INT. J. CONTROL, 1998, VOL. 70, NO. 4, 651–664

Real robustness bounds using generalized stability multipliers

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Sufficient conditions for robust stability using generalized stability multipliers are presented for systems with sector- and norm-bounded, block-structured real uncertainty. Two parametrizations of the multiplier are considered and the robustness criteria are written as linear matrix inequalities. Upper bounds for the peak structured singular value over frequency, which eliminate frequency gridding, are then derived. Numerical examples provide a comparison of the peak upper bounds obtained using the two multiplier parametrizations. These examples show that the conservatism of the peak upper bounds is reduced by increasing the dynamic order of the multiplier.

1. Introduction

Computation of the structured singular value for robust stability analysis is an intractable problem and upper bounds are used in practice. In particular, the upper bound of Fan *et al.* (1991) for real uncertainty is stated in terms of the frequency response of the nominal plant transfer function, and hence is evaluated using frequency gridding. In practice, however, it is often difficult to compute reliably the peak value of the upper bound of Fan *et al.* (1991) over frequency since the upper bound may have sharp peaks or may even be discontinuous (Sparks and Bernstein 1998).

An alternative to frequency gridding for computing robustness bounds was proposed by Sparks and Bernstein (1998). The approach of Sparks and Bernstein (1998) is based on a generalization of the Popov criterion and provides a more reliable test of robust stability than the frequency-domain bound by eliminating frequency gridding. This peak upper bound involves a multiplier that is an affine function of frequency and a scaling that is independent of frequency. However, the peak upper bound is generally conservative since the frequency dependence of the multiplier and scaling is restricted.

In this paper, we reduce the conservatism of the peak upper bound for the structured singular value given by Sparks and Bernstein (1998) by using multipliers and scalings that are more general functions of frequency. The new stability criteria are used to compute peak bounds for the structured singular value for several numerical examples. The reduced conservatism achieved by including additional terms in the multiplier and scaling is demonstrated.

Received 5 August 1997. Revised 12 January 1998.
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2. Preliminaries

For convenience, we recall the following mathematical preliminaries. Define the *paraconjugate transpose* of a real-rational function $Z(s) = C(sI - A)^{-1}B + D$ by $Z^{\sim}(s) \triangleq Z^{\mathrm{T}}(-s) = B^{\mathrm{T}}(-sI - A^{\mathrm{T}})^{-1}C^{\mathrm{T}} + D^{\mathrm{T}}$. Let X^* be the complex conjugate transpose of a complex matrix X, let the Hermitian part of a square matrix X be denoted by He $X \triangleq \frac{1}{2}(X + X^*)$, and note that He $Z(s) = \frac{1}{2}(Z(s) + Z^{\sim}(s))$ for all $s = \mathrm{j}\omega$. Finally, $X > 0$ $(X \geq 0)$ indicates that the Hermitian matrix X is positive definite (non-negative definite).

A square real-rational function $Z(s)$ is *generalized positive real* (Anderson and Moore 1968) if He $Z(s) \geq 0$ for all $s = \mathrm{j}\omega$ such that $\mathrm{j}\omega$ is not a pole of $Z(s)$, while $Z(s)$ is *strictly generalized positive real* if $Z(s)$ has no imaginary axis poles and He $Z(s) > 0$ for all $s = \mathrm{j}\omega$.

3. Sector-bounded uncertainty

Define the set $\mathcal{F}_M$ of real, sector-bounded matrices F by

$$\mathcal{F}_M \triangleq \{F \in \mathcal{F}\colon\ 0 \leq F \leq M\}$$

where $M \in \mathcal{F}$ is positive definite, and $\mathcal{F}$ is the set of real symmetric, block-structured matrices defined by

$$\mathcal{F} \triangleq \{F\colon\ F = \text{block-diag}(I_{l_1} \otimes F_1, \ldots, I_{l_r} \otimes F_r),\ F_i = F_i^{\mathrm{T}} \in \mathbb{R}^{m_i \times m_i},\ i = 1, \ldots, r\}$$

where $\otimes$ denotes a Kronecker product. Now, let $G(s) = C(sI - A)^{-1}B$, where A is asymptotically stable, and consider the negative feedback interconnection of $G(s)$ and F. Noting that this interconnection has the dynamics $\dot{x} = (A - BFC)x$, it follows that the asymptotic stability of the interconnection is equivalent to the asymptotic stability of

$$\dot{x} = (A + \delta A)x \tag{1}$$

where $\delta A = -BFC$. To prove our robust stability criterion for the uncertain system (1), we first recall the following lemma (Haddad and Bernstein 1991).

Lemma 1: *If* $X \in \mathbb{C}^{n\times n}$ *satisfies* He $X \geq 0$ *and* $Y \in \mathbb{C}^{n\times n}$ *satisfies* He $Y > 0$, *then* $\det(I + XY) \neq 0$.

Now, we introduce a robust stability criterion involving a multiplier $N(s)$ and a scaling $Q(s)$ that account for the structure of the sector-bounded matrices in $\mathcal{F}_M$.

Theorem 1: *Let $G(s)$ be asymptotically stable and suppose there exist $m \times m$ real-rational functions $N(s)$ and $Q(s)$ such that the following conditions are satisfied.*

(i) *$N(s)$ and $Q(s)$ have no imaginary poles.*

(ii) *$Q(s) = Q^*(s)$ for all $s = \mathrm{j}\omega$.*

(iii) *$FN(s) = N(s)F$ and $FQ(s) = Q(s)F$ for all $F \in \mathcal{F}$ and for all $s = \mathrm{j}\omega$.*

(iv) He *$N(s) \geq Q(s) > 0$ for all $s = \mathrm{j}\omega$.*

(v) *$Q(s)M^{-1} + N(s)G(s)$ is strictly generalized positive real.*

Then the negative feedback interconnection of $G(s)$ and F is asymptotically stable for all $F \in \mathcal{F}_M$.

