Controller Design with Regional Pole Constraints

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Abstract—**A design procedure is developed that combines linear-quadratic optimal control with regional pole placement. Specifically, a static and dynamic output-feedback control problem is addressed in which the poles of the closed-loop system are constrained to lie in specified regions of the complex plane. These regional pole constraints are embedded within the optimization process by replacing the covariance Lyapunov equation by a modified Lyapunov equation whose solution, in certain cases, leads to an upper bound on the quadratic cost functional. The results include necessary and sufficient conditions for characterizing static output-feedback controllers with bounded performance and regional pole constraints. Sufficient conditions are also presented for the fixed-order (i.e., full- and reduced-order) dynamic output-feedback problem with regional pole constraints. The paper considers circular, elliptical, vertical strip, parabolic, and sector regions.**

I. Introduction

ONE of the fundamental problems in control theory and practice is the design of feedback laws that place the closed-loop poles at desired locations. Much of the pole placement literature focuses on the problem of *exact* pole placement in which closed-loop poles are required to lie at (or arbitrarily close to) prescribed locations. Of course, it is well known that a feedback controller of a given structure may offer design flexibility beyond pole placement alone. Hence the designer may also specify other closed-loop characteristics such as eigenvectors [1]. The present paper, however, is confined to the pole placement problem.

Several pole placement schemes exploit the properties of linear-quadratic regulator theory to move poles to desired locations. For example, by utilizing known relationships between control weightings and asymptotic pole locations in the limit of cheap control, it is possible to select control weightings to achieve certain pole configurations [2]. A different scheme, developed in [3], [4], uses the structure of the Hamiltonian matrix to modify the cost weightings to arbitrarily place the real parts of the closed-loop poles.

It is often the case in practice, however, that exact closed-loop pole locations are not required. Rather, it may suffice to place the closed-loop eigenvalues within a prescribed region in the left-half plane. Beyond the constraint that the poles lie within the given region, the remaining design flexibility can then be used to minimize a performance functional that is relevant to the steady-state aspect of the design. Perhaps the simplest illustration of this idea is the shift technique for achieving a uniform stability margin [5]. By replacing the open-loop dynamics matrix A by $A + \alpha I$, $\alpha > 0$, each closed-loop pole is guaranteed to have real part less than $-\alpha$. The pole-constraint region for this problem is thus $\{\lambda \in \mathbb{C}\colon \operatorname{Re} \lambda < -\alpha\}$.

More general pole constraint regions can also be considered. For the purpose of *analysis*, there is extensive literature concerning tests for determining root clustering, that is, whether a given polynomial or matrix has all of its roots or eigenvalues within a specified region [6]–[11]. For controller *synthesis*, the regional pole placement problem thus involves determining a feedback controller that minimizes a cost functional subject to the requirement that the closed-loop poles lie within a specified pole constraint region. In recent years this problem has been considered for a variety of pole constraint regions, including vertical and horizontal strips, sectors, circles, and hyperbolas [12]–[22]. For each region the basic idea involves constructing an analytic map that transforms the constraint region into the open left-half plane. The structure of this map, which takes the form of a modified Lyapunov equation, then leads to algebraic equations for synthesizing feedback gains for pole placement.

The contribution of the present paper is to extend the regional pole placement approach of [12]–[22] in several ways. First, for several different pole constraint regions we provide an analysis of the modified Lyapunov equation that characterizes the pole constraint regions. By showing that the modified Lyapunov equation is both necessary and sufficient for characterizing the pole constraint region, we provide a more complete foundation for the regional pole placement approach.

Next we show that, in certain cases, the modified Lyapunov equation leads directly to an overbound on the cost functional. By minimizing this "auxiliary cost" we obtain explicit feedback gain expressions for a controller that places the closed-loop poles within the specified pole-constraint region. By minimizing the auxiliary cost, the resulting controller provides a guaranteed upper bound on the original quadratic cost functional.

Finally, the present paper goes beyond earlier work [12]–[22] by providing greater realism with respect to the

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availability of plant measurements and the types of controllers that can be designed. Although the literature on exact pole placement addresses the output feedback problem via both static and dynamic controllers, the available regional pole placement results are confined to static full-state feedback controllers. We remove this constraint by considering the design of static *output*-feedback controllers as well as both full- and reduced-order dynamic output-feedback compensators.

To motivate the regional pole placement approach consider the shifted and "clipped" sector $\mathcal{S}(\mu, \beta, \theta)$ shown in Fig. 1. If the closed-loop poles are confined to this region, then the system modes damp asymptotically at desired rates. Unfortunately, however, it is difficult to address this region directly with current techniques. Hence, as in [12]–[22], we consider a variety of regions that approximate $\mathcal{S}(\mu, \beta, \theta)$, namely, circular, elliptic, parabolic, and vertical strip regions as well as an "unclipped" sector, that is, $\mathcal{S}(\mu, \beta, \theta)$ with $\beta = 0$.

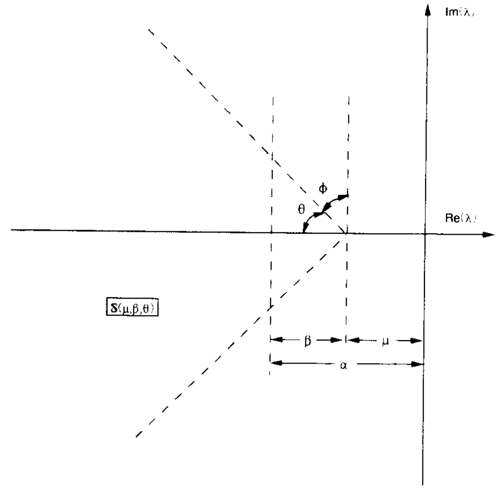

Fig. 1. Sector region $\mathcal{S}(\mu, \beta, \theta)$.

In this paper, we first consider the circular pole constraint region since it is characterized by the simplest modified Lyapunov equation, it leads to an upper bound for the cost, and, in the full-order dynamic compensation case, separation holds. The circular pole constraint region is also easy to treat numerically since it is possible to exploit connections to standard discrete-time design. Having established the approach, we extend the results of the first part of the paper by substituting alternative regions for the circular region and carrying through the steps of the development. It should be noted, however, that the circular pole constraint region has practical value for a variety of reasons. Besides placing an upper bound on the damping ratio ζ, it bounds the natural frequency ω_n and damped natural frequency ω_d. Hence the circular pole constraint region can be used to enforce a variety of practical design specifications.

The content and scope of the paper are as follows. In Section II, we state the static output feedback control problem with regional pole constraints. In Section III, we specialize to the circular pole constraint region and relate the parameters of the circular region to system design parameters such as damping ratio, natural frequency, and damped natural frequency. In Section IV, we show that a sufficient condition involving Kronecker products implies the existence of a unique nonnegative-definite solution to a modified Lyapunov equation that guarantees pole placement within a circular region in the open left-half plane. Section V presents the first-order necessary conditions (Theorem 5.2) for the Auxiliary Minimization Problem. These necessary conditions are in the form of coupled Riccati/Lyapunov equations that characterize static output-feedback controllers. As a special case of Theorem 5.2, we show that in the full-state feedback case our results include those obtained in [16–18]. A partial converse of the necessary conditions shows that solutions of these algebraic equations provide, by construction, a solution to the original modified Lyapunov equation. This result is combined in Theorem 5.3 with a disturbability assumption to guarantee that the poles of the closed-loop system lie within a circular region and that an optimized bound on the closed-loop performance is satisfied. Theorem 5.4 gives an existence result for the Auxiliary Minimization Problem which shows that our *sufficient* conditions are also *necessary* for the static output-feedback problem with pole constraints. In Sections VI and VII, we extend the results of the first part of this paper to fixed-order (i.e., full- and reduced-order) dynamic compensators with pole constraints. As in LQG theory, the full-order control problem with pole constraints involves a system of two separated Riccati equations which shows that regulator/estimator separation holds. To illustrate the results we develop a numerical algorithm for the design equations in Section VIII and apply the algorithm to an illustrative example. In Sections IX and X, we apply the results of the first part of the paper to elliptic, parabolic, and vertical strip regions via static and full-order dynamic output-feedback controllers. In Sections XI, we represent an alternative design approach that captures pole placement within the sector region $\mathcal{S}(\mu, \beta, \theta)$. Finally, in Section XII we numerically solve the design equations for the sector region and demonstrate the approach on a lightly damped flexible beam structure.

Notation

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$.
$\mathbb{C}$, $\mathbb{C}^{r\times s}$, $\mathbb{C}^r$	Complex numbers, $r \times s$ complex matrices, $\mathbb{C}^{r\times 1}$.
$\mathbb{E}$, tr, $0_{r\times s}$	Expectation, trace, $r \times s$ zero matrix.
I_r, $(\ )^{\mathrm{T}}$, $(\ )^*$	$r \times r$ identity, transpose, complex conjugate transpose.
$\otimes$, $\oplus$, U_n	Kronecker product, Kronecker sum, $n \times n$ permutation matrix.
$\sigma(A)$, ∂, $\bar{\lambda}$	Spectrum of A, boundary, complex conjugate of $\lambda \in \mathbb{C}$.
$n, m, l, n_c, d; \tilde{n}$	Positive integers; $n + n_c (n_c \le n)$.
$x, u, y, x_c, \tilde{x}$	$n, m, l, n_c, \tilde{n}$-dimensional vectors; $\tilde{x} = [x^{\mathrm{T}} x_c^{\mathrm{T}}]^{\mathrm{T}}$.

