delta}_\gamma$ by

$$\boldsymbol{\Delta}_\gamma \triangleq \{\Delta \in \boldsymbol{\Delta} : \|\Delta\| \leq \gamma^{-1}\}$$

Henceforth throughout the paper the notation $\|\cdot\|$ denotes the matrix norm appearing in the definitions of $\upsilon(G)$ and $\boldsymbol{\Delta}_\gamma$.

Next we present a necessary and sufficient condition for robust stability of the feedback interconnection of $G(s)$ and Δ for all $\Delta \in \boldsymbol{\Delta}_\gamma$. We assume that the feedback interconnection of $G(s)$ and Δ is well posed (Zhou *et al.* 1996, p. 119), that is, $\det[I + G(\infty)\Delta] \neq 0$ for all $\Delta \in \boldsymbol{\Delta}_\gamma$.

Theorem 1: *Let $\gamma > 0$ and suppose $G(s)$ is asymptotically stable. Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$ if and only if $\upsilon(G(j\omega)) < \gamma$ for all $\omega \in \mathbb{R}$.*

Proof: Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

where A is Hurwitz, and suppose the negative feedback interconnection of $G(s)$ and Δ given by

$$(I + G(s)\Delta)^{-1}G(s) \sim \left[\begin{array}{c|c} A - B\Delta(I + D\Delta)^{-1}C & B - B\Delta(I + D\Delta)^{-1}D \\ \hline (I + D\Delta)^{-1}C & (I + D\Delta)^{-1}D \end{array}\right]$$

is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$. Next, note that, for all $\Delta \in \mathbf{\Delta}_\gamma$ and $\omega \in \mathbb{R}$,

$$\begin{aligned} \det\left[I + G(j\omega)\Delta\right] &= \det\left[I + (C(j\omega I - A)^{-1}B + D)\Delta\right] \\ &= \det(I + D\Delta)\det\left[I + (j\omega I - A)^{-1}B\Delta(I + D\Delta)^{-1}C\right] \\ &= \det(I + D\Delta)\det(j\omega I - A)^{-1}\det\left[j\omega I - (A - B\Delta(I + D\Delta)^{-1}C)\right] \\ &\neq 0 \end{aligned}$$

Hence, $\min_{\Delta\in\mathbf{\Delta}}\{\|\Delta\| : \det\left[I + G(j\omega)\Delta\right] = 0\} > 1/\gamma$ for all $\omega \in \mathbb{R}$ which implies that $\upsilon(G(j\omega)) < \gamma$ for all $\omega \in \mathbb{R}$.

Conversely, suppose $\upsilon(G(j\omega)) < \gamma$ for all $\omega \in \mathbb{R}$ and assume that

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is minimal. Then, by assumption, $\det\left[I + G(\infty)\Delta\right] = \det\left[I + D\Delta\right] \neq 0$ for all $\Delta \in \mathbf{\Delta}_\gamma$. Now, suppose there exists $\Delta \in \mathbf{\Delta}_\gamma$ such that $(I + G(s)\Delta)^{-1}G(s)$ is not asymptotically stable and hence $A - B\Delta(I + D\Delta)^{-1}C$ is not Hurwitz. Since $G(s)$ is assumed to be asymptotically stable it follows that A is Hurwitz and thus there exists $\varepsilon \in (0, 1)$ such that $A - \varepsilon B\Delta(I + \varepsilon D\Delta)^{-1}C$ has an imaginary eigenvalue $j\hat{\omega}$. Hence,

$$\begin{aligned} \det\left[I + \varepsilon G(j\hat{\omega})\Delta\right] &= \det(I + \varepsilon D\Delta)\det(j\hat{\omega} I - A)^{-1}\det\left[j\hat{\omega} I - (A - \varepsilon B\Delta(I + \varepsilon D\Delta)^{-1}C)\right] \\ &= 0 \end{aligned}$$

However, since $\varepsilon\Delta \in \mathbf{\Delta}_\gamma$ and $\upsilon(G(j\omega)) < \gamma$ or, equivalently, $\min_{\Delta\in\mathbf{\Delta}}\{\|\Delta\| : \det\left[I + G\Delta\right] = 0\} > 1/\gamma$ for all $\omega \in \mathbb{R}$, it follows that $\det\left[I + \varepsilon G(j\hat{\omega})\Delta\right] \neq 0$, which is a contradiction. □

Remark 4: If $r = 0$ and $\|\cdot\|$ is either a Hölder norm (p-norm) or a normalized unitarily invariant norm then, using a similar construction given in Theorem 11.8 of Zhou *et al.* (1996), Theorem 1 can be extended to the case in which Δ is a real rational stable matrix transfer function. Extensions to more general norms is a subject of current research.

Finally, the following proposition provides an ordering between different structured matrix norms

Proposition 1: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|'$ and $\|\cdot\|''$ denote matrix norms on $\mathbb{C}^{m\times m}$. Assume that there exists $\Delta \in \boldsymbol{\Delta}$ such that* $\det[I + G\Delta] = 0$ *and let $k_1, k_2 > 0$ satisfy*

$$k_1\|\Delta\|' \leq \|\Delta\|'' < k_2\|\Delta\|' \tag{3}$$

for all $\Delta \in \boldsymbol{\Delta}$ such that $\det[I + G\Delta] = 0$. *Furthermore, let $\upsilon'(G)$ and $\upsilon''(G)$ denote the structured matrix norms with defining norms $\|\cdot\|'$ and $\|\cdot\|''$, respectively. Then*

$$k_1\upsilon''(G) \leq \upsilon'(G) \leq k_2\upsilon''(G) \tag{4}$$

Proof: The existence of k_1 and k_2 satisfying (3) follows from the equivalence of matrix norms (Stewart and Sun 1990, p. 65). Now, it follows from (3) that

$$k_1 \min_{\Delta\in\chi}\|\Delta\|' \leq \min_{\Delta\in\chi}\|\Delta\|'' \leq k_2 \min_{\Delta\in\chi}\|\Delta\|'$$

where $\chi \triangleq \{\Delta \in \boldsymbol{\Delta}: \det[I + G\Delta] = 0\}$, which implies (4). □

Remark 5: Proposition 1 can be used to construct upper bounds for structured matrix norms in terms of alternative structured matrix norms.

The results of Theorem 1 cannot be obtained from the standard small-μ theorem by using the equivalence of matrix norms, that is, the fact that for an arbitrary pair of matrix norms $\|\cdot\|, \|\cdot\|'$ on $\mathbb{C}^{m\times n}$ such that $\|\cdot\| \neq \|\cdot\|'$ there exist $k_1, k_2 > 0$ such that $k_1\|A\| \leq \|A\|' \leq k_2\|A\|$ for all $A \in \mathbb{C}^{m\times n}$ (Stewart and Sun 1990). (Henceforth we assume that k_1, k_2 such that equality is achieved for some $A \in \mathbb{C}^{m\times n}$.) Specifically, using the necessary and sufficient conditions of the standard small-μ theorem for robust stability we can obtain sufficient but *not* necessary conditions for robust stability for the same system with uncertainty bounded by an arbitrary matrix norm $\|\cdot\| \neq \sigma_{\max}(\cdot)$. To see this, let $\|\cdot\|$ be an equi-induced Hölder norm on $\mathbb{C}^{m\times m}$ such that $\|\cdot\| \neq \sigma_{\max}(\cdot)$. In this case there exist $k_1, k_2 > 0$ such that $k_1 < 1 < k_2$ and $k_1\|A\| \leq \sigma_{\max}(A) \leq k_2\|A\|$ for all $A \in \mathbb{C}^{m\times m}$. Now for $\gamma > 0$ it can be shown that $\boldsymbol{\Delta}_{k_2\gamma} \subseteq \{\Delta \in \boldsymbol{\Delta}: \sigma_{\max}(\Delta) \leq \gamma^{-1}\} \subseteq \boldsymbol{\Delta}_{k_1\gamma}$. Next, let G be such that $k_1\mu(G) = \upsilon(G)$ where $\mu(G)$ denotes the structured singular values. Then it follows from the standard small-μ theorem that $\det[I + G\Delta] \neq 0$ for all $\Delta \in \{\Delta \in \boldsymbol{\Delta}: \sigma_{\max}(\Delta) < \gamma^{-1}\}$ which implies that $\det[I + G\Delta] \neq 0$ for all $\Delta \in \boldsymbol{\Delta}_{k_2\gamma}$. Furthermore, it follows from Theorem 1 that $\det[I + G\Delta] \neq 0$ for all $\Delta \in \boldsymbol{\Delta}_{k_1\gamma}$. Hence, since $\boldsymbol{\Delta}_{k_2\gamma} \subseteq \boldsymbol{\Delta}_{k_1\gamma}$ the robust stability predictions via the equivalence of matrix norms may be conservative.

As an illustration of the above discussion let $\beta > 0$, consider the constant matrix

$$G = \beta^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and let $\gamma > 0$ be the maximum allowable uncertainty such that $\det(I + G\Delta) \neq 0$ for all $\Delta \in \{\Delta \in \mathbb{C}^{2\times 2}: \|\Delta\|_\infty < \gamma\}$, where $\|\Delta\|_\infty \triangleq \max_{i,j=1,2}|\Delta_{(i,j)}|$. Since $\sigma_{\max}(G) = \beta^{-1}$, it follows from the small-μ theorem or, equivalently, in this case the small gain theorem, that $\det(I + G\Delta) \neq 0$ for all $\Delta \in \{\Delta \in \mathbb{C}^{2\times 2}: \sigma_{\max}(\Delta) < \beta\}$. Using the equivalence of matrix norms we have $\|\Delta\|_\infty \leq \sigma_{\max}(\Delta) \leq 2\|\Delta\|_\infty$. Hence, the largest value of γ that can be guaranteed by the small-μ theorem to satisfy $\det(I + G\Delta) \neq 0$ for all $\Delta \in \{\Delta \in \mathbb{C}^{2\times 2}: \|\Delta\|_\infty < \gamma\}$ is $\beta/2$. However, a direct computation yields that $\det(I + G\Delta) \neq 0$ for all

$\Delta \in \{\Delta \in \mathbb{C}^{2\times 2} : \|\Delta\|_\infty < \beta\}$. Thus the robust stability prediction of the structured singular value is conservative by a factor of two.