Proof: Let $F \in \mathcal{F}_M$ and $\omega \in \mathbb{R}$ and note from Lemma 2.1 of Bernstein *et al.* (1995) that $0 \le F \le M$ if and only if $F \ge FM^{-1}F$. Since, by (iv), $Q(\mathrm{j}\omega) > 0$, it follows that $(F - FM^{-1}F)^{1/2}Q(\mathrm{j}\omega)(F - FM^{-1}F)^{1/2} \ge 0$. Now, since $F \in \mathcal{F}$ and $M \in \mathcal{F}$, it follows that $FM^{-1}F \in \mathcal{F}$ and $(F - FM^{-1}F)^{1/2} \in \mathcal{F}$, so that, by (iii), $(F - FM^{-1}F)Q(\mathrm{j}\omega) \ge 0$ and thus $FQ(\mathrm{j}\omega) \ge FQ(\mathrm{j}\omega)M^{-1}F$. Next, since, by (iv), $\mathrm{He}\, N(\mathrm{j}\omega) \ge Q(\mathrm{j}\omega)$, it follows that $\mathrm{He}\, F^{1/2}N(\mathrm{j}\omega)F^{1/2} \ge F^{1/2}Q(\mathrm{j}\omega)F^{1/2}$, so that, by (iii), $\mathrm{He}\, FN(\mathrm{j}\omega) \ge FQ(\mathrm{j}\omega) \ge FQ(\mathrm{j}\omega)M^{-1}F$. Hence, $N^{\sim}(\mathrm{j}\omega)F + FN(\mathrm{j}\omega) \ge 2FQ(\mathrm{j}\omega)M^{-1}F$, so that

$$He\left[N^{\sim}(\mathrm{j}\omega)F - FQ(\mathrm{j}\omega)M^{-1}F\right] \ge 0$$

and

$$\mathrm{He}\left[FN^{-1}(\mathrm{j}\omega) - N^{-\sim}(\mathrm{j}\omega)FQ(\mathrm{j}\omega)M^{-1}FN^{-1}(\mathrm{j}\omega)\right] \ge 0$$

Finally, it follows that

$$\mathrm{He}\left[FN^{-1}(\mathrm{j}\omega)\left[I - Q(\mathrm{j}\omega)M^{-1}FN^{-1}(\mathrm{j}\omega)\right]^{-1}\right] \ge 0$$

Next, since $Q(s)M^{-1} + N(s)G(s)$ is strictly generalized positive real, it follows that $\mathrm{He}\left[Q(s)M^{-1} + N(s)G(s)\right] > 0$ for all $s = \mathrm{j}\omega$. Applying Lemma 1 with $X = FN^{-1}(s)\left[I - Q(s)M^{-1}FN^{-1}(s)\right]^{-1}$ and $Y = Q(s)M^{-1} + N(s)G(s)$ for all $s = \mathrm{j}\omega$ yields

$$\begin{aligned}
0 &\ne \det\left[I + FN^{-1}(s)\left[I - Q(s)M^{-1}FN^{-1}(s)\right]^{-1}\left[Q(s)M^{-1} + N(s)G(s)\right]\right] \\
&= \det\left[I + FN^{-1}(s)Q(s)M^{-1}\left[I - FN^{-1}(s)Q(s)M^{-1}\right]^{-1}.\right. \\
&\quad \left. + \left[I - FN^{-1}(s)Q(s)M^{-1}\right]^{-1}FG(s)\right] \\
&= \det\left[\left[I - FN^{-1}(s)Q(s)M^{-1}\right]^{-1} + \left[I - FN^{-1}(s)Q(s)M^{-1}\right]^{-1}FG(s)\right] \\
&= \det\left[I - FN^{-1}(s)Q(s)M^{-1}\right]^{-1}\det\left[I + FG(s)\right]
\end{aligned}$$

for all $s = \mathrm{j}\omega$ and for all $F \in \mathcal{F}_M$, so that $\det(I + FG(s)) \ne 0$ for all $s = \mathrm{j}\omega$ and for all $F \in \mathcal{F}_M$.

Next, note that $\left[I + G(s)F\right]^{-1}G(s) = C(sI - (A - BFC))^{-1}B$. Suppose that there exists $F \in \mathcal{F}_M$ such that $\left[I + G(s)F\right]^{-1}G(s)$ is not asymptotically stable. Since A is Hurwitz, there exists $\varepsilon \in (0, 1]$ such that $A - \varepsilon BFC$ has an eigenvalue $\mathrm{j}\hat{\omega}$ on the imaginary axis. Thus, setting $s = \mathrm{j}\hat{\omega}$ in the identity

$$\begin{aligned}
\det(I + \varepsilon G(s)F) &= \det(I + \varepsilon C(sI - A)^{-1}BF) \\
&= \det(I + \varepsilon(sI - A)^{-1}BFC) \\
&= \det(sI - A)^{-1}\det(sI - (A - \varepsilon BFC))
\end{aligned}$$

implies $\det(I + \varepsilon G(j\hat{\omega})F) = 0$. However, since $\varepsilon F \in \mathcal{F}_M$, it follows that $\det(I + \varepsilon G(j\hat{\omega})F) \ne 0$, which is a contradiction. Hence, the negative feedback interconnection of $G(s)$ and F is asymptotically stable for all $F \in \mathcal{F}_M$. □

4. Norm-bounded uncertainty

Define the set $\boldsymbol{\Delta}_\gamma$ of real, norm-bounded uncertain matrices Δ by

$$\boldsymbol{\Delta}_\gamma \triangleq \left\{\Delta \in \boldsymbol{\Delta}: \sigma_{\max}(\Delta) \le \gamma^{-1}\right\}$$

where $\gamma > 0$ and $\mathbf{\Delta}$ is the set of symmetric, block-structured matrices defined by

$$\mathbf{\Delta} \triangleq \{\Delta : \Delta = \text{block-diag}(I_{l_1} \otimes \Delta_1, \ldots, I_{l_r} \otimes \Delta_r),\ \Delta_i = \Delta_i^{\mathrm{T}} \in \mathbb{R}^{m_i \times m_i},\ i = 1, \ldots, r\}$$