$w(\cdot)$	d-dimensional standard white noise process.
A, B, C, D, D_1, D_2	$n \times n, n \times m, l \times n, n \times d, n \times d, l \times d$ matrices.
$A_c, B_c, C_c, K;\ A_0$	$n_c \times n_c, n_c \times l, m \times n_c, m \times l$ matrices; $A + BKC$.
V, V_1, V_{12}, V_2	$DD^{\mathrm{T}}, D_1D_1^{\mathrm{T}}, D_1D_2^{\mathrm{T}}, D_2D_2^{\mathrm{T}}$; $V_2 > 0$.
V_e	$V_1 - V_{12}B_c^{\mathrm{T}} - B_cV_{12}^{\mathrm{T}} + B_cV_2B_c^{\mathrm{T}}$.
$\tilde{D}, \tilde{V}$	$\begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}, \begin{bmatrix} V_1 & V_{12}B_c^T \\ B_cV_{12}^T & B_cV_2B_c^T \end{bmatrix} = \tilde{D}\tilde{D}^{\mathrm{T}}$.
R_1, R_2	$n \times n, m \times m$ state and control weightings; $R_1 \geq 0, R_2 > 0$.
R_{12}	$n \times m$ cross weighting; $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geq 0$.
R_0	$R_1 + R_{12}KC + C^{\mathrm{T}}K^{\mathrm{T}}R_{12}^{\mathrm{T}} + C^{\mathrm{T}}K^{\mathrm{T}}R_2KC$.
R_r	$R_1 + R_{12}C_c + C_c^{\mathrm{T}}R_{12}^{\mathrm{T}} + C_c^{\mathrm{T}}R_2C_c$.
$\alpha;\ q, r$	Nonnegative constant; positive constants.
$A_\alpha, A_{0\alpha}, A_{c\alpha}$	$A + \alpha I_n, A_0 + \alpha I_n, A_c + \alpha I_{n_c}$.
$A_{e\alpha}, A_{r\alpha}$	$A_\alpha - B_cC, A_\alpha + BC_c$.
$\tilde{A}, \tilde{A}_\alpha, \tilde{R}$	$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \tilde{A} + \alpha I_{\tilde{n}} = \begin{bmatrix} A_\alpha & BC_c \\ B_cC & A_{c\alpha} \end{bmatrix}, \begin{bmatrix} R_1 & R_{12}C_c \\ C_c^TR_{12}^T & C_c^TR_2C_c \end{bmatrix}$.

II. Static Output Feedback Control With Pole Constraints

In this section, we introduce a static output-feedback control problem in which the poles of the closed-loop system are constrained to lie in a specified region in the open left-half plane.

Static Output Feedback Control Problem with Pole Constraints

For the nth-order system

$$\dot{x}(t) = Ax(t) + Bu(t) + Dw(t), \qquad t \in [0, \infty), \tag{2.1}$$

$$y(t) = Cx(t) \tag{2.2}$$

design a static output-feedback control law

$$u(t) = Ky(t) \tag{2.3}$$

that satisfies the following design criteria:

1) the poles of the closed-loop system are constrained to lie within an open region $\mathscr{R}$ contained in the open left-half plane; and

2) the performance functional

$$J(K) \triangleq \lim_{t\to\infty} \mathbb{E}\frac{1}{t}\int_0^t [x^{\mathrm{T}}(t)R_1x(t) + 2x^{\mathrm{T}}(t)R_{12}u(t) + u^{\mathrm{T}}(t)R_2u(t)]\,dt \tag{2.4}$$

is minimized.

Note that the closed-loop system (2.1)–(2.3) is given by

$$\dot{x}(t) = A_0x(t) + Dw(t) \tag{2.5}$$

where $A_0 \triangleq A + BKC$, and that (2.4) can be written as

$$J(K) = \lim_{t\to\infty} \mathbb{E}[x^{\mathrm{T}}(t)R_0x(t)]. \tag{2.6}$$

As noted in 1), we focus on open regions $\mathscr{R}$ that are subsets of the open left-half plane. Accordingly, we can define the open set

$$\mathscr{K}_{\mathscr{R}} \triangleq \{K: \sigma(A_0) \subset \mathscr{R}\}$$

of feedback gains K that place the closed-loop poles in $\mathscr{R}$. Note that if $K \in \mathscr{K}_{\mathscr{R}}$, then A_0 is asymptotically stable.

Of course it is possible that for certain problems $\mathscr{K}_{\mathscr{R}}$ is empty, that is, there do not exist feedback gains that are able to place the closed-loop poles in $\mathscr{R}$. Thus it is desirable to have necessary and sufficient conditions for determining whether $\mathscr{K}_{\mathscr{R}}$ is not empty. In the least restrictive case in which $\mathscr{R}$ is the open left-half plane, this is precisely the output-feedback stabilization problem. Unfortunately, even in this case a complete solution is not currently available, although partial results can be found in [23], [24]. For general regions $\mathscr{R}$, results on exact pole placement can be used to show that $\mathscr{K}_{\mathscr{R}}$ is not empty. Thus we shall assume that $\mathscr{K}_{\mathscr{R}}$ is not empty and examine the consequences of optimality. As will be seen, this approach is effective in practice since if $\mathscr{K}_{\mathscr{R}}$ is not empty and the optimization problem has a solution, then optimal feedback gains in $\mathscr{K}_{\mathscr{R}}$ can be determined by solving the optimality conditions.

Finally, if $\mathscr{K}_{\mathscr{R}}$ is not empty and $K \in \mathscr{K}_{\mathscr{R}}$, then, as noted above, A_0 is asymptotically stable. In this case the performance (2.4) is given by

$$J(K) = \operatorname{tr} Q_0R_0 \tag{2.7}$$

where the $n \times n$ nonnegative-definite steady-state covariance defined by

$$Q_0 \triangleq \lim_{t\to\infty} \mathbb{E}[x(t)x^{\mathrm{T}}(t)] \tag{2.8}$$

satisfies the algebraic Lyapunov equation

$$0 = A_0Q_0 + Q_0A_0^{\mathrm{T}} + V. \tag{2.9}$$

III. The Circular Region

We now give a concrete form to the region $\mathscr{R}$. In the first part of this paper, we consider the circular region $\mathscr{C}(q, r)$ with center at $-q$ and radius $r \leq q$ (Fig. 2). Some observations concerning $\mathscr{C}(q, r)$ are worth noting. For simplicity, in the sector region $\mathscr{S}(\mu, \beta, \theta)$ let $\mu = 0$ so that $\alpha = \beta$. Furthermore, let $\lambda = -\zeta\omega_n \pm j\omega_d$ be a complex pole, where ζ is the damping ratio, $0 \leq \zeta \leq 1$, $\omega_n = |\lambda|$ is the natural (undamped) frequency, and $\omega_d \triangleq \omega_n\sqrt{1 - \zeta^2}$ is the damped natural frequency. Then if $\lambda \in \mathscr{C}(q, r)$ it follows that $\zeta \geq \sqrt{1 - (r/q)^2}$, $\omega_d \leq r$, $q - r \leq \omega_n \leq q + r$, and $q - r \leq \zeta\omega_n \leq q + r$.

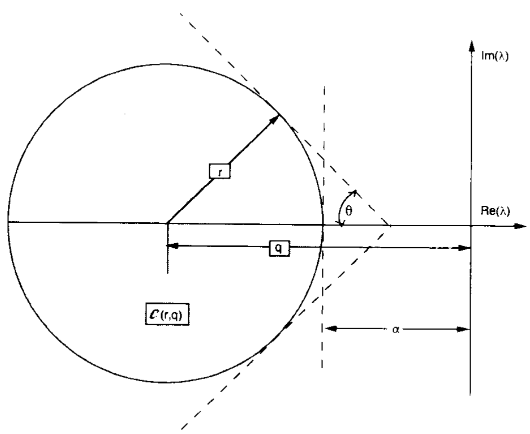

Fig. 2. Circular region $\mathscr{C}(q, r)$.

In practice, however, design specifications are often given in terms of $\zeta \geq \zeta_{\min}$ and $\zeta\omega_n \geq \alpha$. Such specifications will be satisfied by $\mathscr{C}(q, r)$ if $q - r \geq \alpha$ and $\sqrt{1-(r/q)^2} \geq \zeta_{\min}$ or, equivalently

$$r \leq \min\{q-\alpha,\ q\sqrt{1-\zeta_{\min}^2}\}. \tag{3.1}$$

Specifications involving $\zeta_{\min}$ and α are usually expressed in terms of the sector $\mathscr{S}(0, \beta, \theta)$, where $\theta = \cos^{-1}(\zeta_{\min})$ (Fig. 1). We note, however, that the disk $\mathscr{C}(q, r)$ will be contained in the sector $\mathscr{S}(0, \beta, \theta)$ and will be tangent to the sector boundary at three points if

$$q = \frac{\alpha}{1-\sin\theta}, \tag{3.2}$$

$$r = \frac{\alpha \sin\theta}{1-\sin\theta}. \tag{3.3}$$

Hence different values of q and r can be chosen to enforce different bounds on the damping ratio, natural frequency, and damped natural frequency of the closed-loop system. It is also useful to note that the circular region can be obtained by an affine map of the unit disk centered at the origin, that is, a matrix A is discrete-time stable if and only if the eigenvalues of $1/r(A + qI_n)$ lie in $\mathscr{C}(q, r)$. Although this transformation could be utilized as in [16], [22] to obtain some of the results in subsequent sections, the *meaning* of results obtained in this manner is obscure because of the discrete-time origin of the resulting continuous-time controllers.

IV. Necessary and Sufficient Conditions for Pole Placement Within a Circular Region

The key step in constraining the closed-loop poles to lie within the circular region $\mathscr{C}(q, r)$ is to replace the Lyapunov equation (2.9) by a modified Lyapunov equation whose solution overbounds the closed-loop steady-state covariance. Justification for this technique is provided by Theorem 4.1 and Theorem 4.2, the main results of this section. However, before stating these results, it is necessary to introduce the following series of preliminary results.

Recall from [25] that $\lambda \in \sigma(A)$ is a *B-controllable eigenvalue of A* if

$$\operatorname{rank}[A - \lambda I_n,\ B] = n. \tag{4.1}$$

This definition is useful in obtaining an analytical characterization of stabilizability and controllability in terms of the individual modes of A. We recall the following results.

Proposition 4.1: The pair (A, B) is stabilizable if and only if every eigenvalue of A in the closed right-half plane is B-controllable. Furthermore, the pair (A, B) is controllable if and only if every eigenvalue of A is B-controllable.