Although, as shown by Proposition 1, one can use standard mixed-μ upper bounds (Fan *et al.* 1991, Haddad *et al.* 1996) to compute upper bounds for structured matrix norms involving mixed real and complex uncertainty, as noted above, these upper bounds may be conservative. In the next section we construct alternative lower and upper bounds for structured matrix norms.

4. Lower and upper bounds for the structured matrix norm

In this section we provide lower and upper bounds for the structured matrix norm.

4.1. *A lower bound for the structured matrix norm*

In order to develop a lower bound for $\upsilon(G)$ define the *real spectral radius* $\rho_{\mathrm{R}}(G)$ by (Young *et al.* 1991)

$$\rho_{\mathrm{R}}(G) \triangleq \begin{cases} \max\{|\lambda| : \lambda \in \operatorname{spec}(G) \cap \mathbb{R}\}, & \text{if } \operatorname{spec}(G) \cap \mathbb{R} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Theorem 2: *Let $G \in \mathbb{C}^{m\times m}$. Then*

$$\frac{\rho_{\mathrm{R}}(G)}{\|I\|} \leq \upsilon(G) \tag{5}$$

Furthermore, if $r = 0$, then

$$\frac{\rho_{\mathrm{R}}(G)}{\|I\|} \leq \frac{\rho(G)}{\|I\|} \leq \upsilon(G) \tag{6}$$

Proof: The result (5) is immediate if $\rho_{\mathrm{R}}(G) = 0$. Now, suppose $\rho_{\mathrm{R}}(G) > 0$. In this case, it follows that either $\rho_{\mathrm{R}}(G)$ or $-\rho_{\mathrm{R}}(G)$ is an eigenvalue of G. Hence either $\det\left[I + \rho_{\mathrm{R}}^{-1}(G)G\right] = 0$ or $\det\left[I - \rho_{\mathrm{R}}^{-1}(G)G\right] = 0$. Next, since $\rho_{\mathrm{R}}^{-1}(G)I \in \boldsymbol{\Delta}$, it follows that

$$\min_{\Delta\in\boldsymbol{\Delta}}\{\|\Delta\| : \det[I + G\Delta] = 0\} \leq \frac{\|I\|}{\rho_{\mathrm{R}}(G)}$$

which implies (5). Next, (6) is immediate if $\rho(G) = 0$. Now, suppose $\rho(G) > 0$ and let $\lambda \in \operatorname{spec}(G)$ be such that $|\lambda| = \rho(G)$. Then $\det\left[I - \lambda^{-1}G\right] = 0$ and, since $r = 0$, $\lambda^{-1}I \in \boldsymbol{\Delta}$. Now it follows that

$$\min_{\Delta\in\boldsymbol{\Delta}}\{\|\Delta\| : \det[I + G\Delta] = 0\} \leq \frac{\|I\|}{\rho(G)}$$

which implies (6). □

The following corollaries are immediate.

Corollary 4: *Let $G \in \mathbb{C}^{m\times m}$ and suppose $\|I\| = 1$. Then $\rho_{\mathrm{R}}(G) \leq \upsilon(G)$. Furthermore, if $r = 0$ then $\rho(G) \leq \upsilon(G)$.*

Remark 6: Note that if $\|\cdot\|$ is an equi-induced matrix norm then $\|I\| = 1$. However, $\|I\|_\infty = 1$ although $\|\cdot\|_\infty$ is not submultiplicative and thus is not equi-induced.

Corollary 5 (Young *et al.* 1991): *Let $G \in \mathbb{C}^{m\times m}$ and suppose $\|\cdot\| = \sigma_{\max}(\cdot)$. Then $\rho_{\mathrm{R}}(G) \le \mu(G)$. Furthermore, if $r = 0$ then $\rho(G) \le \mu(G)$.*

4.2. *Upper bounds for the structured matrix norm*

In this subsection we provide several upper bounds for $\upsilon(G)$. First, we present sufficient conditions (which are also necessary if $r = 0$) for a constant $\gamma \ge 0$ to provide an upper bound for the structured matrix norm. The following lemmas are needed.

Lemma 7: *Let $G \in \mathbb{C}^{m\times m}$ and $\gamma \ge 0$. If $\rho(G\Delta) \le \gamma\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ then $\upsilon(G) \le \gamma$.*

Proof: Note that if $\gamma = 0$ then $\rho(G\Delta) = 0$ for all $\Delta \in \mathbf{\Delta}$ which implies that $\det[I + G\Delta] \ne 0$ for all $\Delta \in \mathbf{\Delta}$ and hence $\upsilon(G) = 0$. Next, assume $\gamma > 0$. Suppose $\rho(\gamma\Delta) \le \gamma\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ and assume $\gamma < \upsilon(G)$. Then it follows from the definition of $\upsilon(G)$ that there exists $\hat{\Delta} \in \mathbf{\Delta}$ such that $\|\hat{\Delta}\| = \beta^{-1}$ and $\det[I + G\hat{\Delta}] = 0$ where $\beta \triangleq \upsilon(G)$. Since $\rho(G\Delta) \le \gamma\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ it follows that $\rho(G\hat{\Delta}) \le \gamma/\beta < 1$ and hence $\det[I + G\hat{\Delta}] \ne 0$ which is a contradiction. □

Lemma 8: *Let $G \in \mathbb{C}^{m\times m}$, $\gamma \ge 0$, and assume $r = 0$. Then $\rho(G\Delta) \le \gamma\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ if and only if $\upsilon(G) \le \gamma$. Furthermore, $\upsilon(G) = \max_{\Delta\in\mathbf{\Delta}_1}\rho(G\Delta)$.*

Proof: It follows from Lemma 7 that if $\rho(G\Delta) \le \gamma\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ then $\upsilon(G) \le \gamma$. Conversely, assume $\upsilon(G) \le \gamma$ and suppose there exists $\hat{\Delta} \in \mathbf{\Delta}$ such that $\rho(G\hat{\Delta}) > \upsilon(G)\|\hat{\Delta}\|$. If $\upsilon(G) = 0$ then $\det[I + G\Delta] \ne 0$ for all $\Delta \in \mathbf{\Delta}$. However, since $\rho(G\hat{\Delta}) > \upsilon(G)\|\hat{\Delta}\| = 0$ there exists $-\lambda \in \operatorname{spec}(G\hat{\Delta})$ such that $|\lambda| = \rho(G\hat{\Delta})$. Hence, since $r = 0$, $\tilde{\Delta} = \lambda\hat{\Delta} \in \mathbf{\Delta}$ and $\det[I + G\tilde{\Delta}] = 0$ which is a contradiction. Next, assume $\upsilon(G) > 0$ and let $-\lambda \in \operatorname{spec}(G\hat{\Delta})$ such that $|\lambda| = \rho(G\hat{\Delta})$. In this case $\det[I + \lambda^{-1}G\hat{\Delta}] = 0$ and, since $r = 0$, $\lambda^{-1}\hat{\Delta} \in \mathbf{\Delta}$ and $\|\lambda^{-1}\hat{\Delta}\| = \rho^{-1}(G\hat{\Delta})\|\hat{\Delta}\| < 1/\upsilon(G)$. Now, it follows by definition that $\det[I + G\Delta] \ne 0$ for all $\Delta \in \mathbf{\Delta}$ such that $\|\Delta\| < 1/\upsilon(G)$ which is a contradiction. Hence, $\rho(G\Delta) \le \upsilon(G)\|\Delta\| \le \gamma\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$. Next, note that $\rho(G\Delta) \le \upsilon(G)\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ if and only if $\max_{\Delta\in\mathbf{\Delta}_1}\rho(G\Delta) \le \upsilon(G)$. Now suppose $\max_{\Delta\in\mathbf{\Delta}_1}\rho(G\Delta) < \upsilon(G)$ and let η satisfy $\max_{\Delta\in\mathbf{\Delta}_1}\rho(G\Delta) < \eta < \upsilon(G)$. In this case $\rho(G\Delta) < \eta\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ and hence it follows from Lemma 7 that $\upsilon(G) \le \eta$ which is a contradiction. Hence, $\max_{\Delta\in\mathbf{\Delta}_1}\rho(G\Delta) = \upsilon(G)$. □

The following theorem uses Lemma 7 to construct upper bounds for the structured matrix norm in terms of a function $\phi(\cdot)$. In order to account for the structure of $\mathbf{\Delta}$ we introduce the set of scaling matrices $\mathcal{D}$ defined by

$$\mathcal{D} \triangleq \{D \in \mathbb{C}^{m\times m} : \det D \ne 0 \text{ and } D\Delta = \Delta D, \text{ for all } \Delta \in \mathbf{\Delta}\} \tag{7}$$

Now the following theorem is immediate.