Note that $\mathbf{\Delta} = \mathcal{F}$ and that if $\Delta \in \mathbf{\Delta}_\gamma$, then $\sigma_{\max}(\Delta) \le \gamma^{-1}$, so that $-\gamma^{-1}I \le \Delta \le \gamma^{-1}I$ and thus $0 \le \Delta + \gamma^{-1}I \le 2\gamma^{-1}I$. Hence $\Delta + \gamma^{-1}I \in \mathcal{F}_M$ for $M = 2\gamma^{-1}I$. Conversely, if $F \in \mathcal{F}_M$, where $M = 2\gamma^{-1}I$, then $F - \gamma^{-1}I \in \mathbf{\Delta}_\gamma$. Hence, $\mathbf{\Delta}_\gamma + \gamma^{-1}I = \mathcal{F}_{2\gamma^{-1}I}$. Analogous to the previous section, we define $\delta A = B\Delta C$ in (1) so that the asymptotic stability of the feedback interconnection of $G(s)$ and Δ is equivalent to the asymptotic stability of (1). Finally, define the shifted transfer function $G_\gamma(s)$ by

$$G_\gamma(s) \triangleq \left(I - \gamma^{-1}G(s)\right)^{-1}G(s)$$

Now, to make connections with structured singular value theory, we state a robust stability criterion involving a frequency-dependent multiplier and a frequency-dependent scaling that account for the structure of the norm-bounded matrices in $\mathbf{\Delta}_\gamma$.

Theorem 2: *Let $G_\gamma(s)$ be asymptotically stable and suppose there exist $m \times m$ real-rational functions $N(s)$ and $Q(s)$ such that the following conditions are satisfied.*

(i) *$N(s)$ and $Q(s)$ have no imaginary poles.*

(ii) *$Q(s) = Q^*(s)$ for all $s = j\omega$.*

(iii) *$\Delta N(s) = N(s)\Delta$ and $\Delta Q(s) = Q(s)\Delta$ for all $\Delta \in \mathbf{\Delta}$ and for all $s = j\omega$.*

(iv) *$\mathrm{He}\, N(s) \ge Q(s) > 0$ for all $s = j\omega$.*

(v) *$\frac{1}{2}\gamma Q(s) + N(s)G_\gamma(s)$ is strictly generalized positive real.*

Then the negative feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \mathbf{\Delta}_\gamma$.

Proof: The result follows from Theorem 1 by letting $M = 2\gamma^{-1}I$ and by noting from figure 1 that asymptotic stability of the negative feedback interconnection of $G_\gamma(s)$ and F for all $F \in \mathcal{F}_M$ is equivalent to asymptotic stability of the feedback interconnection of $G(s)$ and Δ for all $\Delta \in \mathbf{\Delta}_\gamma$. □

5. Convex robust stability tests

In this section, we consider two parametrizations of the multiplier $N(s)$ and the scaling $Q(s)$ which allow the robust stability test given by Theorem 2 to be written as LMIs and evaluated as convex feasibility problems. Specifically, we parametrize the

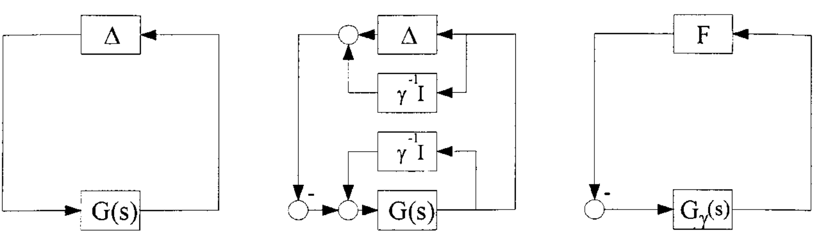

Figure 1. Loop-shifting transformation.

multiplier and scaling as rational functions by considering matrix polynomials and sums of rational functions. For convenience, define the set $\mathcal{X}$ of symmetric matrices X that commute with every element in $\mathbf{\Delta}$ as

$$\mathcal{X} \triangleq \{X : X = \text{block-diag}(X_1 \otimes I_{m_1}, \ldots, X_r \otimes I_{m_r}),\ X_i = X_i^{\mathrm{T}} \in \mathbb{R}^{l_i \times l_i},\ i = 1, \ldots, r\}$$

Hence, if $\Delta \in \mathbf{\Delta}$ and $X \in \mathcal{X}$, then $\Delta X = X\Delta = \text{block-diag}(X_1 \otimes \Delta_1, \ldots, X_r \otimes \Delta_r)$.

The following lemma relates the generalized positive realness of a real-rational function to the feasibility of an LMI.

Lemma 2: *Let $Z(s) = C(sI - A)^{-1}B + D$, where (A, B) is controllable and (A, C) is observable. Then the following statements are equivalent.*

(i) *$Z(s)$ is generalized positive real.*

(ii) *There exist $P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, $\det P \neq 0$, L, and W such that*

$$A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L = 0 \tag{2}$$

$$B^{\mathrm{T}}P + W^{\mathrm{T}}L = C \tag{3}$$

$$D^{\mathrm{T}} + D = W^{\mathrm{T}}W \tag{4}$$

(iii) *There exists $P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, $\det P \neq 0$, such that*

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB - C^{\mathrm{T}} \\ B^{\mathrm{T}}P - C & -(D + D^{\mathrm{T}}) \end{bmatrix} \leq 0 \tag{5}$$

Proof: The equivalence of (i) and (ii) is a standard result, see Anderson and Moore (1968), for example. Next, rewriting (2)–(4) as

$$\begin{bmatrix} A^{\mathrm{T}}P + PA & PB - C^{\mathrm{T}} \\ B^{\mathrm{T}}P - C & -(D + D^{\mathrm{T}}) \end{bmatrix} = -\begin{bmatrix} L^{\mathrm{T}} \\ W^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} L & W \end{bmatrix} \tag{6}$$

implies (5). Conversely, if there exists $P = P^{\mathrm{T}}$, $\det P \neq 0$, such that (5) holds, then there exist L and W such that (6) holds, which implies (ii), as required. □

The following lemma provides a state space realization for the proper series connection of a matrix polynomial and a proper transfer function (Ly *et al.* 1994).