Proof: See [25]. □

The following definition generalizes the notion of stabilizability to an arbitrary open region in the left-half plane.

Definition 4.1: The pair (A, B) is *assignable with respect to the region* $\mathscr{R}$ if every eigenvalue of A that is not in $\mathscr{R}$ is B-controllable.

As noted previously, necessary and sufficient conditions do not currently exist for determining whether $\mathscr{K}_{\mathscr{R}}$ is nonempty even if $\mathscr{R}$ is the entire open left-half plane. However, in the full-state feedback case the problem is solved completely in terms of the assignability of (A, B).

Proposition 4.2: Suppose $C = I_n$. Then (A, B) is assignable with respect to $\mathscr{R}$ if and only if $\mathscr{K}_{\mathscr{R}}$ is not empty.

Proof: (A, B) is assignable with respect to $\mathscr{R}$ if and only if each eigenvalue of A that is not in $\mathscr{R}$ is B-controllable. It follows from standard results that an eigenvalue of A is B-controllable if and only if it can be placed arbitrarily, in which case there exists K such that $\sigma(A + BK) \subset \mathscr{R}$. □

It is useful to provide the following alternative characterizations of the circular region $\mathscr{C}(q, r)$.

Proposition 4.3: Let $q \geq r > 0$, define $\alpha \triangleq q - r$, and let $\lambda \in \mathbb{C}$. Then the following are equivalent:

$$\lambda \in \mathscr{C}(q, r), \tag{4.2}$$

$$|\lambda + q| < r, \tag{4.3}$$

$$2\operatorname{Re}(\lambda + \alpha) + \frac{1}{r}|\lambda + \alpha|^2 < 0, \tag{4.4}$$

$$\lambda + \bar{\lambda} + 2\alpha + \frac{1}{r}\left[\lambda\bar{\lambda} + (\lambda + \bar{\lambda})\alpha + \alpha^2\right] < 0. \tag{4.5}$$

Remark 4.1: Note that if $\sigma(A_0) \subset \mathscr{C}(q, r)$, then both A_0 and $A_{0\alpha} \triangleq A_0 + \alpha I_n$ are asymptotically stable.

Next, we characterize matrices whose eigenvalues lie in $\mathscr{C}(q, r)$ in terms of a modified Lyapunov equation. To do this we first introduce the $n^2 \times n^2$ matrix

$$\mathscr{A} \triangleq A_{0\alpha} \oplus A_{0\alpha} + \frac{1}{r} A_{0\alpha} \otimes A_{0\alpha} \tag{4.6}$$

where $\alpha \triangleq q - r$, $\otimes$ denotes Kronecker product [26], and $A_{0\alpha} \oplus A_{0\alpha} \triangleq A_{0\alpha} \otimes I_n + I_n \otimes A_{0\alpha}$ is the Kronecker sum. The following result relates the spectrum of A_0 to the spectrum of $\mathscr{A}$.

Proposition 4.4: Let $q \geq r > 0$ and define $\alpha \triangleq q - r$. If $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of A_0, then the n^2 eigenvalues of $\mathscr{A}$ are given by

$$\lambda_i + \lambda_j + 2\alpha + \frac{1}{r}(\lambda_i + \alpha)(\lambda_j + \alpha), \qquad i, j = 1, \cdots, n. \tag{4.7}$$

Proof: The result can be seen by transforming A_0 into its Jordan form. Alternatively, the result follows from a more general result of Stephanos quoted in [9]. □

The following result shows that the stability of $\mathscr{A}$ is both necessary and sufficient for determining whether the eigenvalues of A_0 lie in $\mathscr{C}(q, r)$. Recall that the condition $K \in \mathscr{K}_{\mathscr{C}(q,r)}$ is equivalent to $\sigma(A_0) \subset \mathscr{C}(q, r)$.

Proposition 4.5: Let $K \in \mathbb{R}^{m \times l}$, let $q \geq r > 0$, and define $\alpha \triangleq q - r$. Then $\mathscr{A}$ is asymptotically stable if and only if $\sigma(A_0) \subset \mathscr{C}(q, r)$.

Proof: Suppose $\mathscr{A}$ is asymptotically stable and let $\lambda \in \sigma(A_0)$. Then, with $\lambda_i = \lambda$ and $\lambda_j = \bar{\lambda}$, it follows from Proposition 4.4 that $\mu \triangleq \lambda + \bar{\lambda} + 2\alpha + 1/r(\lambda + \alpha)(\bar{\lambda} + \alpha)$ is an eigenvalue of $\mathscr{A}$. Since $\mathscr{A}$ is asymptotically stable, it follows that μ (which is real) is negative. Noting (4.5), it follows from Proposition 4.3 that $\lambda \in \mathscr{C}(q, r)$, as required. Conversely, suppose each eigenvalue $\lambda_1, \cdots, \lambda_n$ of A_0 is contained in $\mathscr{C}(q, r)$. Then by (4.3) of Proposition 4.3, $|\lambda_i + q| < r$, $i = 1, \cdots, n$. Hence, with $\theta_{ij} \triangleq \arg[(\lambda_i + q)(\lambda_j + q)]$, it follows that

$$\begin{aligned}
\operatorname{Re}\left[\lambda_i + \lambda_j + 2\alpha + \frac{1}{r}(\lambda_i + \alpha)(\lambda_j + \alpha)\right]
&= \operatorname{Re}\left[(\lambda_i + q)(\lambda_j + q)\right] - r^2 \\
&= |\lambda_i + q|\,|\lambda_j + q| \cos\theta_{ij} - r^2 \\
&\leq |\lambda_i + q|\,|\lambda_j + q|\,|\cos\theta_{ij}| - r^2 \\
&< r^2 |\cos\theta_{ij}| - r^2 \leq 0
\end{aligned}$$

which shows that $\mathscr{A}$ is asymptotically stable. □

We now use $\mathscr{A}$ to construct a modified Lyapunov equation for characterizing $\mathscr{C}(q, r)$. First, we consider existence, uniqueness, and definiteness of its solution when $\mathscr{A}$ is asymptotically stable.

Theorem 4.1: Let $K \in \mathbb{R}^{m \times l}$, let $q \geq r > 0$, and define $\alpha \triangleq q - r$. If $\mathscr{A}$ is asymptotically stable (or, equivalently, $\sigma(A_0) \subset \mathscr{C}(q, r)$), then there exists a unique $n \times n$ matrix Q satisfying

$$0 = A_{0\alpha} Q + Q A_{0\alpha}^{\mathrm{T}} + \frac{1}{r} A_{0\alpha} Q A_{0\alpha}^{\mathrm{T}} + V \tag{4.8}$$

and, furthermore, Q is nonnegative definite. If, in addition, $(A_{0\alpha}, D)$ is controllable, then Q is positive definite.

Proof: Note that (4.8) is equivalent to

$$0 = \mathscr{A} \operatorname{vec} Q + \operatorname{vec} V \tag{4.9}$$

where "vec" is the column-stacking operator [26]. Since $\mathscr{A}$ is invertible, (4.9) implies

$$Q = -\operatorname{vec}^{-1}\left[\mathscr{A}^{-1} \operatorname{vec} V\right] \tag{4.10}$$

so that existence and uniqueness hold. To show that Q is nonnegative definite, note that (4.10) can be written as [26]

$$\begin{aligned}
Q &= \int_0^\infty \operatorname{vec}^{-1}\left[e^{\mathscr{A} t} \operatorname{vec} V\right] \mathrm{d}t \\
&= \int_0^\infty \operatorname{vec}^{-1}\left[e^{(A_{0\alpha} \oplus A_{0\alpha})t} e^{1/r(A_{0\alpha} \otimes A_{0\alpha})t} \operatorname{vec} V\right] \mathrm{d}t \\
&= \int_0^\infty \operatorname{vec}^{-1}\left[e^{(A_{0\alpha} \oplus A_{0\alpha})t} \operatorname{vec} \cdot \left\{\sum_{k=0}^\infty (k!)^{-1} A_{0\alpha}^k V A_{0\alpha}^{k\mathrm{T}} (t/r)^k\right\}\right] \mathrm{d}t \\
&= \int_0^\infty e^{A_{0\alpha} t}\left[\sum_{k=0}^\infty (k!)^{-1} A_{0\alpha}^k V A_{0\alpha}^{k\mathrm{T}} (t/r)^k\right] e^{A_{0\alpha}^{\mathrm{T}} t} \mathrm{d}t \\
&\geq 0.
\end{aligned}$$

If $(A_{0\alpha}, D)$ is controllable, then $(A_{0\alpha}, [(1/r) A_{0\alpha} Q A_{0\alpha}^{\mathrm{T}} + V]^{1/2})$ is also controllable. Hence, it follows from Lemma 12.2 of [27] that Q is positive definite. □

Next, we state the converse of Theorem 4.1 which guarantees that if (4.8) has a solution then the closed-loop poles lie in $\mathscr{C}(q, r)$ along with a bound on the performance criterion.