Theorem 3: *Let $G \in \mathbb{C}^{m\times m}$ and let $\phi : \mathbb{C}^{m\times m} \to \mathbb{R}$ be such that $\rho(A\Delta) \le \phi(A)\|\Delta\|$ for all $\Delta \in \mathbf{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. Then*

$$\upsilon(G) \le \inf_{D\in\mathcal{D}} \phi(DGD^{-1}) \tag{8}$$

Proof: Note that $\upsilon(G) = \upsilon(DGD)^{-1}$ for all $D \in \mathscr{D}$. Hence, since $\rho(DGD^{-1}\Delta) \leq \phi(DGD^{-1})\|\Delta\|$ for all $\Delta \in \boldsymbol{\Delta}$ and $D \in \mathscr{D}$ it follows from Lemma 7 that $\upsilon(G) \leq \phi(GDG^{-1})$ which implies (8). □

An immediate application of Lemma 7 is the following result involving G that commute with uncertainties Δ.

Corollary 6: *Let $G \in \mathbb{C}^{m\times m}$, let $\|\cdot\|$ be a submultiplicative matrix norm such that $\|I\| = 1$, and assume $G\Delta = \Delta G$ for all $\Delta \in \boldsymbol{\Delta}$. Then*

$$\rho_{\mathrm{R}}(G) \leq \upsilon(G) \leq \rho(G) \tag{9}$$

Furthermore, if $r = 0$, then

$$\upsilon(G) = \rho(G) \tag{10}$$

Proof: Since $G\Delta = \Delta G$ for all $\Delta \in \boldsymbol{\Delta}$, it follows that $\rho(G\Delta) \leq \rho(G)\rho(\Delta) \leq \rho(G)\|\Delta\|$ for all $\Delta \in \boldsymbol{\Delta}$. Then it follows from Lemma 7 that $\upsilon(G) \leq \rho(G)$. Finally, (9) and (10) follow from (5) and (6). □

The next result uses Theorem 3 to construct an upper bound for the structured matrix norm.

Corollary 7: *Let $G \in \mathbb{C}^{m\times m}$ and $k \geq \max\{\|AB\| : A, B \in \mathbb{C}^{m\times m}, \ \|A\| \leq 1, \|B\| \leq 1\}$. Then*

$$\upsilon(G) \leq k^2 \inf_{D\in\mathscr{D}} \|DGD^{-1}\| \tag{11}$$

Proof: It follows from Lemma 4 that the matrix norm $k\|\cdot\|$ is submultiplicative. Next, using Corollary 3 we obtain

$$\rho(A\Delta) \leq k\|A\Delta\| \leq k^2\|A\|\|\Delta\|$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. The result is now a direct consequence of Theorem 3 with $\phi(A) = k^2\|A\|$ for all $A \in \mathbb{C}^{m\times m}$. □

Note that if $\|\cdot\|$ is submultiplicative then the assumptions of Corollary 7 are satisfied with $k = 1$ and hence the following result is immediate.

Corollary 8: *Let $G \in \mathbb{C}^{m\times m}$ and assume $\|\cdot\|$ is submultiplicative. Then $\upsilon(G) \leq \inf_{D\in\mathscr{D}}\|DGD^{-1}\|$.*

Finally we provide upper bounds for the structured matrix norm for the case in which $\|\cdot\|$ is an induced matrix norm on $\mathbb{C}^{m\times m}$.

Corollary 9: *Let $G \in \mathbb{C}^{m\times m}$ and assume $\|\cdot\|$ is induced by vector norms $\|\cdot\|'$ and $\|\cdot\|''$. Then $\upsilon(G) \leq \inf_{D\in\mathscr{D}}\|DGD^{-1}\|'''$, where $\|\cdot\|'''$ is induced by $\|\cdot\|''$ and $\|\cdot\|'$.*

Proof: Let $\|\cdot\|''''$ be equi-induced by $\|\cdot\|'$. Then it follows from Corollary 1 that $(\|\cdot\|'''', \|\cdot\|''', \|\cdot\|)$ is a submultiplicative triple. Hence,

$$\rho(A\Delta) \leq \|A\Delta\|'''' \leq \|A\|'''\|\Delta\|$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. Now the result is a direct consequence of Theorem 3 with $\phi(A) = \|A\|'''$ for all $A \in \mathbb{C}^{m\times m}$. □

5. Specializations to Hölder, unitarily invariant, and induced matrix norms

In this section we specialize the results of §4 to the cases in which $\|\cdot\|$ represents Hölder norms (p-norms), unitarily invariant norms, and induced norms. First we consider the case in which $\|\cdot\|$ is a p-norm.

Proposition 2: *Let $G \in \mathbb{C}^{m\times m}$, and let $1 \le p \le \infty$ and $1 \le q \le \infty$ be such that $1/p + 1/q = 1$. If $\|\cdot\| = \|\cdot\|_p$ then*

$$\frac{\rho_{\mathrm{R}}(G)}{m^{1/p}} \le \upsilon(G) \le \inf_{D\in\mathcal{D}} \|DGD^{-1}\|_q \tag{12}$$

Proof: The lower bound is a direct consequence of Theorem 2 and the fact that $\|I\|_p = m^{1/p}$. Next, it follows from Lemmas 1, 3, and Corollaries 2, 3 that

$$\rho(A\Delta) \le \|A\Delta\|_{q,q} \le \|A\|_{q,p}\|\Delta\|_{p,q} \le \|A\|_q\|\Delta\|_p$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. The result now follows immediately from Theorem 3 with $\phi(A) = \|A\|_q$ for all $A \in \mathbb{C}^{m\times m}$. □

Next, we consider the case in which $\|\cdot\|$ is unitarily invariant matrix norm.

Proposition 3: *Let $G \in \mathbb{C}^{m\times m}$ and assume $\|\cdot\|$ is a unitarily invariant matrix norm on $\mathbb{C}^{m\times m}$. Then*

$$\frac{\rho_{\mathrm{R}}(G)}{\|I\|} \le \upsilon(G) \le \frac{1}{\|E_{11}\|} \inf_{D\in\mathcal{D}} \sigma_{\max}(DGD^{-1}) \tag{13}$$

Proof: The lower bound is a restatement of Theorem 2. Next, it follows from Lemma 2 that

$$\rho(A\Delta) \le \sigma_{\max}(A)\sigma_{\max}(\Delta) \le \frac{1}{\|E_{11}\|}\sigma_{\max}(A)\|\Delta\|$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. The result is now a direct consequence of Theorem 3 with $\phi(A) = \sigma_{\max}(A)/\|E_{11}\|$ for all $A \in \mathbb{C}^{m\times m}$. □

The following corollary is a direct consequence of Proposition 3 by noting that $\|E_{11}\| = 1$ for all normalized unitarily invariant matrix norms.

Corollary 10: *Let $G \in \mathbb{C}^{m\times m}$ and assume $\|\cdot\|$ is a normalized unitarily invariant matrix norm on $\mathbb{C}^{m\times m}$. Then*

$$\frac{\rho_{\mathrm{R}}(G)}{\|I\|} \le \upsilon(G) \le \inf_{D\in\mathcal{D}} \sigma_{\max}(DGD^{-1}) \tag{14}$$

For $\|\cdot\| = \|\cdot\|_{\sigma p}$, where $1 \le p \le \infty$, the following corollary is an immediate consequence of Corollary 10 since $\|I\|_{\sigma p} = m^{1/p}$ and $\|\cdot\|_{\sigma p}$ is a normalized unitarily invariant matrix norm.

Corollary 11: *Let $G \in \mathbb{C}^{m\times m}$, $1 \le p \le \infty$, and assume $\|\cdot\| = \|\cdot\|_{\sigma p}$. Then*

$$\frac{\rho_{\mathrm{R}}(G)}{m^{1/p}} \le \upsilon(G) \le \inf_{D\in\mathcal{D}} \sigma_{\max}(DGD^{-1}) \tag{15}$$

Furthermore, if $\|\cdot\| = \|\cdot\|_{\sigma\infty} = \sigma_{\max}(\cdot)$ then

$$\rho_{\mathrm{R}}(G) \le \mu(G) \le \inf_{D\in\mathcal{D}} \sigma_{\max}(DGD^{-1}) \tag{16}$$

Finally, we consider two special cases in which $\|\cdot\|$ is an induced norm.

Proposition 4: *Let $G \in \mathbb{C}^{m\times m}$. If $\|\cdot\| = \|\cdot\|_{1,1}$ then*

$$\rho_{\mathrm{R}}(G) \le \upsilon(G) \le \inf_{D\in\mathcal{D}} \|DGD^{-1}\|_{1,1} \tag{17}$$

Furthermore, if $\|\cdot\| = \|\cdot\|_{\infty,\infty}$ then

$$\rho_{\mathrm{R}}(G) \le \upsilon(G) \le \inf_{D\in\mathcal{D}} \|DGD^{-1}\|_{\infty,\infty} \tag{18}$$

Proof: The lower bounds are a direct consequence of Theorem 2. Next, it follows from Corollary 2 that $\|\cdot\|_{1,1}$ and $\|\cdot\|_{\infty,\infty}$ are submultiplicative on $\mathbb{C}^{m\times m}$. The result now follows from Corollary 8. □

Remark 7: Note that the cases in which $\|\cdot\| = \|\cdot\|_{\infty,1} = \|\cdot\|_{\infty}$ and $\|\cdot\| = \|\cdot\|_{2,2} = \sigma_{\max}(\cdot)$ correspond to particular p-norms and σp-norms already discussed in the previous subsections.