Lemma 3: *Let $G(s) = C(sI - A)^{-1}B + D$ and $X(s) = X_0 + sX_1 + s^2X_2 + \cdots + s^rX_r$, where $X_i \in \mathbb{R}^{m \times m}$, $i = 1, \ldots, r$. If $X(s)G(s)$ is proper, then*

$$X(s)G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline \sum_{i=0}^{r} X_i C A^i & X_0 D + \sum_{i=1}^{r} X_i C A^{i-1} B \end{array}\right] \tag{7}$$

5.1. *Matrix polynomial parametrization*

We first choose $N(s)$ and $Q(s)$ to be polynomial functions of s. Let n be a positive integer, q be a positive even integer, $N_i \in \mathcal{X}$, $i = 0, 1, 2, \ldots, n$, and $Q_j \in \mathcal{X}$, $j = 0, 2, 4, \ldots, q$, and define

$$N(s) \triangleq N_0 + sN_1 + \cdots + s^n N_n = \sum_{i=0}^{n} s^i N_i$$

$$Q(s) \triangleq Q_0 + s^2 Q_2 + \cdots + s^q Q_q = \sum_{\substack{j=0 \\ j \text{ even}}}^{q} s^j Q_j$$

Furthermore, define $N_{\text{even}}(s) \triangleq \sum_{\substack{i=0 \\ i \text{ even}}}^{n} s^i N_i$.

Next, we rewrite the robust stability test of the previous section as a convex feasibility problem by representing the strict generalized positivity of $\frac{1}{2}\gamma Q(s) + N(s)G_\gamma(s)$ and the constraints $N_{\text{even}}(s) \geq Q(s) > 0$ for all $s = \mathrm{j}\omega$ as LMIs. For convenience, let

$$\frac{1}{p^\sim(s)p(s)} I \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right] \tag{8}$$

where $p(s)$ is a polynomial with real coefficients. Next, note that the shifted transfer function $G_\gamma(s) = (I - \gamma^{-1}G(s))^{-1}G(s)$ has the realization $G_\gamma(s) = C(sI - A_\gamma)^{-1}B$, where $A_\gamma = A + \gamma^{-1}BC$ and $G(s) = C(sI - A)^{-1}B$. Finally, define

$$\widetilde{A} \triangleq \begin{bmatrix} A_\gamma & 0 & 0 \\ B_p C & A_p & 0 \\ 0 & 0 & A_p \end{bmatrix}, \qquad \widetilde{B} \triangleq \begin{bmatrix} B \\ 0 \\ B_p \end{bmatrix}$$

$$\widetilde{C} \triangleq \left[\sum_{i=1}^{n}\sum_{k=1}^{i} N_i C_p A_p^{i-k} B_p C A_\gamma^{k-1} \quad \sum_{i=0}^{n} N_i C_p A_p^i \quad \tfrac{1}{2}\gamma \sum_{\substack{j=0 \\ j \text{ even}}}^{q} Q_j C_p A_p^j \right]$$

$$\widetilde{D} \triangleq \sum_{i=2}^{n}\sum_{k=2}^{i} N_i C_p A_p^{i-k} B_p C A_\gamma^{k-2} B + \tfrac{1}{2}\gamma \sum_{\substack{j=2 \\ j \text{ even}}}^{q} Q_j C_p A_p^{j-1} B_p$$

The following proposition is a reformulation of the robust stability test of Theorem 2 using LMIs.

Proposition 1: *Let $\gamma > 0$, $G(s) = C(sI - A)^{-1}B$, and $p(s)$ be a Hurwitz polynomial such that $1/p^\sim(s)p(s)(N_{\text{even}}(s) - Q(s))$ and $\gamma/2p^\sim(s)p(s)Q(s) + 1/p^\sim(s)p(s)N(s)G_\gamma(s)$ are proper, and assume that $G_\gamma(s)$ is asymptotically stable. If there exist $P_Q = P_Q^{\mathrm{T}}$, $P_N = P_N^{\mathrm{T}}$, $P = P^{\mathrm{T}}$, $\varepsilon_Q > 0$, $\varepsilon > 0$, $N_i \in \mathcal{X}$, $i = 0, 1, \ldots, n$, and $Q_j \in \mathcal{X}$, $j = 0, 2, 4, \ldots, q$, such that*

$$\begin{bmatrix} A_p^{\mathrm{T}} P_Q + P_Q A_p & P_Q B_p - \sum_{\substack{j=0 \\ j \text{ even}}}^{q} A_p^{\mathrm{T}j} C_p^{\mathrm{T}} Q_j \\ B_p^{\mathrm{T}} P_Q - \sum_{\substack{j=0 \\ j \text{ even}}}^{q} Q_j C_p A_p^j & \varepsilon_Q I - \sum_{\substack{j=2 \\ j \text{ even}}}^{q} Q_j C_p A_p^{(j-1)} B_p + B_p^{\mathrm{T}} A_p^{\mathrm{T}(j-1)} C_p^{\mathrm{T}} Q_j \end{bmatrix} \leq 0 \tag{9}$$

$$\begin{bmatrix} A_p^{\mathrm{T}} P_N + P_N A_p & \begin{array}{c} P_N B_p - \displaystyle\sum_{\substack{i=0\\ i \text{ even}}}^{n} A_p^{\mathrm{T}i} C_p^{\mathrm{T}} N_i \\ + \displaystyle\sum_{\substack{j=0\\ j \text{ even}}}^{q} A_p^{\mathrm{T}j} C_p^{\mathrm{T}} Q_j \end{array} \\ \begin{array}{c} B_p^{\mathrm{T}} P_N - \displaystyle\sum_{\substack{i=0\\ i \text{ even}}}^{n} N_i C_p A_p^i \\ + \displaystyle\sum_{\substack{j=0\\ j \text{ even}}}^{q} Q_j C_p A_p^j \end{array} & \begin{array}{c} \displaystyle\sum_{\substack{j=2\\ j \text{ even}}}^{q} Q_j C_p A_p^{(j-1)} B_p + B_p^{\mathrm{T}} A_p^{\mathrm{T}(j-1)} C_p^{\mathrm{T}} Q_j \\ - \displaystyle\sum_{\substack{i=2\\ i \text{ even}}}^{n} N_i C_p A_p^{(i-1)} B_p + B_p^{\mathrm{T}} A_p^{\mathrm{T}(i-1)} C_p^{\mathrm{T}} N_i \end{array} \end{bmatrix} \leq 0 \qquad (10)$$