Theorem 4.2: Let $K \in \mathbb{R}^{m \times l}$ and suppose there exists nonnegative-definite $Q \in \mathbb{R}^{n \times n}$ satisfying (4.8). Then the following conditions are equivalent:

$$(A_0, D) \text{ is assignable with respect to } \mathscr{C}(q, r), \tag{4.11}$$

$$\sigma(A_0) \subset \mathscr{C}(q, r), \tag{4.12}$$

$$\mathscr{A} \text{ is asymptotically stable}. \tag{4.13}$$

Furthermore, if (4.11)–(4.13) are satisfied, then the steady-state covariance Q_0 given by (2.9) exists and satisfies

$$Q_0 \leq Q \tag{4.14}$$

and, consequently

$$J(K) \leq \mathscr{J}(K) \tag{4.15}$$

where

$$\mathscr{J}(K) \triangleq \operatorname{tr} Q R_0. \tag{4.16}$$

Proof: First, we show that (4.11) implies (4.12). Suppose (4.12) is false, that is, suppose there exists $\lambda \in \sigma(A_0)$ such that $\lambda \notin \mathscr{C}(q, r)$. Since $\bar{\lambda} \in \sigma(A_0^{\mathrm{T}})$, let $\eta \in \mathbb{C}^n$, $\eta \neq 0$, be an eigenvector of A_0^{T} associated with $\bar{\lambda}$, that is, $A_0^{\mathrm{T}} \eta = \bar{\lambda}\eta$. Computing $\eta^*(4.8)\eta$ yields

$$0 = \mu \eta^* Q \eta + \eta^* V \eta \tag{4.17}$$

where $\mu \triangleq \lambda + \bar{\lambda} + 2\alpha + 1/r[\lambda\bar{\lambda} + (\lambda + \bar{\lambda})\alpha + \alpha^2]$. Since $\lambda \notin \mathscr{C}(q, r)$ it follows from Proposition 4.3 that $\mu \geq 0$. Since, furthermore, $\eta^* Q \eta \geq 0$ and $\eta^* V \eta \geq 0$, it follows from (4.17) that $\eta^* V \eta = 0$ or, equivalently, $\eta^* D = 0$. Combining this fact with $A_0^{\mathrm{T}} \eta = \bar{\lambda}\eta$ yields

$$\eta^*[A_0 - \lambda I_n, D] = 0 \tag{4.18}$$

which implies

$$\operatorname{rank}\left[A_0 - \lambda I_n, D\right] < n. \tag{4.19}$$

Thus, λ is not D-controllable. Since $\lambda \notin \mathscr{C}(q,r)$, it follows that (A_0, D) is not assignable with respect to $\mathscr{C}(q,r)$, which contradicts (4.11).

Conversely, if (4.12) is satisfied, then there are no eigenvalues of A_0 that are not in $\mathscr{C}(q,r)$. Hence (4.11) is trivially satisfied. Finally, the equivalence of (4.12) and (4.13) is a restatement of Proposition 4.5.

Next, to prove (4.14) subtract (2.9) from (4.8) to obtain

$$0 = A_0(Q - Q_0) + (Q - Q_0)A_0^{\mathrm{T}} + 2\alpha Q + \frac{1}{r}A_{0\alpha}QA_{0\alpha}^{\mathrm{T}} \quad (4.20)$$

which, since A_0 is asymptotically stable, is equivalent to

$$Q - Q_0 = \int_0^\infty e^{A_0 t}\left[2\alpha Q + \frac{1}{r}A_{0\alpha}QA_{0\alpha}^{\mathrm{T}}\right]e^{A_0^{\mathrm{T}}t}\,dt \geq 0. \quad (4.21)$$

Finally, (4.15) follows immediately from (4.14). □

Remark 4.2: Note that (4.11) is a closed-loop disturbability condition which guarantees that the system does not possess any hidden undisturbed poles outside of $\mathscr{C}(q,r)$. Of course, if V is positive definite or (A_0, D) is controllable, then (4.11) and thus (4.12) and (4.13) are automatically satisfied. If pole placement is of primary interest rather than the performance bound (4.15), then one can set $V = I_n$ so that (4.11) is satisfied.

V. The Auxiliary Minimization Problem and Necessary and Sufficient Conditions for Optimality

As discussed in the previous section, replacing (2.9) by (4.8) effectively constrains the closed-loop poles to lie within $\mathscr{C}(q,r)$ while yielding an upper bound for the performance criterion. That is, given a controller K for which there exists a nonnegative-definite solution (4.8), the actual performance $J(K)$ is guaranteed to be no worse than the bound $\mathscr{J}(K)$. Hence, $\mathscr{J}(K)$ can be viewed as an auxiliary cost which leads to the following optimization problem.

Auxiliary Minimization Problem

Determine $K \in \mathscr{K}_{\mathscr{C}(q,r)}$ that minimizes $\mathscr{J}(K)$ where $Q \geq 0$ is given by (4.8).

A question that arises immediately is whether or not the Auxiliary Minimization Problem possesses a solution. Note that this question is nontrivial since $\mathscr{K}_{\mathscr{C}(q,r)}$ is an open set. To this end we impose slightly stronger hypotheses to obtain the following existence result.

Theorem 5.1: Assume $R_1 > 0$, $V > 0$ and suppose $\mathscr{K}_{\mathscr{C}(q,r)}$ is not empty. Then there exists a solution to the Auxiliary Minimization Problem.

Proof: Let $\{K_i\}_{i=1}^\infty$ be a sequence of gains with $K_i \in \mathscr{K}_{\mathscr{C}(q,r)}$, and define $A_{0i} \triangleq A + BK_iC$. Furthermore, assume that $K_i \to \partial\mathscr{K}_{\mathscr{C}(q,r)}$. Hence, for each i there exists an eigenvalue $\lambda_i \in \sigma(A_{0i})$ such that the sequence $\lambda_1, \lambda_2, \cdots$, approaches the boundary of $\mathscr{C}(q,r)$.

Now define $\mathscr{A}_i \triangleq A_{0i\alpha} \oplus A_{0i\alpha} + 1/rA_{0i\alpha} \otimes A_{0i\alpha}$, where $A_{0i\alpha} \triangleq A_{0i} + \alpha I_n$, and let $\mu_i \in \sigma(\mathscr{A}_i)$ be the real, negative eigenvalue given by $\mu_i = \lambda_i + \bar{\lambda}_i + 2\alpha + 1/r(\lambda_i + \alpha)(\bar{\lambda}_i + \alpha)$. Clearly, $\mu_i \to 0$ since $\lambda_i \to \partial\mathscr{C}(q,r)$. Now, repeating the development in the proof of Theorem 4.2, we obtain, in analogy to (4.17)

$$0 = \mu_i \eta_i^* Q_i \eta_i + \eta_i^* V \eta_i$$

where η_i is a unit norm eigenvector of A_{0i}^{T} corresponding to $\bar{\lambda}_i$, and Q_i is the solution to (4.8) with K replaced by K_i. Hence

$$\eta_i^* Q_i \eta_i = -\frac{1}{\mu_i}\eta_i^* V \eta_i \to \infty \qquad \text{as } i \to \infty$$

since $V > 0$ and $\mu_i \to 0$. Since $R_1 > 0$, we have thus shown that $\mathscr{J}(K) \to \infty$ as K approaches the boundary of $\mathscr{K}_{\mathscr{C}(q,r)}$. Now let $\beta > 0$ be sufficiently large that the closed set $\kappa_\beta \triangleq \{K \in \mathscr{K}_{\mathscr{C}(q,r)}: \mathscr{J}(K) \leq \beta\}$ is not empty. Since $\mathscr{J}(K) = O(\|K\|^2)$, it follows that κ_β is bounded and hence compact. Hence $\mathscr{J}$ has a global minimum of κ_β and hence on $\mathscr{K}_{\mathscr{C}(q,r)}$. □

The following result presents the necessary conditions for optimality in the Auxiliary Minimization Problem. For convenience in stating this result define the notation

$$R_{2a} \triangleq R_2 + \frac{1}{r}B^{\mathrm{T}}PB, \qquad P_a \triangleq B^{\mathrm{T}}P + \frac{1}{r}B^{\mathrm{T}}PA_\alpha + R_{12}^{\mathrm{T}}$$

for arbitrary $P \in \mathbb{R}^{n \times n}$.

Theorem 5.2: Suppose $K \in \mathscr{K}_{\mathscr{C}(q,r)}$ solves the Auxiliary Minimization Problem and assume that $CQC^{\mathrm{T}} > 0$, where Q satisfies (4.8). Then K is given by

$$K = -R_{2a}^{-1}P_aQC^{\mathrm{T}}(CQC^{\mathrm{T}})^{-1} \quad (5.1)$$

where the $n \times n$ nonnegative-definite matrices Q, P satisfy

$$0 = (A_\alpha - BR_2^{-1}P_a\nu)Q + Q(A_\alpha - BR_{2a}^{-1}P_a\nu)^{\mathrm{T}} + \frac{1}{r}(A_\alpha - BR_{2a}^{-1}P_a\nu)Q(A_\alpha - BR_{2a}^{-1}P_a\nu)^{\mathrm{T}} + V, \quad (5.2)$$

$$0 = A_\alpha^{\mathrm{T}}P + PA_\alpha + R_1 + \frac{1}{r}A_\alpha^{\mathrm{T}}PA_\alpha - P_a^{\mathrm{T}}R_{2a}^{-1}P_a + \nu_\perp^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}P_a\nu_\perp, \quad (5.3)$$

$$\nu \triangleq QC^{\mathrm{T}}(CQC^{\mathrm{T}})^{-1}C, \qquad \nu_\perp \triangleq I_n - \nu. \quad (5.4)$$

Furthermore, the auxiliary cost is given by

$$\mathscr{J}(K) = \mathrm{tr}\left[Q(R_1 - 2R_{12}R_{2a}^{-1}P_a\nu + \nu^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}R_2R_{2a}^{-1}P_a\nu)\right]. \quad (5.5)$$

Conversely, if there exist $n \times n$ nonnegative-definite matrices Q, P satisfying (5.2) and (5.3), then Q satisfies (4.8) with K given by (5.11), and, furthermore, $\mathscr{J}(K)$ is given by (5.5).