6. Extensions to block-norm uncertainty characterization with mixed spatial norms

In this section we specialize the structured matrix norm to the case in which the uncertainty is characterized by mixed spatial norms which allows the size of the uncertain blocks to be characterized by different spatial norms. Now let $\|\cdot\|$ be given by

$$\|A\| = \|[\|A_{ij}\|_{i,j}]\|_{\infty} \tag{19}$$

for all $A \in \mathbb{C}^{m\times m}$ where A is partitioned as $A = [A_{ij}]$, $i,j = 1,\ldots,c$, $A_{ij} \in \mathbb{C}^{m_i\times m_j}$, $\sum_{i=1}^{c} m_i = m$ where $\|\cdot\|_{(i,j)}$ is a specified matrix norm on $\mathbb{C}^{m_i\times m_j}$. Next, let $r = 0$, and let $l_i = 1$, $i = 1\ldots,c$, so that

$$\mathbf{\Delta} = \{\Delta \in \mathbb{C}^{m\times m} : \Delta = \text{block-diag}(\Delta_1,\ldots,\Delta_c), \Delta_i \in \mathbb{C}^{m_i\times m_j}, i = 1,\ldots,c\} \tag{20}$$

where the dimension m_i of each block is given such that $\sum_{i=1}^{n} m_i = m$. We assume that G is conformally partitioned with the elements of $\mathbf{\Delta}$ as $[G_{ij}]$, where $G_{ij} \in \mathbb{C}^{m_i\times m_j}, i,j = 1,\ldots,c$. Now the structured matrix norm $\upsilon(G)$ defined with respect to the norm given by (19) and uncertainty set $\mathbf{\Delta}$ can be written as

$$\upsilon(G) = \left(\min_{\Delta\in\mathbf{\Delta}} \left\{ \max_{i=1,\ldots,c} \|\Delta_i\|_{(i)} : \det[I + G\Delta] = 0 \right\} \right)^{-1}, \tag{21}$$

where $\|\cdot\|_{(i)}$ is a given matrix norm on $\mathbb{C}^{m_i\times m_i}$ for $i = 1,\ldots,c$, and if there does not exist $\Delta \in \mathbf{\Delta}$ such that $\det[I + G\Delta] = 0$ then $\upsilon(G) = 0$. Furthermore, in this case $\mathbf{\Delta}_\gamma$ can be equivalently characterized by

$$\mathbf{\Delta}_\gamma = \{\Delta \in \mathbf{\Delta} : \|\Delta_i\|_{(i)} \le \gamma^{-1}, i = 1,\ldots,c\} \tag{22}$$

Note that Theorems 1, 2, 3, and Corollary 7 hold for the block-norm uncertainty characterization given by (22). This uncertainty characterization allows for different spatial norms in capturing the size of the respective uncertainty blocks.

Next we consider several cases of the above uncertainty characterization and develop upper bounds for the structured matrix norm $\upsilon(G)$. As in §3 we can introduce scaling matrices to account for structure of the uncertainty and hence reduce conservatism. However, in order to facilitate the presentation we shall not do so in this section. First, we consider the case in which $\|\cdot\|_{(i)} = \|\cdot\|'$, $i = 1,\ldots,c$.

Proposition 5: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|_{(i)} = \|\cdot\|'$, $i = 1,\ldots,c$. Suppose there exists a submultiplicative triple of matrix norms $(\|\cdot\|''',\|\cdot\|'',\|\cdot\|')$ such that $\|\cdot\|'''$ is submultiplicative. Then $\upsilon(G) \le \rho([\|G_{ij}\|''])$.*

Proof: It follows from Lemma 6 that

$$\rho(A\Delta) \le \rho([\|(A\Delta)_{ij}\|''']) \tag{23}$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. Next since by definition $\|\Delta_i\|' \le \|\Delta\|$ note that

$$[\|(A\Delta)_{ij}\|'''] \le\le [\|A_{ij}\|''][\|\Delta_i\|'] \le\le [\|A_{ij}\|'']\|\Delta\|$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. Hence, it follows from Lemma 5 that

$$\rho([\|(A\Delta)_{ij}\|''']) \le \rho([\|A_{ij}\|''])\|\Delta\| \tag{24}$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. The result now follows from (23), (24), and Theorem 3 with $\phi(A) = \rho([\|A_{ij}\|''])$ for all $A \in \mathbb{C}^{m\times m}$. □

Next we specialize Proposition 5 to the case in which $\|\cdot\|_{(i)} = \|\cdot\|'$, $i = 1,\ldots,c$, is submultiplicative.

Corollary 12: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|_{(i)} = \|\cdot\|'$, $i = 1,\ldots,c$ be a submultiplicative matrix norm. Then*

$$\upsilon(G) \le \rho([\|G_{ij}\|']) \tag{25}$$

Remark 8: Note that if $\|\cdot\|' = \sigma_{\max}(\cdot)$ then $\|\Delta\| = \max_i \sigma_{\max}(\Delta_i) = \sigma_{\max}(\Delta)$, $i = 1,\ldots,c$, and hence $\upsilon(G)$ becomes the structured singular value. In this case the upper bound given by (25) specializes to the upper bound given by Corollary 4.3 of Hyland and Collins (1989) and Equation (15) of Safonov (1982) for the case of diagonal uncertainty. If, alternatively, $\|\cdot\|' = \|\cdot\|_{\infty,\infty}$ then $\|\Delta\| = \max_i \|\Delta\|_{\infty,\infty} = \|\Delta\|_{\infty,\infty}$, $i = 1,\ldots,c$, so that Corollary 12 specializes to the results of Khammash and Pearson (1993) for the case where $\Delta_i \in \mathbb{C}^{m_i\times m_i}$, $i = 1,\ldots,c$.

Remark 9: In order to connect the robust stability bounds for structured uncertainty involving structured matrix norms and the robust stability bounds given by Hyland and Collins (1989) via majorant analysis consider the block-structured uncertainty characterized by majorant bounds given by

$$\boldsymbol{\Delta}_\gamma = \{\Delta \in \mathbb{C}^{m\times m} : [\|\Delta_{ij}\|'] \le\le \gamma^{-1}M\},$$

where $\|\cdot\|'$ is a given submultiplicative norm on $\mathbb{C}^{m_i\times m_j}$ and $M \in \mathbb{R}^{c\times c}$, $M \gg 0$. Now, note that $\boldsymbol{\Delta}_\gamma$ can be equivalently written as

$$\{\Delta \in \mathbb{C}^{m\times m} : \|[\|\Delta_{ij}\|'] \circ M^{\mathrm{HI}}\|_\infty \le \gamma^{-1}\}$$

where M^{HI} denotes the Hadamard inverse of M and $[\|\Delta_{ij}\|'] \circ M^{\mathrm{HI}}$ denotes the Hadamard product of $[\|\Delta_{ij}\|']$ and M^{HI}. Next using Lemmas 5 and 6 it follows that

$$\rho(G(j\omega)\Delta) \le \rho([\|G_{ij}(j\omega)\|'][\|\Delta_{ij}\|']) \le \rho([\|G_{ij}(j\omega)\|']M)\|[\|\Delta_{ij}\|'] \circ M^{\mathrm{HI}}\|_\infty$$

for all $\Delta \in \mathbb{C}^{m\times m}$ and hence Lemma 7 yields $\upsilon(G(j\omega)) \le \rho(\lceil \|G_{ij}(j\omega)\| \rceil M)$, where $\upsilon(G(j\omega))$ denotes the structured matrix norm with the defining norm $\|\Delta\| = \|\lceil\|\Delta_{ij}\|\rceil \circ M^{\mathrm{HI}}\|_\infty$. Furthermore, it follows from Theorem 1 that if $\rho(\lceil\|G_{ij}(j\omega)\|\rceil M) < \gamma$ for all $G \in \mathbb{C}^{m\times m}$ then the feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}_\gamma$ which yields Theorem 4.1 of Hyland and Collins (1989) with $\|\cdot\|' = \sigma_{\max}(\cdot)$ and $\gamma = 1$.

Next we let $\|\cdot\|_{(i)} = \|\cdot\|_{q_i,p}$, where $p \ge 1$ and $q_i \ge 1$, $i = 1,\ldots,c$.

Proposition 6: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|_{(i)} = \|\cdot\|_{q_i,p}$, where $p \ge 1$ and $q_i \ge 1$, $i = 1,\ldots,c$. Then $\upsilon(G) \le \rho(\lceil\|G_{ij}\|'_{(p,q_j)}\rceil)$.*

Proof: Since $\|\cdot\|_{p,p}$ is submultiplicative it follows from Lemma 6 that

$$\rho(A\Delta) \le \rho(\lceil\|(A\Delta)_{ij}\|_{p,p}\rceil) \tag{26}$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. Next note that

$$\lceil\|(A\Delta)_{ij}\|_{p,p}\rceil \le\le \lceil\|A_{ij}\|'_{(ij)}\rceil\lceil\|\Delta_{ij}\|_{(i,j)}\rceil \le\le \lceil\|A_{ij}\|'_{(i,j)}\rceil\|\Delta\| \tag{27}$$

for all $\Delta \in \boldsymbol{\Delta}$ and $A \in \mathbb{C}^{m\times m}$. Now $\upsilon(G) \le \rho(\lceil\|G_{ij}\|'_{(p,q_j)}\rceil)$ follows as a direct consequence of (26), (27), Lemma 5, and Theorem 3 with $\phi(A) = \rho(\lceil\|A_{ij}\|'_{(i,j)}\rceil)$ for all $A \in \mathbb{C}^{m\times m}$. □

Next, we specialize the above results to the case in which $\|\cdot\|_{(i)}$, $i = 1,\ldots,c$, is either a Hölder norm or a unitarily invariant norm. The following result considers the case where $\|\cdot\|_{(i)} = \|\cdot\|_p$, $i = 1,\ldots,c$.