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}} P + P \tilde{A} & \tilde{B}^{\mathrm{T}} P - \tilde{C} \\ \tilde{B} P - \tilde{C}^{\mathrm{T}} & \varepsilon I - (\tilde{D} + \tilde{D})^{\mathrm{T}} \end{bmatrix} \leq 0 \qquad (11)$$

then the feedback interconnection of $G(s)$ *and* Δ *is asymptotically stable for all* $\Delta \in \Delta_\gamma$.

Proof: First, using Lemma 3, it follows that

$$\frac{1}{p^{\sim}(s)p(s)} Q(s) \sim \left[\begin{array}{c|c} A_p & B_p \\ \hline \displaystyle\sum_{\substack{j=0\\ j \text{ even}}}^{q} Q_j C_p A_p^j & \displaystyle\sum_{\substack{j=2\\ j \text{ even}}}^{q} Q_j C_p A_p^{j-1} B_p \end{array} \right]$$

and

$$\frac{1}{p^{\sim}(s)p(s)} (N_{\text{even}}(s) - Q(s))$$

$$\sim \left[\begin{array}{c|c} A_p & B_p \\ \hline \displaystyle\sum_{\substack{i=0\\ i \text{ even}}}^{n} N_i C_p A_p^i + \displaystyle\sum_{\substack{j=0\\ j \text{ even}}}^{q} Q_j C_p A_p^j & \displaystyle\sum_{\substack{i=2\\ i \text{ even}}}^{n} N_i C_p A_p^{i-1} B_p + \displaystyle\sum_{\substack{j=2\\ j \text{ even}}}^{q} Q_j C_p A_p^{j-1} B_p \end{array} \right]$$

Next, using implication (iii) $\Rightarrow$(i) of Lemma 2, (9) and (10) imply that

$$\frac{1}{p^{\sim}(s)p(s)} Q(s) \geq \varepsilon_Q I, \qquad \frac{1}{p^{\sim}(s)p(s)} (N_{\text{even}}(s) - Q(s)) \geq 0$$

for all $s = \mathrm{j}\omega$. Since $p(s)$ is Hurwitz, $p^{\sim}(s)p(s) > 0$ for all $s = \mathrm{j}\omega$, so that $N_{\text{even}}(s) \geq Q(s) > 0$ for all $s = \mathrm{j}\omega$.

Next, note that

$$\frac{\gamma}{2p^{\sim}(s)p(s)} Q(s) + \frac{1}{p^{\sim}(s)p(s)} N(s) G_\gamma(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right]$$

Using implication (iii) $\Rightarrow$(i) of Lemma 2, it follows from (11) that

$$\mathrm{He} \left[\frac{\gamma}{2p^{\sim}(s)p(s)} Q(s) + \frac{1}{p^{\sim}(s)p(s)} N(s) G_\gamma(s) \right] \geq \frac{\varepsilon}{2} I$$

for all $s = \mathrm{j}\omega$. Hence, it follows that $\frac{1}{2}\gamma Q(s) + N(s)G_\gamma(s)$ is strictly generalized positive real. The result follows from Theorem 2 by noting that $N(s)\Delta = \Delta N(s)$ and $Q(s)\Delta = \Delta Q(s)$ for all $\Delta \in \boldsymbol{\Delta}$ and for all $s \in \mathbb{C}$ and that $\operatorname{He} N(s) = \operatorname{He} N_{\text{even}}(s) = N_{\text{even}}(s) \geq Q(s)$ for all $s = \mathrm{j}\omega$. □

5.2. *Rational function parametrization*

We next let $N(s)$ and $Q(s)$ be rational functions involving sums of first-order rational transfer functions. Let n and q be positive integers, $N_i \in \mathcal{X}$, $i = 0, 1, \ldots, n$, $Q_j \in \mathcal{X}$, $j = 0, 1, \ldots, q$, $\beta_i \in \mathbb{R}$, $\beta_i \neq 0$, $i = 0, 1, \ldots, n$ and $\alpha_j \in \mathbb{R}$, $\alpha_j \neq 0$, $j = 0, 1, \ldots, q$ and define

$$N(s) \triangleq N_0 + \frac{1}{s+\beta_1}N_1 + \cdots + \frac{1}{s+\beta_n}N_n$$

$$Q(s) \triangleq Q_0 + \left(\frac{1}{s+\alpha_1} + \frac{1}{-s+\alpha_1}\right)Q_1 + \cdots + \left(\frac{1}{s+\alpha_q} + \frac{1}{-s+\alpha_q}\right)Q_q$$