Proof: To optimize (4.16) over the open set $\mathscr{K}_{\mathscr{C}(q,r)}$ subject to the constraint (4.8), form the Lagrangian

$$\mathscr{L}(K,Q) \triangleq \operatorname{tr}\left[\lambda Q R_0 + \left(A_{0\alpha}Q + QA_{0\alpha}^{\mathrm{T}} + \frac{1}{r}A_{0\alpha}QA_{0\alpha}^{\mathrm{T}} + V\right)P\right]$$

where the Lagrange multipliers $\lambda \geq 0$ and $P \in \mathbb{R}^{n\times n}$ are not both zero. Setting $\partial\mathscr{L}/\partial Q = 0$, $\lambda = 0$ implies $P = 0$ since $\mathscr{A}$ is asymptotically stable. Hence, without loss of generality set $\lambda = 1$. Thus the stationarity conditions are given by

$$\frac{\partial\mathscr{L}}{\partial Q} = A_{0\alpha}^{\mathrm{T}}P + PA_{0\alpha} + \frac{1}{r}A_{0\alpha}^{\mathrm{T}}PA_{0\alpha} + R_0 = 0, \quad (5.6)$$

$$\frac{\partial\mathscr{L}}{\partial K} = R_{2a}KCQC^{\mathrm{T}} + P_aQC^{\mathrm{T}} = 0. \quad (5.7)$$

Since by assumption CQC^{T} is invertible, (5.7) implies (5.1). Finally, (5.2) and (5.3) are equivalent to (4.8) and (5.6) with K given by (5.1). □

Remark 5.1: The definiteness condition $CQC^{\mathrm{T}} > 0$ holds if C has full-row rank and Q is positive definite. Conversely, note that if $CQC^{\mathrm{T}} > 0$ then C must have full-row rank but Q need not be positive definite. A sufficient condition for $CQC^{\mathrm{T}} > 0$ that is weaker than $Q > 0$ is $CVC^{\mathrm{T}} > 0$. This can easily be seen by rewriting (2.9) as

$$Q_0 = \int_0^\infty e^{A_0 t}Ve^{A_0^{\mathrm{T}}t}\,dt$$

and expanding the exponentials. The first term of the series expansion shows that $CVC^{\mathrm{T}} > 0$ implies $CQ_0C^{\mathrm{T}} > 0$. Finally, since $Q \geq Q_0$, it follows that $CQC^{\mathrm{T}} \geq CQ_0C^{\mathrm{T}} > 0$.

Remark 5.2: The matrix ν defined by (5.4) is idempotent since $\nu^2 = \nu$. Note that ν is an oblique projection since it is not necessarily symmetric.

Remark 5.3: Theorem 5.2 is a direct generalization of optimal static output feedback theory originally developed in [28]. To recover the result of [28] let $r \to \infty$ and $\alpha \to 0$ so that $A_\alpha \to A$ and all terms premultiplied by $1/r$ disappear.

Remark 5.4: Several special cases can be recovered from Theorem 5.2. For example, when the full state is available, that is, $C = I_n$, the projection $\nu = I_n$ so that $\nu_\perp = 0$. In this case (5.1) becomes

$$K = -R_{2a}^{-1}P_a \quad (5.8)$$

and (5.2) and (5.3) specialize to

$$0 = A_\alpha^{\mathrm{T}}P + PA_\alpha + \frac{1}{r}A_\alpha^{\mathrm{T}}PA_\alpha + R_1 - P_a^{\mathrm{T}}R_{2a}^{-1}P_a. \quad (5.9)$$

This corresponds to results obtained in [16]–[18]. Finally, to recover the standard LQR result let $r \to \infty$ and $\alpha \to 0$ so that (5.9) corresponds to the standard regulator Riccati equation.

We now combine Theorem 4.2 with the converse of Theorem 5.1 to obtain our main result guaranteeing that the closed-loop poles lie in $\mathscr{C}(q,r)$ along with an optimized performance bound.

Theorem 5.3: Suppose there exist $n \times n$ nonnegative-definite Q, P satisfying (5.2) and (5.3) and let K be given by (5.1). Then (A_0, D) is assignable with respect to $\mathscr{C}(q,r)$ if and only if the closed-loop poles lie in $\mathscr{C}(q,r)$. In this case the performance criterion (2.4) satisfies the bound

$$J(K) \leq \operatorname{tr}\left[Q\left(R_1 - 2R_{12}R_{2a}^{-1}P_a\nu + \nu^{\mathrm{T}}P_a^{\mathrm{T}}R_{2a}^{-1}R_2R_{2a}^{-1}P_a\nu\right)\right]. \quad (5.10)$$

Proof: The converse portion of Theorem 5.2 implies that Q satisfies (4.8) with K given by (5.1) and the auxiliary cost given by (5.5). It now follows from Theorem 4.2 that the assignability condition (4.11) is equivalent to the pole constraint (4.12). Furthermore, the performance bound (4.15), which is equivalent to (5.10), holds. □

In applying Theorem 5.3 the principal issue concerns conditions on the problem data under which the coupled Riccati/Lyapunov equations (5.2) and (5.3) possess nonnegative-definite solutions. Next, we show that our sufficient conditions are also necessary in the sense that if $\mathscr{K}_{\mathscr{C}(q,r)}$ is not empty then (5.2) and (5.3) must have a solution.

Theorem 5.4: Assume $R_1 > 0$, $V > 0$ and $CVC^{\mathrm{T}} > 0$, and suppose $\mathscr{K}_{\mathscr{C}(q,r)}$ is not empty. Then there exist nonnegative-definite matrices Q, P satisfying (5.2) and (5.3).

Proof: The result is a direct consequence of Theorem 5.1 and Theorem 5.2. □

VI. Dynamic Output Feedback Control With Pole Constraints

In this section, we introduce the dynamic output-feedback control problem with regional pole constraints. For simplicity in this section we restrict our attention to controllers of order $n_c = n$, that is, controllers whose order is equal to the dimension of the plant. This constraint is removed in Section VII where controllers of reduced order are considered.

Dynamic Output Feedback Control with Pole Constraints

Given the nth-order stabilizable and detectable plant

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t), \quad (6.1)$$

$$y(t) = Cx(t) + D_2w(t) \quad (6.2)$$

determine an nth-order dynamic compensator

$$\dot{x}_c(t) = A_cx_c(t) + B_cy(t), \quad (6.3)$$

$$u(t) = C_cx_c(t) \quad (6.4)$$

that satisfies design criteria 1) and 2), with $J(K)$ denoted by $\tilde{J}(A_c, B_c, C_c)$.

As in Section II we define the open set

$$\tilde{\mathscr{K}}_{\mathscr{R}} \triangleq \{(A_c, B_c, C_c) : \sigma(\tilde{A}) \subset \mathscr{R}\}$$

of dynamic compensators that place the closed-loop poles in $\mathscr{R}$. Again, we focus on the circular region $\mathscr{C}(q,r)$ with center $-q$ and radius r. If $\tilde{\mathscr{K}}_{\mathscr{C}(q,r)}$, is not empty and $(A_c, B_c, C_c) \in \tilde{\mathscr{K}}_{\mathscr{C}(q,r)}$, then $\tilde{A}$ is asymptotically stable. In this case the performance is given by

$$\tilde{\mathscr{J}}(A_c, B_c, C_c) = \operatorname{tr}\tilde{Q}_0\tilde{R} \quad (6.5)$$

where the $\tilde{n} \times \tilde{n}$ nonnegative-definite steady-state covariance defined by

$$\tilde{Q}_0 \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{6.6}$$

satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q}_0 + \tilde{Q}_0\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{6.7}$$

Next, we proceed as in Section IV where we replace the Lyapunov equation (6.7) for the dynamic problem with a modified Lyapunov equation that guarantees that the closed-loop poles lie within the circular region $\mathscr{C}(q, r)$ with an optimized performance bound. Thus, for the dynamic output feedback problem, Propositions 4.4 and 4.5 and Theorem 4.1 and 4.2 follow with A_0, $A_{0\alpha}$, R_0, V replaced by $\tilde{A}$, $\tilde{A}_\alpha$, $\tilde{R}$, $\tilde{V}$. For clarity we state the Auxiliary Minimization Problem for the dynamic problem.

Dynamic Auxiliary Minimization Problem

Determine $(A_c, B_c, C_c) \in \tilde{K}_{\mathscr{C}(q,r)}$ that minimizes

$$\tilde{\mathscr{J}}(A_c, B_c, C_c) \triangleq \operatorname{tr} \tilde{Q}\tilde{R} \tag{6.8}$$

where $\tilde{Q} \geq 0$ satisfies

$$0 = \tilde{A}_\alpha\tilde{Q} + \tilde{Q}\tilde{A}_\alpha^{\mathrm{T}} + \frac{1}{r}\tilde{A}_\alpha\tilde{Q}\tilde{A}_\alpha^{\mathrm{T}} + \tilde{V} \tag{6.9}$$

and such that (A_c, B_c, C_c) is controllable and observable.

By deriving necessary conditions for the Auxiliary Minimization Problem as in Section V we can obtain sufficient conditions for characterizing full-order dynamic output feedback controllers guaranteeing pole placement in $\mathscr{C}(q, r)$ with an optimized performance bound. For convenience in stating this result recall the definitions of R_{2a} and P_a and define the additional notation

$$V_{2a} \triangleq V_2 + \frac{1}{r}CQC^{\mathrm{T}}, \qquad Q_a \triangleq QC^{\mathrm{T}} + \frac{1}{r}A_\alpha QC^{\mathrm{T}} + V_{12}$$

for arbitrary $Q \in \mathbb{R}^{n\times n}$.

Theorem 6.1: Suppose there exist $n \times n$ nonnegative-definite matrices Q, P satisfying

$$0 = A_\alpha Q + QA_\alpha^{\mathrm{T}} + V_1 + \frac{1}{r}A_\alpha QA_\alpha^{\mathrm{T}} - Q_aV_{2a}^{-1}Q_a^{\mathrm{T}}, \tag{6.10}$$

$$0 = A_\alpha^{\mathrm{T}}P + PA_\alpha + R_1 + \frac{1}{r}A_\alpha^{\mathrm{T}}PA_\alpha - P_a^{\mathrm{T}}R_{2a}^{-1}P_a \tag{6.11}$$

and let A_c, B_c, C_c be given by

$$A_c = A - BR_{2a}^{-1}P_a - Q_aV_{2a}^{-1}C, \tag{6.12}$$

$$B_c = Q_aV_{2a}^{-1}, \tag{6.13}$$

$$C_c = -R_{2a}^{-1}P_a. \tag{6.14}$$

Then $(\tilde{A}, \tilde{D})$ is assignable with respect to $\mathscr{C}(q, r)$ if and only if the closed-loop poles lie in $\mathscr{C}(q, r)$. In this case the performance criterion (2.4) satisfies the bound

$$\tilde{\mathscr{J}}(A_c, B_c, C_c) \leq \operatorname{tr}[QR_1 + PQ_aV_{2a}^{-1}Q_a^{\mathrm{T}}]. \tag{6.15}$$

Proof: The proof of this result follows as a special case of the corresponding result for reduced-order dynamic compensation given in Section VII. □

Theorem 6.1 presents sufficient conditions for the LQG control problem with the closed-loop poles constrained to lie in $\mathscr{C}(q, r)$. These sufficient conditions comprise a system of two decoupled modified Riccati equations similar to the estimator and regulator Riccati equations of LQG theory with additional terms arising due to the enforcement of pole placement in the circular region. Note that since the Q and P equations are decoupled they can be solved independently of each other. Since regulator/estimator separation holds, the certainty equivalence principle is valid for the LQG problem with pole constraints in $\mathscr{C}(q, r)$. Finally, note that if we sufficiently relax the pole constraint requirement, that is, $r \to \infty$ and $\alpha \to 0$, then the standard LQG result is recovered.