Corollary 13: *Let $G \in \mathbb{C}^{m\times m}$ and let $1 \le p \le \infty$ and $1 \le q \le \infty$ be such that $1/p + 1/q = 1$. If $\|\cdot\|_{(i)} = \|\cdot\|_p$, $i = 1,\ldots,c$, then $\upsilon(G) \le \rho(\lceil\|G_{ij}\|_q\rceil)$.*

Proof: Since $\|\cdot\|_{q,q}$ is submultiplicative the result follows from Remark 2 and Proposition 5. □

Now we consider the case in which $\|\cdot\|_{(i)} = \|\cdot\|'$, $i = 1,\ldots,c$, is a normalized unitarily invariant matrix norm.

Corollary 14: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|'$ be a normalized unitarily invariant matrix norm. If $\|\cdot\|_{(i)} = \|\cdot\|'$, $i = 1,\ldots,c$, then $\upsilon(G) \le \rho(\lceil\sigma_{\max}(G_{ij})\rceil)$.*

Proof: It follows from Corollary 1 that for all $A, B \in \mathbb{C}^{m_i\times m_j}$

$$\sigma_{\max}(AB) \le \sigma_{\max}(A)\sigma_{\max}(B) \le \sigma_{\max}(A)\|B\|'$$

so that $(\sigma_{\max}(\cdot), \sigma_{\max}(\cdot), \|\cdot\|')$ is a submultiplicative triple of matrix norms. Now, since $\sigma_{\max}(\cdot)$ is submultiplicative, the result follows immediately from Proposition 5. □

Remark 10: Note that the cases $\|\cdot\|_{(i)} = \|\cdot\|_{1,1}$ and $\|\cdot\|_{(i)} = \|\cdot\|_{\infty,\infty}$ are special cases of Corollary 12.

7. Robust performance

In this section we consider the problem of robust performance within the structured matrix norm framework. In order to do this consider the nominal square transfer function matrix $G(s) \in \mathbb{C}^{\hat{m}\times\hat{m}}$ in a negative feedback interconnection with structured uncertainty $\Delta \in \boldsymbol{\Delta} \subseteq \mathbb{C}^{m\times m}$ and external disturbance inputs $w(s)$ and per-

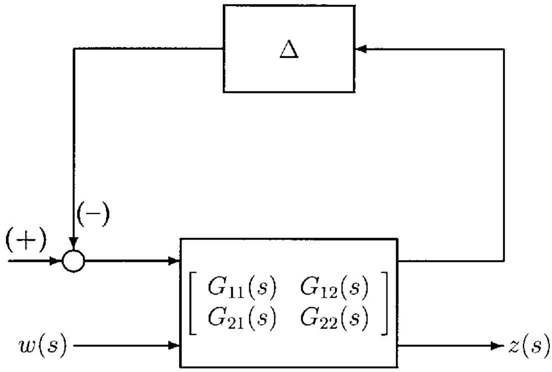

Figure 2. Nominal closed-loop system with feedback uncertainty.

formance outputs $z(s)$ shown in figure 2. Here we assume that $w(s) \in \mathrm{H}_\infty^{m_\mathrm{p}}$ so that every element of the input vector $w(s)$ in a stable function. Note that since the Laplace transform of L_p signals, $1 \leq p < \infty$, is in H_∞ the above assumption allows for a general class of disturbance inputs. Furthermore, we partition $G(s) \in \mathbb{C}^{\hat{m}\times\hat{m}}$ as

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

where $G_{11}(s) \in \mathbb{C}^{m\times m}$, $G_{21}(s) \in \mathbb{C}^{m\times m_\mathrm{p}}$, $G_{21}(s) \in \mathbb{C}^{m_\mathrm{p}\times m}$, and $G_{22}(s) \in \mathbb{C}^{m_\mathrm{p}\times m_\mathrm{p}}$ such that $m + m_\mathrm{p} = \hat{m}$. Here $G(s)$ may denote a nominal closed-loop system. Next, the output $z(s)$ is related to the input $w(s)$ by $z(s) = \mathcal{G}(s)w(s)$ where

$$\mathcal{G}(s) \triangleq G_{22}(s) - G_{21}(s)\Delta(I + G_{11}(s)\Delta)^{-1}G_{12}(s) \tag{28}$$

Next we give several definitions for a class of subharmonic functions which prove useful in assigning signal norms on $\mathrm{H}_\infty^{m_\mathrm{p}}$. Let $\mathbb{C}^+$ denote the open right half complex plane. Recall that a function $f : \mathbb{C}^+ \to [-\infty, \infty)$ is *subharmonic* (Boyd and Desoer 1985) if $f(\cdot)$ is continuous and

$$f(s) \leq \frac{1}{2\pi}\int_0^{2\pi} f(s + \alpha e^{j\theta})\,\mathrm{d}\theta$$

for all $s \in \mathbb{C}^+$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha < \operatorname{Re} s$. Furthermore, define a subset $\mathbf{SH}$ of subharmonic functions by (Boyd and Desoer 1985).

$\mathbf{SH} \triangleq \{f : \mathbb{C}^+ \to [-\infty, \infty) : f(\cdot)$ is subharmonic, $f(\cdot)$ is bounded from above, and $\lim_{\sigma\to 0} f(\sigma + j\omega) = f(j\omega),\ \omega \in \mathbb{R}\}$.

Now it follows from Boyd and Desoer (1985) that

$$\sup_{\omega\in\mathbb{R}} f(j\omega) = \sup_{\operatorname{Re} s\geq 0} f(s), \qquad f(\cdot) \in \mathbf{SH} \tag{29}$$

In order to address worst-case robust performance, i.e. the magnitude of the output corresponding to the worst case input, define the signal norms $|||\cdot|||'$ and $|||\cdot|||''$ on $\mathrm{H}_\infty^{m_\mathrm{p}}$ such that $|||w(s)|||' \triangleq \sup_{\omega\in\mathbb{R}} \|w(j\omega)\|'$, and $|||z(s)|||'' \triangleq \sup_{\omega\in\mathbb{R}} \|z(j\omega)\|''$, for all $w(s), z(s) \in \mathrm{H}_\infty^{m_\mathrm{p}}$ where $\|\cdot\|'$ and $\|\cdot\|''$ are given vector norms on $\mathbb{C}^{m_\mathrm{p}}$. Note that it follows from Theorem 2.1 of Boyd and Desoer (1985) that $\|f(\cdot)\|', \|f(\cdot)\|'' \in \mathbf{SH}$ for all $f(\cdot) \in \mathrm{H}_\infty^{m_\mathrm{p}}$ and hence it follows form (29) that

$|||\cdot|||'$ and $|||\cdot|||''$ are valid signal norms on $\mathrm{H}_\infty^{m_\mathrm{p}}$. In this case we can define the worst case robust performance as $\max_{\Delta\in\Delta_\gamma} |||\mathcal{G}(s)|||'''$ where the signal norm $|||\cdot|||'''$ is defined as

$$|||\mathcal{G}(s)|||''' \triangleq \max_{|||w(s)|||'\leq 1} |||z(s)|||'' \tag{30}$$

The following results are needed in obtaining robust performance bounds.

Theorem 4: *Let $\|\cdot\|'''$ be the matrix norm induced by the vector norms $\|\cdot\|'$ and $\|\cdot\|''$. Then $|||\mathcal{G}(s)|||''' = \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|'''$.*

Proof: Note that for all $w(s) \in \mathrm{H}_\infty^{m_\mathrm{p}}$

$$|||z(s)|||'' = \sup_{\omega\in\mathbb{R}} \|z(j\omega)\|'' \leq \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|'''\|w(j\omega)\|' \leq \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|'''|||w(s)|||'$$

and hence $|||\mathcal{G}(s)|||''' \leq \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|'''$. Next, note that for all $\varepsilon > 0$ there exists $\hat{\omega} \in \mathbb{R}$ such that $\sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|''' - \|\mathcal{G}(j\hat{\omega})\|''' \leq \varepsilon$. Furthermore, let $\hat{w}(j\hat{\omega}) \in \mathbb{C}^{m_\mathrm{p}}$ be such that $\|\mathcal{G}(j\hat{\omega})\hat{w}(j\hat{\omega})\|'' = \|\mathcal{G}(j\hat{\omega})\|'''\|\hat{w}(j\hat{\omega})\|'$. Now define $w(s) \in \mathrm{H}_\infty^{m_\mathrm{p}}$ such that $w(j\omega) = \hat{w}(j\hat{\omega})$ for all $w \in \mathbb{R}$ so that $|||w(s)|||' = \|\hat{w}(j\hat{\omega})\|'$. In this case

$$|||z(s)|||'' = |||\mathcal{G}(s)w(s)|||'' = \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\hat{w}(j\omega)\|'' \geq \|\mathcal{G}(j\hat{\omega})\hat{w}(j\hat{\omega})\|''$$

$$= \|\mathcal{G}(j\hat{\omega})\|'''\|\hat{w}(j\hat{\omega})\|' \geq \left(\sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|''' - \varepsilon\right) |||w(s)|||',$$

which implies that $|||\mathcal{G}(s)|||''' \geq \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|''' - \varepsilon$ for all $\varepsilon > 0$. Now, since $\sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|''' - \varepsilon \leq |||\mathcal{G}(s)|||''' \leq \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|'''$ for all $\varepsilon > 0$ it follows that $|||\mathcal{G}(s)|||''' = \sup_{\omega\in\mathbb{R}} \|\mathcal{G}(j\omega)\|'''$. □

Remark 11: It follows from Theorem 4 that it suffices to compute upper bounds for $\max_{\Delta\in\Delta_\gamma} \|\mathcal{G}(j\omega)\|'''$ for all $\omega \in \mathbb{R}$ in order to obtain the robust performance measure $\max_{\Delta\in\Delta_\gamma} |||\mathcal{G}(s)|||'''$. Hence the following results are focused on obtaining upper bounds for $\max_{\Delta\in\Delta_\gamma} \|\mathcal{G}(j\omega)\|'''$ using the structured matrix norm framework.