Next, note that

$$\operatorname{He} N(s) = N_0 + \frac{1}{2}\left(\frac{1}{s+\beta_1} + \frac{1}{-s+\beta_1}\right)N_1 + \cdots + \frac{1}{2}\left(\frac{1}{s+\beta_n} + \frac{1}{-s+\beta_n}\right)N_n$$

for all $s = \mathrm{j}\omega$ and let

$$\operatorname{He} N(s) \sim \left[\begin{array}{c|c} A_N & B_N \\ \hline C_N & D_N \end{array}\right], \quad Q(s) \sim \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$$

and

$$\operatorname{He} N(s) - Q(s) \sim \left[\begin{array}{c|c} A_{NQ} & B_{NQ} \\ \hline C_{NQ} & D_{NQ} \end{array}\right]$$

where

$$A_N \triangleq \begin{bmatrix} -\beta_1 I & & & \\ & \beta_1 I & & \\ & & \ddots & \\ & & & \beta_n I \end{bmatrix}, \quad B_N \triangleq \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}$$

$$C_N \triangleq \left[\tfrac{1}{2}N_1 \quad -\tfrac{1}{2}N_1 \quad \cdots \quad -\tfrac{1}{2}N_n\right], \quad D_N \triangleq N_0$$

$$A_Q \triangleq \begin{bmatrix} -\alpha_1 I & & & \\ & \alpha_1 I & & \\ & & \ddots & \\ & & & \alpha_q I \end{bmatrix}, \quad B_N \triangleq \begin{bmatrix} I \\ I \\ \vdots \\ I \end{bmatrix}$$

$$C_N \triangleq [\,Q_1 \quad -Q_1 \quad \cdots \quad -Q_q\,], \quad D_N \triangleq Q_0$$

$$A_{NQ} = \begin{bmatrix} A_Q & 0 \\ 0 & A_N \end{bmatrix}, \quad B_{NQ} = \begin{bmatrix} B_Q \\ B_N \end{bmatrix}$$

$$C_{NQ} = \begin{bmatrix} -C_Q & C_N \end{bmatrix}, \quad D_{NQ} = D_N - D_Q$$

Furthermore, for convenience, let

$$\tilde{A} \triangleq \begin{bmatrix} A_\gamma & 0 & 0 \\ B_N C & A_N & 0 \\ 0 & 0 & A_Q \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B \\ 0 \\ B_Q \end{bmatrix}$$

$$\tilde{C} \triangleq \begin{bmatrix} D_N C & C_N & \tfrac{1}{2}\gamma C_Q \end{bmatrix}, \quad \tilde{D} \triangleq \tfrac{1}{2}\gamma D_Q$$

Now, we specialize the robust stability test of Theorem 2 using the rational multipliers and scaling defined above.

Proposition 2: *Let $G(s) = C(sI - A)^{-1}B$ and $\gamma > 0$, and assume that $G_\gamma(s)$ is asymptotically stable. If there exist $P_Q = P_Q^{\mathrm{T}}$, $P_{NQ} = P_{NQ}^{\mathrm{T}}$, $P = P^{\mathrm{T}}$, $\varepsilon_Q > 0$, $\varepsilon > 0$, $N_i \in \mathcal{X}$, $i = 0, 1, \ldots, n$, and $Q_j \in \mathcal{X}$, $j = 0, 1, \ldots, q$, such that*

$$\begin{bmatrix} A_Q^{\mathrm{T}} P_Q + P_Q A_Q & B_Q^{\mathrm{T}} P_Q - C_Q \\ B_Q P_Q - C_Q^{\mathrm{T}} & \varepsilon_Q I - (D_Q + D_Q^{\mathrm{T}}) \end{bmatrix} \le 0 \tag{12}$$

$$\begin{bmatrix} A_{NQ}^{\mathrm{T}} P_{NQ} + P_{NQ} A_{NQ} & B_{NQ}^{\mathrm{T}} P_{NQ} - C_{NQ} \\ B_{NQ} P_{NQ} - C_{NQ}^{\mathrm{T}} & -(D_{NQ} + D_{NQ}^{\mathrm{T}}) \end{bmatrix} \le 0 \tag{13}$$

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}} P + P\tilde{A} & \tilde{B}^{\mathrm{T}} P - \tilde{C} \\ \tilde{B}P - \tilde{C}^{\mathrm{T}} & \varepsilon I - (\tilde{D} + \tilde{D}^{\mathrm{T}}) \end{bmatrix} \le 0 \tag{14}$$

then the feedback interconnection of $G(s)$ and Δ is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}_\gamma$.

Proof: First note that

$$\tfrac{1}{2}\gamma Q(s) + N(s) G_\gamma(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

Hence, using implication (iii) $\Rightarrow$ (i) of Lemma 2, (12) and (13) imply that $\operatorname{He} N(s) \ge Q(s) > 0$, while (14) implies that $\tfrac{1}{2}\gamma Q(s) + N(s) G_\gamma(s)$ is strictly generalized positive real. The result follows by applying Theorem 2. □

6. Peak structured singular value bounds

In this section, we use Propositions 1 and 2 to recast the peak upper bounds of the structured singular value as convex feasibility tests. Recall that the structured singular value of $G(\mathrm{j}\omega)$ for real, block-structured uncertainty is defined as

$$\mu(G(\mathrm{j}\omega)) \triangleq \frac{1}{\min\{\sigma_{\max}(\Delta) : \Delta \in \boldsymbol{\Delta}_0\}}$$

where

$$\Delta_0 \triangleq \{\Delta \in \boldsymbol{\Delta}\colon \ \det(I + G(\mathrm{j}\omega)\Delta) = 0\}$$

If Δ_0 is empty, then $\mu(G(\mathrm{j}\omega)) \triangleq 0$. Hence, the real structured singular value $\mu(G(\mathrm{j}\omega))$ is the inverse of the smallest real perturbation having the specified block structure that moves a pole to $\mathrm{j}\omega$ on the imaginary axis.

The peak value of $\mu(G(\mathrm{j}\omega))$ over frequency provides a measure of the smallest destabilizing perturbation having the specified block structure. However, it is often difficult to compute reliably the peak value of the frequency-domain upper bound by frequency gridding since sharp peaks or discontinuities may be overlooked (Sparks and Bernstein 1998). Hence, it is preferable to compute the peak of the upper bound directly.