VII. Reduced-Order Dynamic Output Feedback Control With Pole Constraints

In this section, we extend Theorem 6.1 by expanding the formulation of Section VI to allow the compensator to be of fixed dimension n_c that may be less than the plant order n. Hence, in this section define $\tilde{n} = n + n_c$, where $n_c \leq n$. As in [29] this constraint leads to an oblique projection that introduces additional coupling in the design equations along with an additional pair of design equations. This coupling shows that regulator/estimator separation breaks down in the reduced-order controller case. The following lemma is required for the statement of the main theorem.

Lemma 7.1: Let $\hat{Q}$, $\hat{P}$ be $n \times n$ nonnegative-definite matrices and suppose that rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ G, Γ, and $n_c \times n_c$ invertible M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\hat{Q}\hat{P} = G^{\mathrm{T}}M\Gamma, \tag{7.1}$$

$$\Gamma G^{\mathrm{T}} = I_{n_c}. \tag{7.2}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{\mathrm{T}}\Gamma, \tag{7.3}$$

$$\tau_\perp \triangleq I_n - \tau \tag{7.4}$$

are idempotent and have rank n_c and $n - n_c$, respectively.

Proof: The result is a direct consequence of [30, Theorem 6.2.5]. □

We now state the main result of this section concerning reduced-order controllers.

Theorem 7.1: Let $n_c \leq n$, suppose there exist $n \times n$ nonnegative-definite matrices Q, P, $\hat{Q}$, $\hat{P}$ satisfying

$$\begin{aligned}0 = {} & A_\alpha Q + QA_\alpha^{\mathrm{T}} + V_1 + \frac{1}{r}A_\alpha QA_\alpha^{\mathrm{T}} - Q_aV_{2a}^{-1}Q_a^{\mathrm{T}} \\ & + \tau_\perp\left[\frac{1}{r}(A_\alpha - BR_{2a}^{-1}P_a)\hat{Q}(A_\alpha - BR_{2a}^{-1}P_a)^{\mathrm{T}} \right. \\ & \left. + Q_aV_{2a}^{-1}Q_a^{\mathrm{T}}\right]\tau_\perp^{\mathrm{T}},\end{aligned} \tag{7.5}$$

$$0 = A_\alpha^{\mathrm{T}} P + PA_\alpha + R_1 + \frac{1}{r} A_\alpha^{\mathrm{T}} PA_\alpha - P_a^{\mathrm{T}} R_{2a}^{-1} P_a + \tau_\perp^{\mathrm{T}} \left[\frac{1}{r}(A_\alpha - Q_a V_{2a}^{-1} C)^{\mathrm{T}} \hat{P} (A_\alpha - Q_a V_{2a}^{-1} C) + P_a^{\mathrm{T}} R_{2a}^{-1} P_a \right] \tau_\perp, \tag{7.6}$$

$$0 = (A_\alpha - BR_{2a}^{-1} P_a)\hat{Q} + \hat{Q}(A_\alpha - BR_{2a}^{-1} P_a)^{\mathrm{T}} + \frac{1}{r}(A_\alpha - BR_{2a}^{-1} P_a)\hat{Q}(A_\alpha - BR_{2a}^{-1} P_a)^{\mathrm{T}} + Q_a V_{2a}^{-1} Q_a^{\mathrm{T}} - \tau_\perp \left[\frac{1}{r}(A_\alpha - BR_{2a}^{-1} P_a)\hat{Q}(A_\alpha - BR_{2a}^{-1} P_a)^{\mathrm{T}} + Q_a V_{2a}^{-1} Q_a^{\mathrm{T}} \right] \tau_\perp^{\mathrm{T}}, \tag{7.7}$$

$$0 = (A_\alpha - Q_a V_{2a}^{-1} C)^{\mathrm{T}} \hat{P} + \hat{P}(A_\alpha - Q_a V_{2a}^{-1} C) + \frac{1}{r}(A_\alpha - Q_a V_{2a}^{-1} C)^{\mathrm{T}} \hat{P}(A_\alpha - Q_a V_{2a}^{-1} C) + P_a^{\mathrm{T}} R_{2a}^{-1} P_a - \tau_\perp^{\mathrm{T}} \left[\frac{1}{r}(A_\alpha - Q_a V_{2a}^{-1} C)^{\mathrm{T}} \hat{P}(A_\alpha - Q_a V_{2a}^{-1} C) + P_a^{\mathrm{T}} R_{2a}^{-1} P_a \right] \tau_\perp, \tag{7.8}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{7.9}$$

and let A_c, B_c, C_c be given by

$$A_c = \Gamma(A - BR_{2a}^{-1} P_a - Q_a V_{2a}^{-1} C) G^{\mathrm{T}}, \tag{7.10}$$

$$B_c = \Gamma Q_a V_{2a}^{-1}, \tag{7.11}$$

$$C_c = -R_{2a}^{-1} P_a G^{\mathrm{T}}. \tag{7.12}$$

Then $(\tilde{A}, \tilde{D})$ is assignable with respect to $\mathscr{C}(q, r)$ if and only if the closed-loop poles lie in $\mathscr{C}(q, r)$. In this case the performance criterion (2.4) satisfies the bound

$$J(A_c, B_c, C_c) \le \operatorname{tr}\left[QR_1 + P(Q_a V_{2a}^{-1} Q_a^{\mathrm{T}} - \tau_\perp Q_a V_{2a}^{-1} Q_a^{\mathrm{T}} \tau_\perp^{\mathrm{T}}) \right]. \tag{7.13}$$

Proof: The proof follows as in the proof of Theorem 5.2 with additional terms arising due to the reduced-order dynamic compensation structure. For details of a similar proof see [29]. □

Remark 7.1: It is easy to see that Theorem 7.1 is a direct generalization of Theorem 6.1. To recover Theorem 6.1, set $n_c = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (7.5)–(7.8) is zero and (7.7) and (7.8) become superfluous. Furthermore, (7.5) and (7.6) now reduce to (6.10) and (6.11) as expected. Alternatively, letting $r \to \infty$, $\alpha \to 0$, and retaining the reduced-order constraint $n_c < n$ yields the result of [29].

VIII. Numerical Results For the Circular Region

As noted in Section III, the circular region can be obtained by means of a simple transformation of the unit disk. Hence, by employing appropriate substitutions it is possible to recast the continuous-time pole constraint design equations as discrete-time design equations. To see this we note that the full-order dynamic compensation design equations (6.10) and (6.11) can be rewritten as (assuming $V_{12} = 0$ and $R_{12} = 0$ for convenience)

$$Q = A_{q,r} Q A_{q,r}^{\mathrm{T}} - A_{q,r} Q C_r^{\mathrm{T}} (V_2 + C_r Q C_r^{\mathrm{T}})^{-1} C_r Q A_{q,r}^{\mathrm{T}} + V_{1r}, \tag{8.1}$$

$$P = A_{q,r}^{\mathrm{T}} P A_{q,r} - A_{q,r}^{\mathrm{T}} P B_r (R_2 + B_r^{\mathrm{T}} P B_r)^{-1} B_r^{\mathrm{T}} P A_{q,r} + R_{1r} \tag{8.2}$$

where

$$A_{q,r} \triangleq \frac{1}{r}(A + qI_n), \qquad B_r \triangleq \frac{1}{\sqrt{r}} B, \qquad C_r \triangleq \frac{1}{\sqrt{r}} C, \tag{8.3}$$

$$R_{1r} \triangleq \frac{1}{r} R_1, \qquad V_{1r} \triangleq \frac{1}{r} V_1. \tag{8.4}$$

Hence, by employing (8.3) and (8.4), it is possible to solve (6.10) and (6.11) by means of a standard discrete-time Riccati solver. Similar transformations can be utilized for static output feedback and reduced-order dynamic compensation. Since software for standard discrete-time Riccati equations is readily available, we shall focus on the full-order dynamic compensation problem.

We consider the coupled disk problem originating in [31]. The problem data are as follows:

$$n = n_c = 8, \qquad m = l = d = 1,$$

$$A = \begin{bmatrix} -0.161 & \\ -6.004 & \\ -0.5822 & \\ -9.9835 & I_7 \\ -0.4073 & \\ -3.982 & \\ 0 & \\ 0 & 0_{1\times 7} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0.0064 \\ 0.00235 \\ 0.00713 \\ 1.0002 \\ 0.1045 \\ 0.9955 \end{bmatrix},$$

$$C = [1 \quad 0_{1\times 7}], \qquad V_1 = BB^{\mathrm{T}}, \qquad V_{12} = 0, \qquad V_2 = 1,$$

$$E_1 = 10^{-3}[0_{1\times 4} \quad 0.55 \quad 11 \quad 1.32 \quad 18],$$

$$R_1 = E_1^{\mathrm{T}} E_1, \quad R_{12} = 0, \quad R_2 = 1.$$

Fig. 3 shows the closed-loop pole locations for LQG and the case $q = 12$, $r = 11.98$. Finally, Fig. 4 shows the impulse response of the performance variable $z = E_1 x$ for LQG and the case $q = 12$, $r = 11.98$.