Lemma 9: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|'$ be a matrix norm on $\mathbb{C}^{m\times m}$ such that for all $A \in \mathbb{C}^{m\times m}$ there exists $B \in \mathbb{C}^{m\times m}$ such that $\rho(AB) = \|A\|'\|B\|$ and $\upsilon(A) \leq \|A\|'$. If $r = 0$, $c = 1$, and $l_1 = 1$ then $\upsilon(G) = \|G\|'$.*

Proof: Note that there exists $\Delta \in \mathbb{C}^{m\times m}$ such that $\rho(G\Delta) = \|G\|'\|\Delta\|$ and hence $\max_{\Delta\in\Delta_\gamma} \rho(G) \geq \|G\|'$. However, $\upsilon(G) \leq \|G\|'$ and hence it follows from Lemma 8 that $\upsilon(G) = \|G\|'$. □

Now we introduce a key definition which is used in the following lemmas. Let $\|\cdot\|$ denote a vector norm on $\mathbb{C}^m$ and define the *dual norm* $\|\cdot\|_\mathrm{D}$ of $\|\cdot\|$ as $\|y\|_\mathrm{D} \triangleq \max_{\|x\|=1} |y^*x|$, where $y \in \mathbb{C}^m$ (Stewart and Sun 1990). Note that $\|\cdot\|_\mathrm{DD} = \|\cdot\|$ (Stewart and Sun 1990, p. 56). The following key lemmas are needed for the main results of this section.

Lemma 10: *Let $\|\cdot\|$ denote the matrix norm on $\mathbb{C}^{m\times m}$ induced by vector norms $\|\cdot\|'$ and $\|\cdot\|''$ on $\mathbb{C}^m$ and let $x, y \in \mathbb{C}^m$. Then $\|xy^*\| = \|x\|''\|y\|'_\mathrm{D}$.*

Proof: It need only be noted that $\|xy^*\| = \max_{\|z\|'=1} \|xy^*z\|'' = \max_{\|z\|'=1} \|x\|''|y^*z| = \|x\|''\|y\|'_\mathrm{D}$. □

Lemma 11: *Let $G \in \mathbb{C}^{m\times m}$ and let $\|\cdot\|$ be a matrix norm on $\mathbb{C}^{m\times m}$ induced vector norms $\|\cdot\|'$ and $\|\cdot\|''$ and let $\|\cdot\|'''$ denote the matrix norm on $\mathbb{C}^{m\times m}$ induced by vector norms $\|\cdot\|''$ and $\|\cdot\|'$. If $r = 0$, $c = 1$, and $l_1 = 1$ then $\upsilon(G) = \|G\|'''$.*

Proof: First note that Corollary 9 implies $\upsilon(G) \leq \|G\|'''$. Now let $x \in \mathbb{C}^m$ be such that $\|Gx\|' = \|G\|'''\|x\|''$ and let $y \in \mathbb{C}^m$ be such that $|y^*Gx| = \|y\|'_{\mathrm{D}}\|Gx\|'$ (note that the existence of such a y follows from the fact that $\|\cdot\|'_{\mathrm{DD}} = \|\cdot\|'$). Next choose $\Delta = xy^*$. In this case it follows from Lemma 10 that $\|\Delta\| = \|x\|''\|y\|'_{\mathrm{D}}$. Hence,

$$\rho(G\Delta) = |y^*Gx| = \|y\|'_{\mathrm{D}}\|G\|'''\|x\|'' = \|G\|'''\|\Delta\|$$

The result now follows from Lemma 9. □

In order to address robust performance within the structured matrix norm framework we introduce an additional uncertainty block Δ_{p} between $w(s)$ and $z(s)$ so that $w(s) = \Delta_{\mathrm{p}}z(s)$ and requite stability robustness in the face of all perturbations, including the block Δ_{p}. Now, define the set

$$\tilde{\boldsymbol{\Delta}} \triangleq \left\{\tilde{\Delta} = \text{block-diag}(\Delta, \Delta_{\mathrm{p}}) : \Delta \in \boldsymbol{\Delta}, \Delta_{\mathrm{p}} \in \mathbb{C}^{m_{\mathrm{p}}\times m_{\mathrm{p}}}\right\},$$

and define the associated structured matrix norm by

$$\tilde{\upsilon}(G(j\omega)) \triangleq \left(\min_{\Delta \in \tilde{\boldsymbol{\Delta}}}\left\{\max\left(\|\Delta\|, \|\Delta_{\mathrm{p}}\|'\right) : \det\left[I + G(j\omega)\tilde{\Delta}\right] = 0\right\}\right)^{-1}$$

and if there does not exist $\tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}$ such that $\det\left[I + G(j\omega)\tilde{\Delta}\right] = 0$, then $\tilde{\upsilon}(G(j\omega)) \triangleq 0$ where $\|\cdot\|, \|\cdot\|'$ are given matrix norms on $\mathbb{C}^{m\times m}$ and $\mathbb{C}^{m_{\mathrm{p}}\times m_{\mathrm{p}}}$, respectively. Furthermore, define

$$\upsilon_{\mathrm{p}}(\mathscr{G}(j\omega), \Delta) \triangleq \left(\min_{\Delta_{\mathrm{p}} \in \mathbb{C}^{m_{\mathrm{p}}\times m_{\mathrm{p}}}}\left\{\|\Delta_{\mathrm{p}}\|' : \det\left[I + \mathscr{G}(j\omega)\Delta_{\mathrm{p}}\right] = 0\right\}\right)^{-1},$$

and if there does not exist $\Delta_{\mathrm{p}} \in \mathbb{C}^{m\times m}$ such that $\det\left[I + \mathscr{G}(j\omega)\Delta_{\mathrm{p}}\right] = 0$, then $\upsilon_{\mathrm{p}}(\mathscr{G}(j\omega), \Delta) \triangleq 0$.

Lemma 12: *Let $\omega, \gamma \in \mathbb{R}$, $\gamma > 0$. Then $\tilde{\upsilon}(G(j\omega)) < \gamma$ if and only if $\upsilon(G_{11}(j\omega)) < \gamma$ and $\upsilon_{\mathrm{p}}(\mathscr{G}(j\omega), \Delta) < \gamma$ for all $\Delta \in \boldsymbol{\Delta}_\gamma$.*

Proof: Note that it follows from the definition that if $\tilde{\upsilon}(G(j\omega)) < \gamma$ then $\det\left[I + G(j\omega)\tilde{\Delta}\right] \neq 0$ for all $\tilde{\Delta} = \text{block-diag}(\Delta, 0)$, $\Delta \in \boldsymbol{\Delta}_\gamma$. Hence, $\det\left[I + G_{11}(j\omega)\Delta\right] \neq 0$, $\Delta \in \boldsymbol{\Delta}_\gamma$, and hence $\upsilon(G_{11}(j\omega)) < \gamma$. Next, since

$$\det\left[I + G(j\omega)\tilde{\Delta}\right] = \det\left[I + G_{11}(j\omega)\Delta\right]\det\left[I + \mathscr{G}(j\omega)\Delta_{\mathrm{p}}\right]$$

it follows that $\max_{\Delta\in\boldsymbol{\Delta}_\gamma} \upsilon_{\mathrm{p}}(\mathscr{G}(j\omega), \Delta) < \gamma$. The converse follows by reversing these steps. □

The following corollary is immediate.

Corollary 15: *Let $\omega, \gamma \in \mathbb{R}, \gamma > 0$. If $\tilde{\upsilon}(G(j\omega)) < \gamma$ then*

$$\max\left\{\upsilon(G_{11}(j\omega)), \max_{\Delta\in\boldsymbol{\Delta}_\gamma} \upsilon_{\mathrm{p}}(\mathscr{G}(j\omega), \Delta)\right\} \leq \tilde{\upsilon}(G(j\omega)) \tag{31}$$

Now we present the main result of this section involving robust stability and performance.

Theorem 5: *Let $\omega, \gamma \in \mathbb{R}, \gamma > 0$ and let $\|\cdot\|'$ denote the matrix norm on $\mathbb{C}^{m_p \times m_p}$ (corresponding to the defining norm on Δ_p) induced by vector norms $\|\cdot\|''$ and $\|\cdot\|'''$. If $\sup_{\omega \in \mathbb{R}} \tilde{\upsilon}(G(j\omega)) < \gamma$ then the negative feedback interconnection of $G_{11}(s)$ and Δ is asymptotically stable for all $\Delta \in \Delta_\gamma$. Furthermore,*

$$\|z(j\omega)\|'' \leq \tilde{\upsilon}(G(j\omega))\|w(j\omega)\|''' \tag{32}$$

for all $\Delta \in \Delta_\gamma$.