Using Proposition 1, we can define

$$\begin{aligned}\bar{\mu}_1 = \inf\{\gamma > 0\colon \ & \text{there exist } N_i, Q_j \in \mathcal{X},\ i = 0,1,2,\ldots,n,\ j = 0,2,4,\ldots,q,\\ & P_Q = P_Q^{\mathrm{T}}, P_N = P_N^{\mathrm{T}}, P = P^{\mathrm{T}}, \varepsilon_Q > 0,\\ & \text{and } \varepsilon > 0 \text{ such that (9), (10), (11) hold}\}\end{aligned}$$

while using Proposition 2 we can define

$$\begin{aligned}\bar{\mu}_2 = \inf\{\gamma > 0\colon \ & \text{there exist } N_i, Q_j \in \mathcal{X},\ i = 0,1,\ldots,n,\ j = 0,1,\ldots,q,\ P_Q = P_Q^{\mathrm{T}},\\ & P_{NQ} = P_{NQ}^{\mathrm{T}}, P = P^{\mathrm{T}}, \varepsilon_Q > 0,\\ & \text{and } \varepsilon > 0 \text{ such that (12), (13), (14) hold}\}\end{aligned}$$

Theorem 2 guarantees asymptotic stability of an uncertain system for norm-bounded, structured uncertainty. Using the definition of the structured singular value, it follows that the peak structured singular value over frequency is bounded by the norm bound of Theorem 2. Since Propositions 1 and 2 are tests that utilize a particular form of the multiplier and the scaling to satisfy the conditions in Theorem 2, it thus follows that

$$\mu(G(\mathrm{j}\omega)) \leq \bar{\mu}_1, \bar{\mu}_2$$

for all $\omega \in \mathbb{R}$.

7. Numerical examples

We now present three numerical examples to demonstrate the robust stability tests of Propositions 1 and 2 by computing $\bar{\mu}_1$ and $\bar{\mu}_2$. Each peak upper bound is computed by using an interior point technique (Boyd *et al.* 1994) to solve the LMIs in a bisection search for the smallest γ. For each value of γ, the appropriate system of LMIs, (9)–(11) or (12)–(14), is checked for feasibility. The value of γ is then increased or reduced to determine the smallest γ such that the system of LMIs is feasible.

7.1. *Example* 1

Consider the plant (Balakrishnan *et al.* 1995)

	$n = 0$	1	2
$q = 0$	4.8027	4.5491	4.5004
1	4.8027	4.1435	4.1105
2	4.8027	4.1435	4.0988

Table 1. Peak upper bounds $\bar{\mu}_2$ for Example 1.

$$G(s) = \begin{bmatrix} 2 & \dfrac{-10s-8}{5(s+1)} \\ \dfrac{-2s+8}{s+1} & 2 \end{bmatrix}$$

in a feedback interconnection with the uncertain matrix $\Delta = \mathrm{diag}(\delta_1, \delta_2)$. The problem is to determine the largest norm bound on the uncertain matrix Δ for which the system is guaranteed to be asymptotically stable.

Using LMIs (9)–(11), $\bar{\mu}_1$ was computed as 4.0988 using a multiplier and scaling with $n = q = 2$ and $p(s) = s + 1$. Then, using LMIs (12)–(14), $\bar{\mu}_2$ was computed for multipliers and scalings with $n = 0, 1, 2$ and $q = 0, 1, 2$ and $\alpha_i = \beta_i = -i$. The results appear in table 1. Note that as the order of the multiplier or scaling increases, the conservatism of $\bar{\mu}_2$ decreases. However, letting the number of scaling terms be greater than the number of multiplier terms yields no improvement.

In Balakrishnan *et al.* (1995), the authors obtained a robust stability bound of 4.1425 for this example using a second-order multiplier. Hence, by exploiting frequency-dependent scaling, the upper bounds $\bar{\mu}_1$ and $\bar{\mu}_2$ provide a less conservative robust stability measure than the technique of Balakrishnan *et al.* (1995) for this example.

7.2. *Example* 2

Next, consider the plant of Haddad *et al.* (1994)

$$\dot{x} = \begin{bmatrix} -2 & -400 & 0.1 & 0.2 \\ 1 & 0 & 0.5 & 0 \\ 0 & 2 & -3 & -80 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 & 0.8 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1.5 & 0 & 1 & 0 \\ 0 & 1 & 2 & 2 \end{bmatrix} x$$

in a feedback interconnection with the uncertain matrix $\Delta = \mathrm{diag}(\delta_1, \delta_2)$.

As before, $\bar{\mu}_1$ was computed for multipliers $N(s)$ and scalings $Q(s)$ of various orders. Specifically, a peak upper bound of 2.7176 was computed using constant scaling and a multiplier with $n = 1$ and $p(s) = 1$, of 2.2336 using constant scaling and a multiplier with $n = 3$ and $p(s) = s + 1$, of 1.9952 using scaling and a multiplier with $n = q = 2$ and $p(s) = s + 1$, and of 1.6930 using scaling and a multiplier with $n = 3$, $q = 2$, and $p(s) = s + 1$. The peak upper bounds are shown in figure 2 along with the frequency-domain upper bound $\mu(G(j\omega))$.