IX. Static Controller Synthesis for Elliptical, Vertical Strip, and Parabolic Regions

In this section, the results of the first part of the paper are rederived for alternative pole placement regions. Specifically, we characterize static output feedback controllers that guarantee closed-loop pole placement within elliptical, vertical strip, and parabolic regions in the open left-half plane. We denote the elliptical region with center at $-q$ and semi-axes of length r_1 and r_2 by $\mathscr{E}(q, r_1, r_2)$ (see Fig. 5). The vertical

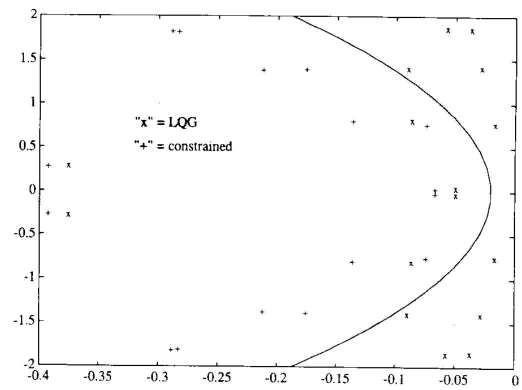

Fig. 3. Closed-loop pole locations for LQC and constrained design.

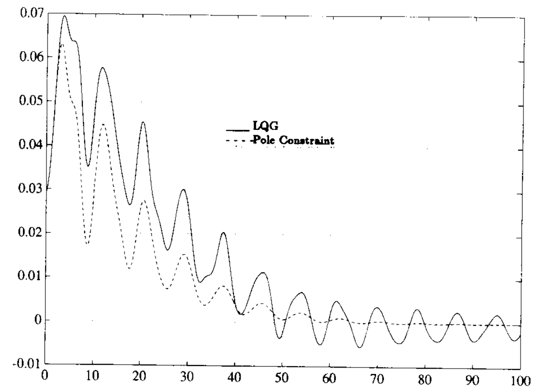

Fig. 4. Impulse response of the performance variable.

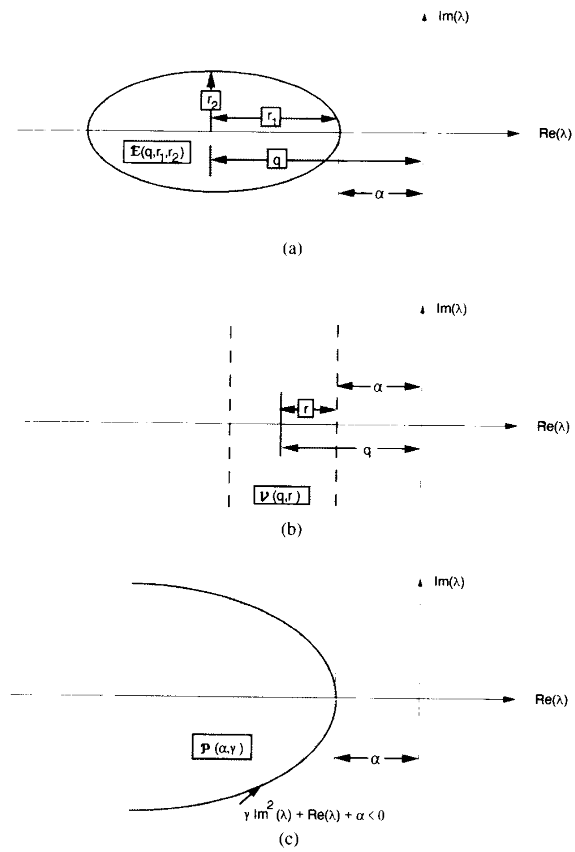

Fig. 5. (a) Elliptical region, (b) vertical strip region, and (c) parabolic region.

strip region centered at $-q$ with half width r is denoted by $\mathscr{V}(q, r)$. Finally, the parabolic region with vertex at $-\alpha$ and parameter $\gamma > 0$ is denoted by $\mathscr{P}(\alpha, \gamma)$.

In order to facilitate the presentation we shall first consider the elliptical region $\mathscr{E}(q, r_1, r_2)$ and then consider the vertical strip and parabolic regions. We now provide a characterization of the elliptical region.

Proposition 9.1: Let $q \geq r_1 > 0$, $r_2 > 0$, define $\alpha \triangleq q - r_1$, $\gamma \triangleq 1/2r_1(1 + (r_1^2/r_2^2))$, and $\delta \triangleq 1/2r_1(1 - (r_1^2/r_2^2))$, and let $\lambda \in \mathbb{C}$. Then the following are equivalent:

$$\lambda \in \mathscr{E}(q, r_1, r_2), \tag{9.1}$$

$$2\,\mathrm{Re}\,(\lambda + \alpha) + \delta\,\mathrm{Re}\,(\lambda + \alpha)^2 + \gamma\,|\,\lambda + \alpha\,|^2 < 0. \tag{9.2}$$

The circular region $\mathscr{C}(q, r)$ and the vertical strip region $\mathscr{V}(q, r)$ can be obtained as special cases of the elliptical region $\mathscr{E}(q, r_1, r_2)$. Specifically, setting $r_1 = r_2 = r$ or, equivalently, $\delta = 0$ and $\gamma = 1/r$, we recover the circular region, that is, $\mathscr{C}(q, r) = \mathscr{E}(q, r, r)$. Alternatively, letting $r_1 = r$, $r_2 \to \infty$ or, equivalently, $\delta = \gamma = 1/2r_1$, we obtain the vertical strip region $\mathscr{V}(q, r)$, that is, $\mathscr{V}(q, r) = \mathscr{E}(q, r, \infty)$. Although the parabolic region $\mathscr{P}(\alpha, \gamma)$ is not a special case of the elliptical region, we note that $\mathscr{P}(\alpha, \gamma)$ can be characterized by replacing δ by $-\gamma$ in (9.2). Hence, by setting $\delta = 0$, γ, $-\gamma$ the subsequent development yields results for the circular, vertical strip, and parabolic regions, respectively.

Next, we characterize matrices whose eigenvalues lie in $\mathscr{E}(q, r_1, r_2)$ in terms of a modified Lyapunov equation. First, we introduce the $n^2 \times n^2$ matrix

$$\mathscr{A}_0 \triangleq \left(A_{0\alpha} + \frac{\delta}{2}A_{0\alpha}^2\right) \oplus \left(A_{0\alpha} + \frac{\delta}{2}A_{0\alpha}^2\right) + \gamma A_{0\alpha} \otimes A_{0\alpha}. \tag{9.3}$$

The following result shows that the stability of $\mathscr{A}_0$ is necessary and sufficient for determining whether the eigenvalues of A_0 lie in the elliptical region $\mathscr{E}(q, r_1, r_2)$. Recall that the condition $K \in \mathscr{K}_{\mathscr{E}(q, r_1, r_2)}$ is equivalent to $\sigma(A_0) \subset \mathscr{E}(q, r_1, r_2)$.

Proposition 9.2: Let $K \in \mathbb{R}^{m \times l}$. Then $\mathscr{A}_0$ is asymptotically stable if and only if $\sigma(A_0) \subset \mathscr{E}(q, r_1, r_2)$.

Proof: The proof is similar to the proof of Proposition 4.5. □

Next, we use $\mathscr{A}_0$ to construct a modified Lyapunov equation that characterizes $\mathscr{E}(q, r_1, r_2)$. First, we consider existence, uniqueness, and definiteness of its solution when $\mathscr{A}_0$ is asymptotically stable.

Theorem 9.1: Let $K \in \mathbb{R}^{m \times l}$. If $\mathscr{A}_0$ is asymptotically stable, then there exists a unique $n \times n$ matrix Q satisfying

$$0 = A_{0\alpha}Q + QA_{0\alpha}^{\mathrm{T}} + \frac{\delta}{2}A_{0\alpha}^{2}Q + \frac{\delta}{2}QA_{0\alpha}^{2\mathrm{T}} + \gamma A_{0\alpha}QA_{0\alpha}^{\mathrm{T}} + V \quad (9.4)$$

and, furthermore, Q is nonnegative definite.

Proof: The proof follows as a minor generalization of the proof of Theorem 4.1. □

Finally, we state the converse of Theorem 9.1 which guarantees that if (9.4) has a nonnegative-definite solution then the closed-loop poles of the dynamic system lie in $\mathscr{E}(q, r_1, r_2)$.

Theorem 9.2: Let $K \in \mathbb{R}^{m \times l}$ and suppose there exists a nonnegative-definite matrix $Q \in \mathbb{R}^{n \times n}$ satisfying (9.4). Then the following are equivalent:

$$(A_0, D) \text{ is assignable with respect to } \mathscr{E}(q, r_1, r_2), \quad (9.5)$$

$$\sigma(A_0) \subset \mathscr{E}(q, r_1, r_2), \quad (9.6)$$

$$\mathscr{A}_0 \text{ is asymptotically stable}. \quad (9.7)$$

Furthermore, if (9.5)–(9.7) are satisfied and

$$\frac{\delta}{2}A_{0\alpha}^{2}Q + \frac{\delta}{2}QA_{0\alpha}^{2\mathrm{T}} + \gamma A_{0\alpha}QA_{0\alpha}^{\mathrm{T}} \geq 0 \quad (9.8)$$

then the steady-state covariance Q_0 given by (2.9) exists and satisfies (4.14)–(4.16) with Q given by (9.4).

Proof: The proof is similar to the proof of Theorem 4.2. □

Now we proceed as in Section V where we replace the Lyapunov equation (2.9) with the modified Lyapunov equation (9.4) which guarantees that the closed-loop poles lie within the elliptical region. Furthermore, if (9.8) is satisfied then the optimization procedure involves a bound on the quadratic performance functional (2.4). We can thus present sufficient conditions for characterizing static output feedback controllers guaranteeing pole placement in $\mathscr{E}(q, r_1, r_2)$. For convenience in stating this result define the notation

$$P_a \triangleq B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + \gamma B^{\mathrm{T}}PA_{\alpha}, \qquad R_{2\alpha} \triangleq R_2 + \gamma B^{\mathrm{T}}PB,$$

$$\Delta \triangleq CQC^{\mathrm{T}} \otimes R_{2\alpha} + \frac{\delta}{2}(CB \otimes B^{\mathrm{T}}PQC^{\mathrm{T}})U_{ml} + \frac{\delta}{2}(CQPB \otimes B^{\mathrm{T}}C^{\mathrm{T}})U_{ml},$$

$$\Omega \triangleq P_aQC^{\mathrm{T}} + \frac{\delta}{2}B^{\mathrm{T}}A_{\alpha}^{\mathrm{T}}PQC^{\mathrm{T}} + \frac{\delta}{2}B^{\mathrm{T}}PQA_{\alpha}^{\mathrm{T}}C^{\mathrm{T}},$$

for arbitrary $Q, p \in \mathbb{R}^{n \times n}$.