Proof: It follows from Lemma 12 that if $\tilde{\upsilon}(G(j\omega)) < \gamma$ then $\upsilon(G_{11}(j\omega)) < \gamma$ for all $\omega \in \mathbb{R}$. Hence, it follows from Theorem 1 that the negative feedback interconnection of $G_{11}(s)$ and Δ is asymptotically stable for all $\Delta \in \Delta_\gamma$. Next, let $\|\cdot\|''''$ denote the matrix norm on $\mathbb{C}^{m_p \times m_p}$ induced by $\|\cdot\|'''$ and $\|\cdot\|''$. Then, it follows from Lemma 11 and Corollary 15 that

$$\max_{\Delta \in \Delta_\gamma} \upsilon_p(\mathcal{G}(j\omega), \Delta) = \max_{\Delta \in \Delta_\gamma} \|\mathcal{G}(j\omega)\|'''' \leq \tilde{\upsilon}(G(j\omega))$$

which implies (32). □

Remark 12: Note that it follows from Theorems 4 and 5 that $\tilde{\upsilon}(G(j\omega))$ provides an upper bound on the worst case performance. For example, if $\|\cdot\|' = \|\cdot\|_{p,q}$ where $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ then $|||\mathcal{G}(s)|||_{q,p} \leq \sup_{\omega \in \mathbb{R}} \tilde{\upsilon}(G(j\omega))$. Specifically, if $\|\cdot\|' = \sigma_{\max}(\cdot)$ then $|||\mathcal{G}(s)|||' = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\mathcal{G}(j\omega)) \leq \tilde{\upsilon}(G(j\omega))$ addresses H_∞ performance. In general $\sup_{\omega \in \mathbb{R}} \tilde{\upsilon}(\mathcal{G}(j\omega))$ characterizes the allowable size on the nominal transfer function for both robust stability and performance for specified spatial norms.

8. Illustrative examples

In this section we consider three examples to demonstrate the usefulness of structured matrix norms.

Example 1: Consider the block diagonal matrix $G = \text{block-diag}(1, 0_{9\times 9})$ and let $\gamma > 0$ be the maximum allowable uncertainty level such that $\det(I + G\Delta) \neq 0$ for all $\Delta \in \{\Delta \in \mathbb{C}^{10\times 10} : \|\Delta\|_\infty < \gamma\}$. In this case it follows from Lemma 11 and Remark 1 that the structured matrix norm $\upsilon(G) = \|G\|_{1,\infty} = \|G\|_1 = 1$. Hence, using Theorem 1, the maximum allowable uncertainty level γ is equal to 1. Alternatively, this problem can be *equivalently* formulated as a μ problem. In particular, let $X, Y \in \mathbb{C}^{10\times 100}$ be such that $\Delta = X\tilde{\Delta}Y^*$, where $\tilde{\Delta} \in \mathbb{C}^{100\times 100}$ is a diagonal matrix with $\tilde{\Delta}_{(i,i)} = (\text{vec}(\Delta))_{(i)}$, where $i = 1, \ldots, 100$ and $\text{vec}(\cdot)$ denotes the column stacking operator. Next, note that $\|\Delta\|_\infty < \gamma$ is equivalent to $\sigma_{\max}(\tilde{\Delta}) < \gamma$. Since $\det(I + G\Delta) = \det(I + GX\tilde{\Delta}Y^*) = \det(I + Y^*GX\tilde{\Delta})$ it follows that $\det(I + G\Delta) \neq 0$ for all $\Delta \in \{\Delta \in \mathbb{C}^{10\times 10} : \|\Delta\|_\infty < \gamma\}$ if and only if $\det(I + \tilde{G}\tilde{\Delta}) \neq 0$ for all $\tilde{\Delta} \in \{\tilde{\Delta} \in \tilde{\mathbf{\Delta}} : \sigma_{\max}(\tilde{\Delta}) < \gamma\}$, where $\tilde{G} \triangleq Y^*GX$ and $\tilde{\mathbf{\Delta}} \triangleq \{\tilde{\Delta} \in \mathbb{C}^{100\times 100} : \tilde{\Delta} = \text{diag}(\delta_1, \ldots, \delta_{100}), \ \delta_i \in \mathbb{C}, \ i = 1, \ldots, 100\}$. Now, it follows from the small-μ theorem that the maximum allowable uncertainty is given by $1/\mu(\tilde{G})$ where $\mu(\cdot)$ is evaluated with respect to the uncertainty structure given by $\tilde{\mathbf{\Delta}}$ Since $\mu(\tilde{G})$ cannot be computed exactly for the given block-structured uncertainty, using the μ-toolbox (Balas *et al.* 1991) we compute the upper bound $\inf_{D \in \mathcal{D}} \sigma_{\max}(D\tilde{G}D^{-1})$ where $\mathcal{D}$ is the set of scaling matrices compatible with the elements of $\tilde{\mathbf{\Delta}}$ For this example the upper bound coincides with $\mu(\tilde{G})$, and hence $\upsilon(G)$ and $\mu(\tilde{G})$ give the same robust stability predictions. However, the number of floating point operations (flops) required for computing $\upsilon(G)$ is 100 while number of flops required for computing $\mu(\tilde{G})$ is 58 269 912. It can be shown that the number of flops for computing $\upsilon(G)$ is proportional to m^2 while the number of flops for computing $\mu(\tilde{G})$ is proportional to m^6, where m is the size of uncertainty Δ. To reduce the computational complexity of the structured singular value one can consider a subset of $\mathcal{D}$ in the optimization of $\mu(\tilde{G})$. Specifically, choosing $\mathcal{D} \triangleq \{D \in \mathbb{C}^{100\times 100} : D = \text{block-diag}(d_1 I_{20}, \ldots, d_5 I_{20}),\ d_i > 0,\ i = 1, \ldots, 5\} \subset \mathcal{D}$ and using the μ-toolbox it follows that $\mu(\tilde{G}) \leq 4.4721$ so that the maximum allowable uncertainty predicted is 0.2236. In this case the number of flops required for computing $\mu(\tilde{G})$ is reduced to 16 711 487, however, at the significant expense of robust stability predictions.

Example 2: Let $\mu(G(j\omega))$ and $\upsilon_1(G(j\omega))$ denote the structured matrix norms with defining norms $\sigma_{\max}(\cdot)$ and $\|\cdot\|_1$, respectively, and assume structured uncertainty $\Delta = \{\Delta \in \mathbb{C}^{2\times 2} : \Delta = \text{diag}(\delta_1, \delta_2),\ \delta_1, \delta_2 \in \mathbb{C}\}$. Furthermore, let

$$G(s) \sim \left[\begin{array}{cc|cc} -0.25 & 1.3 & 1 & 0 \\ -1.3 & -0.25 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Since the exact computation of $\mu(G(j\omega))$ and $\upsilon(G(j\omega))$ is difficult we compute the upper bounds given by $\inf_{D(j\omega)\in\mathcal{D}} \sigma_{\max}(D(j\omega)G(j\omega)D(j\omega)^{-1})$ and $\inf_{D(j\omega)\in\mathcal{D}} \|D(j\omega)G(j\omega)D(j\omega)^{-1}\|_\infty$, respectively, where $\mathcal{D} = \{D \in \mathbb{R}^{2\times 2} : D = \text{diag}(d_1, d_2),\ d_1 \neq 0,\ d_2 \neq 0\}$. The upper bound $\inf_{D(j\omega)\in\mathcal{D}} \sigma_{\max}(D(j\omega)G(j\omega)D(j\omega)^{-1})$ is evaluated using LMI techniques (Gahinet and Nemirovskii 1993) while it can be shown that

$$\inf_{D(j\omega)\in\mathcal{D}} \|D(j\omega)G(j\omega)D^{-1}(j\omega)\|_\infty = \max\left\{|G_{1,1}(j\omega)|, \sqrt{|G_{(1,2)}(j\omega)G_{(2,1)}(j\omega)|}, |G_{(2,2)}(j\omega)|\right\}$$

These upper bounds are shown in figure 3 and the predictions of robust stability for the two uncertainty characterizations are shown in figure 4. This example demon-

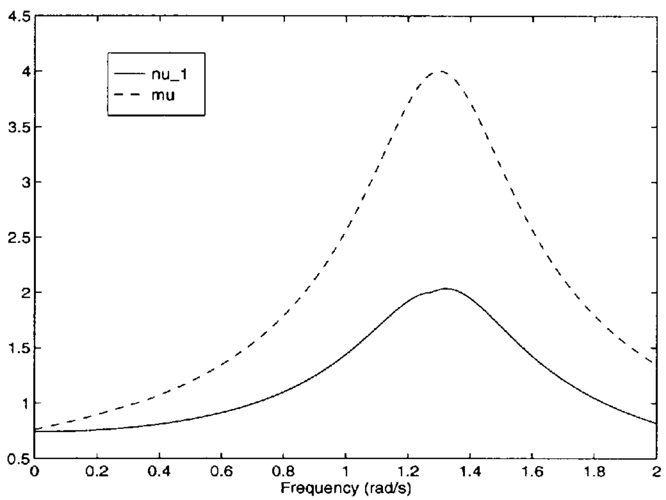

Figure 3. Upper bounds to the structured matrix norm for Example 2.

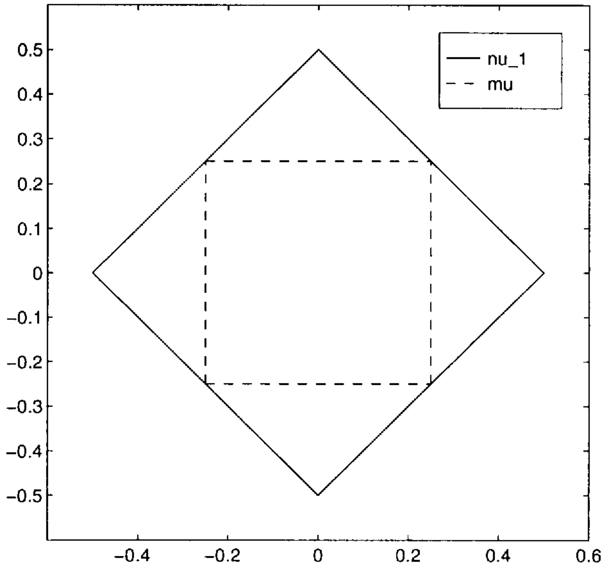

Figure 4. Robust stability predictions for Example 2.

strates that bounding uncertainties by alternative spatial norms not consistent with the geometry of singular value bounds can increase robust stability predictions.