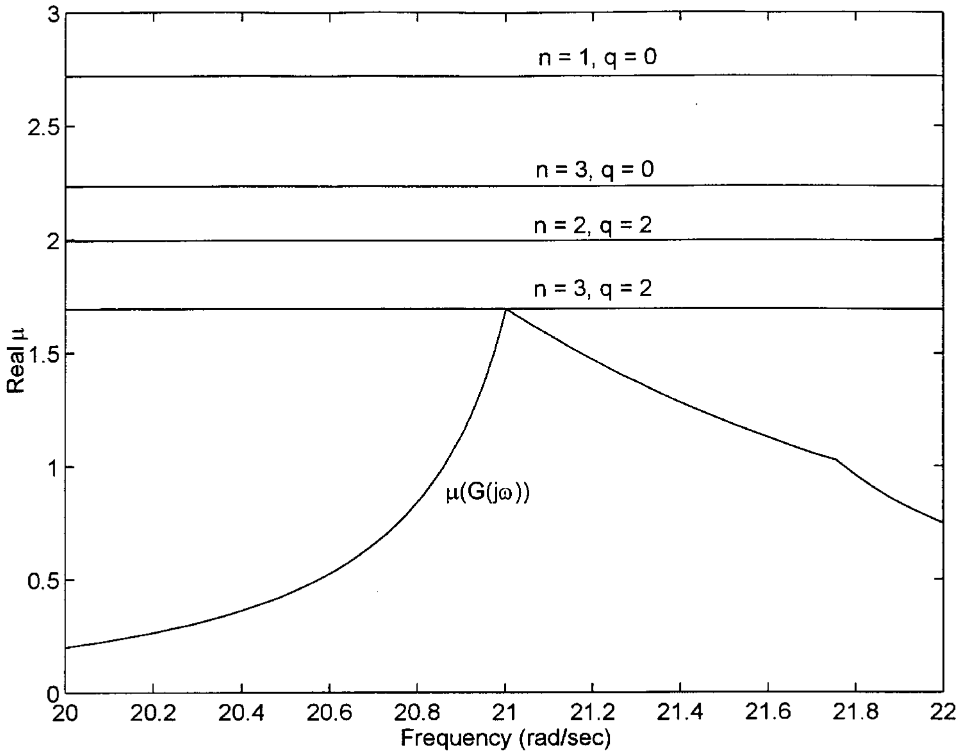

Figure 2. Peak upper bounds $\bar{\mu}_1$ for Example 2.

	$n = 0$	1	2	3
$q = 0$	3.0866	2.8160	2.6655	2.6258
1	3.0866	1.9817	1.9694	1.9321
2	3.0866	1.9817	1.8724	1.8248
3	3.0866	1.9817	1.8724	1.7242

Table 2. Peak upper bounds $\bar{\mu}_2$ for Example 2.

Next, $\bar{\mu}_2$ was computed using LMIs (12)–(14) with multipliers and scaling with $\alpha_i = \beta_i = -i$ and the results are shown in table 2. As in the previous example, the conservatism of the robust stability test is reduced by increasing the number of multiplier and scaling terms, and there is no advantage to increasing the order of the scaling to be greater than the order of the multiplier.

7.3. *Example* 3

Finally, consider the plant $G = C(sI - A)^{-1}B$, where (Haddad *et al.* 1994)

$$A = \text{block-diag}\left(\begin{bmatrix} -4 & -7 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1.5 & -4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & -2.5 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 1 & 0 \end{bmatrix}\right)$$

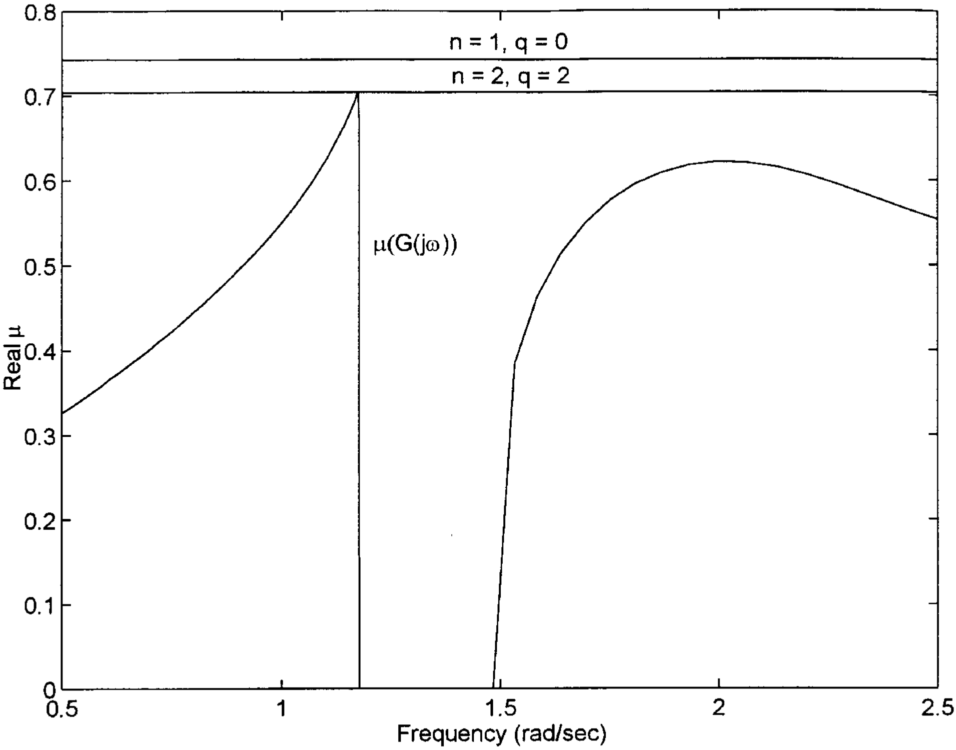

Figure 3. Peak upper bounds $\bar{\mu}_1$ for Example 3.

	$n = 0$	1	2
$q = 0$	0.8679	0.8177	0.7034
1	0.8679	0.7172	0.7034
2	0.8679	0.7172	0.7034

Table 3. Peak upper bounds $\bar{\mu}_2$ for Example 3.

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad C = \begin{bmatrix} 0 & 1 & 2.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 & 0 & 1 \end{bmatrix}$$

in a feedback interconnection with the uncertain matrix $\Delta = \mathrm{diag}(\delta_1, \delta_2)$. Peak upper bounds $\bar{\mu}_1$ and $\bar{\mu}_2$ were computed using multipliers and scaling of different orders, and the results are shown in table 3 and figure 3.

8. Summary and conclusions

Robust stability tests were given for systems with block-structured, sector- and norm-bounded uncertainty. The robust stability tests each used a frequency-dependent stability multiplier and scaling to reduce conservatism. The robust stability tests for block-structured, norm-bounded uncertainty yield upper bounds for the peak of the structured singular value over frequency. To translate the robustness tests into a form that is easily computable, the frequency-dependent multiplier and scaling were parametrized first as matrix polynomials with arbitrary polynomial denominators and then as sums of first-order rational matrix functions. In both cases, the resulting tests were written as linear matrix inequalities, and thus were evaluated as convex feasibility problems. Numerical examples showed the reduction in conservatism achieved by increasing the number of terms in the parametrizations of the multiplier and scaling, and the improvement in the robust stability tests by including both a stability multiplier and a scaling matrix.

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