Example 3: In this example, we demonstrate the utility of the proposed framework for robust performance. Let $\upsilon(G(j\omega))$ denote the structured matrix norm with defining norm $\|\cdot\|_1$. Furthermore, let

$$G_{11}(s) \sim \left[\begin{array}{cc|cc} -0.25 & 1.3 & 1 & 0 \\ -1.3 & -0.25 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \qquad G_{11}(s) \sim \left[\begin{array}{cc|c} -0.25 & 1.3 & 0 \\ -1.3 & -0.25 & 1 \\ \hline 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

$$G_{21}(s) \sim \left[\begin{array}{cc|cc} -0.25 & 1.3 & 1 & 0 \\ -1.3 & -0.25 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \end{array}\right] \qquad G_{22}(s) \sim \left[\begin{array}{cc|c} -0.25 & 1.3 & 0 \\ -1.3 & -0.25 & 1 \\ \hline 1 & 0 & 0 \end{array}\right]$$

and $\boldsymbol{\Delta} = \left\{\Delta \in \mathbb{C}^{2\times 2} : \Delta = \mathrm{diag}(\delta_1, \delta_2),\ \delta_1, \delta_2 \in \mathbb{C}\right\}$. Note that $\|\Delta\|_1 = \|\Delta\|_{1,\infty}$ for all $\Delta \in \boldsymbol{\Delta}$ Now, introducing a performance block it follows from Theorem 5 that $\|z(j\omega)\|_\infty \leq \tilde{\upsilon}(G(j\omega))\|w(j\omega)\|_1$, $\omega \in \mathbb{R}$, where

$$G(s) \sim \left[\begin{array}{cc|ccc} -0.25 & 1.3 & 1 & 0 & 0 \\ -1.3 & -0.25 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

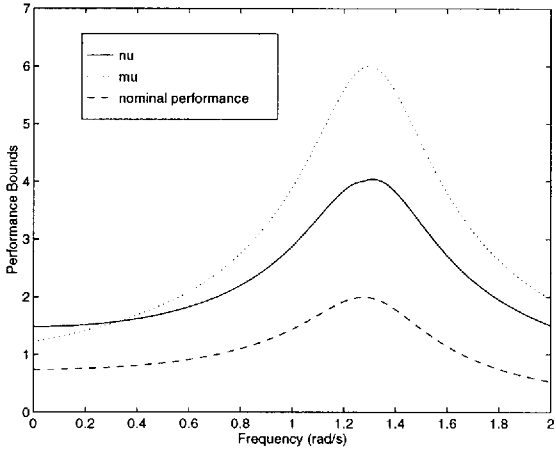

Figure 5. Robust performance bound for Example 3.

and where $\tilde{\nu}(G(j\omega))$ is defined with respect to the uncertainty set $\tilde{\boldsymbol{\Delta}} = \{\tilde{\Delta} \in \mathbb{C}^{3\times 3} : \tilde{\Delta} = \text{block-diag}(\Delta, \delta_{\rm p}),\ \Delta \in \boldsymbol{\Delta}, \delta_{\rm p} \in \mathbb{C}\}$, with defining norm given by $\|\tilde{\Delta}\| = \max\{\|\Delta\|_1, |\delta_{\rm p}|\}$. Next it follows from Proposition 6 with $\|\cdot\|_{(i)} = \|\cdot\|_{1,\infty}$ and $\|\cdot\|_{(i,j)} = \|\cdot\|_{\infty,1} = \|\cdot\|_{\infty}$, $i,j = 1,2$, that $\tilde{\nu}(G(j\omega)) \le \rho(\tilde{G}(j\omega))$ where

$$\tilde{G}(j\omega) = \begin{bmatrix} \left\| \begin{matrix} G_{11}(j\omega) \\ G_{21}(j\omega) \end{matrix} \right\|_{\infty} & \left\| \begin{matrix} G_{12}(j\omega) \\ G_{22}(j\omega) \end{matrix} \right\|_{\infty} \end{bmatrix}$$

Hence, we obtain a computable upper bound on robust performance given by

$$\|z(j\omega)\|_{\infty} \le \rho(\tilde{G}(j\omega))\|w(j\omega)\|_1, \quad \omega \in \mathbb{R}$$

Alternatively, we can provide an upper bound for robust performance using Proposition 1. Specifically, it can be shown that $0.5\|\tilde{\Delta}\| \le \sigma_{\max}(\tilde{\Delta}) \le \|\tilde{\Delta}\|$, $\tilde{\Delta} \in \tilde{\boldsymbol{\Delta}}$ and hence it follows from Proposition 1 that $0.5\mu(G(j\omega)) \le \nu(G(j\omega)) \le \mu(G(j\omega))$, $\omega \in \mathbb{R}$. Hence, we can compute an upper bound to $\mu(G(j\omega))$ in terms $\inf_{D(j\omega)\in\mathcal{D}} \sigma_{\max}(D(j\omega)G(j\omega)D^{-1}(j\omega))$ using standard LMI techniques (Gahinet and Nemirovskii 1993). The nominal performance and the two upper bounds are shown in figure 5.

9. Conclusion

The goal of this paper has been to extend the notion of the structured singular value and introduce lower and upper bounds for robust stability and performance for structured uncertainty involving alternative spatial norms. In particular, we considered a norm-bounded, block-structured uncertainty characterization wherein the defining norm is not the maximum singular value. To this end we introduced the notion of structured matrix norms as a generalization of the structured singular value for characterizing the size of the nominal transfer function. Finally, we demonstrated the usefulness of the proposed framework on several examples wherein the plant uncertainty characterization was not amenable to singular value bounds.

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References

Balas, G. J., Doyle, J. C., Glover, K., Packard, A., and Smith, R., 1991, *μ-Analysis and Synthesis Toolbox* (Natick, MA: The Mathworks Inc.).

Bernstein, D. S., Haddad, W. M., and Sparks, A. G. 1995, A Popov criterion for uncertain linear multivariable systems. *Automatica*, **31**, 1061–1064.

Boyd, S. P., and Desoer, C. A., 1985, Subharmonic functions and performance bounds on linear time-invariant feedback systems. *IMA Journal of Mathematics Control Information*, **2**, 153–170.

Chen, J., and Nett, C. N., 1992, Bounds on generalized structured singular value via the Perron root of matrix majorants. *Systems Control Letters*, **19**, 439–449.

Chen, J., Fan, M. K. H., and Nett, C. N., 1996a, Structured singular values with nondiagonal structures—Part I: Characterizations. *IEEE Transactions Automatic Control*, **41**, 1507–1511.

Chen, J., Fan, M. K. H., and Nett, C N., 1996b, Structured singular values with nondiagonal structures—Part II: Computation. *IEEE Transactions on Automatic Control*, **41**, 1511–1516.

Fan, M. K. H., Tits, A. L., and Doyle, J. C., 1991, Robustness in the presence of mixed parametric uncertainty and unmodelled dynamics. *IEEE Transactions on Automatic Control*, **36**, 25–38.

Gahinet, P., and Nemirovskii, A., 1993, *LMI Lab: A Package for Manipulation and Solving LMI's* (France: INRIA).

Haddad, W. M., Bernstein, D. S., and Chellaboina, V., 1996, Generalized mixed-μ bounds for real and complex multiple-block uncertainty with internal matrix structure. *International Journal of Control*, **64**, 789–806.

Horn, R. A., and Johnson, R. C., 1985, *Matrix Analysis* (Cambridge: Cambridge University Press).

Hyland, D. C., and Collins, Jr., E. G., 1989, An M-matrix and majorant approach to robust stability and performance analysis for systems with structured uncertainty. *IEEE Transactions on Automatic Control*, **34**, 691–710.

Kahan, W., 1966, Numerical linear algebra. *Canadian Mathematics Bulletin*, **9**, 757–801.

Khammash, M., and Pearson, J. B., 1993, Analysis and design for robust performance with structured uncertainty. *Systems Control Letters*, **20**, 179–187.

Ostrowski, A. M., 1975, On some metrical properties of operator matrices and matrices partitioned into blocks. *Journal of Mathematics Analysis Applications*, **10**, 161–209.

Packard, A., and Doyle, J. C., 1993, The complex structured singular value. *Automatica*, **29**, 71–109.

Safonov. M. G., 1982, Stability margins of diagonally perturbed multivariable feedback systems. *IEE Proceedings*, **129-D**, 251–256.

Stewart, G. W., and Sun, J., 1990, *Matrix Perturbation Theory* (San Diego: Academic Press).

Young, P. M., and Dahleh, M. A., 1995, Robust ℓ_p stability and performance. *Systems Control Letters*, **26**, 305–312.

Young, P. M., Newlin, M. P., and Doyle, J. C., 1991, μ Analysis with real parametric uncertainty. *Proceedings of the IEEE Conference on Decision and Control*, Brighton, UK, pp. 1251–1256.

Zhou, K., Doyle, J. C., and Glover, K., 1996, *Robust and Optimal Control* (Englewood Cliffs: Prentice-Hall).