Theorem 9.3: Suppose there exist $n \times n$ nonnegative-definite matrices Q, P satisfying

$$0 = A_{0\alpha}Q + QA_{0\alpha}^{\mathrm{T}} + \frac{\delta}{2}A_{0\alpha}^{2}Q + \frac{\delta}{2}QA_{0\alpha}^{2\mathrm{T}} + \gamma A_{0\alpha}QA_{0\alpha}^{\mathrm{T}} + V, \quad (9.9)$$

$$0 = A_{0\alpha}^{\mathrm{T}}P + PA_{0\alpha} + \frac{\delta}{2}A_{0\alpha}^{2\mathrm{T}}P + \frac{\delta}{2}PA_{0\alpha}^{2} + \gamma A_{0\alpha}^{\mathrm{T}}PA_{0\alpha} + R_0, \quad (9.10)$$

and

$$\det \Delta \neq 0 \quad (9.11)$$

and let K be given by

$$K = -\operatorname{vec}^{-1}(\Delta^{-1}\operatorname{vec}\Omega). \quad (9.12)$$

Then (A_0, D) is assignable with respect to $\mathscr{E}(q, r_1, r_2)$ if and only if the closed-loop poles lie in $\mathscr{E}(q, r_1, r_2)$. If, in addition, (9.8) is satisfied, then the performance criterion (2.4) satisfies the bound

$$J(K) \leq \operatorname{tr}\Big\{Q\Big(R_1 - 2R_{12}\operatorname{vec}^{-1}[\Delta^{-1}\operatorname{vec}\Omega]C + C^{\mathrm{T}}\operatorname{vec}^{-1}[\Delta^{-1}\operatorname{vec}\Omega]^{\mathrm{T}}R_2 \cdot \operatorname{vec}^{-1}[\Delta^{-1}\operatorname{vec}\Omega]C\Big)\Big\}. \quad (9.13)$$

Proof: The proof is an extension of the proof of Theorem 5.2. □

Remark 9.1: As mentioned earlier, by modifying δ and γ in the design equations (9.9), (9.10), we also obtain pole placement in the circular, vertical strip, and parabolic regions. Specifically, letting $\alpha > 0$ and $\gamma > \delta > 0$ yields $\mathscr{E}(\alpha + 1/(\gamma + \delta), 1/(\gamma + \delta), 1/\sqrt{\gamma^2 - \delta^2})$, letting $\alpha > 0$, $\gamma > 0$, and $\delta = 0$ yields $\mathscr{C}(\alpha + 1/\gamma, 1/\gamma)$, letting $\alpha > 0$ and $\gamma = \delta > 0$ yields $\mathscr{V}(\alpha + 1/2\gamma, 1/2\gamma)$, and, finally, letting $\alpha > 0$ and $\gamma = -\delta < 0$ yields $\mathscr{P}(\alpha, \gamma)$.

X. Dynamic Controller Synthesis for Elliptical, Vertical Strip, and Parabolic Regions

In this section we present sufficient conditions that characterize *dynamic* output feedback controllers that enforce regional pole placement within elliptical, vertical strip, and parabolic regions. As in Section VI, we consider the nth-order stabilizable and detectable plant (6.1), (6.2) and determine an nth-order dynamic compensator of the form (6.3), (6.4). Although we could proceed exactly as we did in Section VI for the circular region, it turns out that the additional terms $\delta/2\,\tilde{A}_{\alpha}^{2}\tilde{Q} + \delta/2\,\tilde{Q}\tilde{A}_{\alpha}^{2\mathrm{T}}$ give rise to extremely complex optimality conditions. The complexity of these conditions is due to the matrix A_c and thus appears to be directly related to the lack of controller/estimator separation in the optimal compensator. To simplify matters, we thus consider a minor variation of the approach of Section VI wherein we now *enforce* regulator/estimator separation in the compensator structure. By separately designing a regulator and estimator each of which has its poles in the desired constraint region, the resulting closed-loop system has all of its poles in the same region.

Thus, we consider dynamic compensators having the structure

$$\dot{x}_c(t) = Ax_c(t) + Bu(t) + B_c[y(t) - Cx_c(t)], \quad (10.1)$$

$$u(t) = C_c x_c(t) \quad (10.2)$$

or, equivalently,

$$\dot{x}_c(t) = (A - B_cC + BC_c)x_c(t) + B_c y(t) \quad (10.3)$$

which effectively constrains the compensator dynamics matrix A_c to be of the form

$$A_c = A - B_e C + BC_c. \tag{10.4}$$

Now we specialize the results of Section IX to full-state feedback so that K corresponds to C_c. Similarly, by introducing the error states $e(t) \triangleq x(t) - x_c(t)$, where $e(t)$ satisfies

$$\dot{e}(t) = (A - B_e C)e(t) + (D_1 - B_e D_2)w(t) \tag{10.5}$$

we can develop a dual estimation theory that constrains the observer poles $A - B_e C$ to lie in $\mathscr{E}(q, r_1, r_2)$, $\mathscr{V}(q, r)$, or $\mathscr{P}(\alpha, \gamma)$.

For conciseness we omit the technical details and present the main result. We note, however, that the necessary conditions for the dual estimation problem with pole constraints are obtained by minimizing the error criterion $\lim_{t\to\infty} \mathbb{E}[e^{\mathrm{T}}(t)Ne(t)]$, where N is an $n \times n$ positive-definite weighting matrix. For arbitrary $Q_e, P_e, Q_r, P_r \in \mathbb{R}^{n\times n}$ define the notation

$$Q_{ra} \triangleq Q_r C^{\mathrm{T}} + V_{12} + \gamma A_\alpha Q_r C^{\mathrm{T}}, \; V_{2ra} \triangleq V_2 + \gamma C Q_r C^{\mathrm{T}},$$

$$P_{ra} \triangleq B^{\mathrm{T}} P_r + R_{12}^{\mathrm{T}} + \gamma B^{\mathrm{T}} P_r A_\alpha, \; R_{2ra} \triangleq R_2 + \gamma B^{\mathrm{T}} P_r B,$$

$$\Delta_e \triangleq (V_{2ra} \otimes P_e) + \frac{\delta}{2}(C \otimes P_e Q_e C^{\mathrm{T}})U_{nl} + \frac{\delta}{2}(C Q_e P_e \otimes C^{\mathrm{T}})U_{nl},$$

$$\Delta_r \triangleq (Q_r \otimes R_{2ra}) + \frac{\delta}{2}(B \otimes B^{\mathrm{T}} P_r Q_r)U_{mn} + \frac{\delta}{2}(Q_r P_r B \otimes B^{\mathrm{T}})U_{mn},$$

$$\Omega_e \triangleq P_e Q_{ra} + \frac{\delta}{2} P_e Q_e A_\alpha^{\mathrm{T}} C^{\mathrm{T}} + \frac{\delta}{2} A_\alpha^{\mathrm{T}} P_e Q_e C^{\mathrm{T}},$$

$$\Omega_r \triangleq P_{ra} Q_r + \frac{\delta}{2} B^{\mathrm{T}} A_\alpha^{\mathrm{T}} P_r Q_r + \frac{\delta}{2} B^{\mathrm{T}} P_r Q_r A_\alpha^{\mathrm{T}}.$$

Theorem 10.1: Suppose there exist $n \times n$ nonnegative-definite matrices Q_e, P_e, Q_r, P_r satisfying

$$0 = A_{e\alpha} Q_e + Q_e A_{e\alpha}^{\mathrm{T}} + \frac{\delta}{2} A_{e\alpha}^2 Q_e + \frac{\delta}{2} Q_e A_{e\alpha}^{2\mathrm{T}} + \gamma A_{e\alpha} Q_e A_{e\alpha}^{\mathrm{T}} + V_e, \tag{10.6}$$

$$0 = A_{e\alpha}^{\mathrm{T}} P_e + P_e A_{e\alpha} + \frac{\delta}{2} A_{e\alpha}^{2\mathrm{T}} P_e + \frac{\delta}{2} P_e A_{e\alpha}^2 + \gamma A_{e\alpha}^{\mathrm{T}} P_e A_{e\alpha} + N, \tag{10.7}$$

$$0 = A_{r\alpha} Q_r + Q_r A_{r\alpha}^{\mathrm{T}} + \frac{\delta}{2} A_{r\alpha}^2 Q_r + \frac{\delta}{2} Q_r A_{r\alpha}^{2\mathrm{T}} + \gamma A_{r\alpha} Q_r A_{r\alpha}^{\mathrm{T}} + V_1, \tag{10.8}$$

$$0 = A_{r\alpha}^{\mathrm{T}} P_r + P_r A_{r\alpha} + \frac{\delta}{2} A_{r\alpha}^{2\mathrm{T}} P_r + \frac{\delta}{2} P_r A_{r\alpha}^2 + \gamma A_{r\alpha}^{\mathrm{T}} P_r A_{r\alpha} + R_r, \tag{10.9}$$

and such that

$$\det \Delta_e \neq 0, \qquad \det \Delta_r \neq 0. \tag{10.10}$$

Furthermore, let A_c, B_c, C_c be given by

$$A_c = A - B_c C + BC_c, \tag{10.11}$$

$$B_c = \mathrm{vec}^{-1}\left[\Delta_e^{-1} \,\mathrm{vec}\, \Omega_e\right], \tag{10.12}$$

$$C_c = -\mathrm{vec}^{-1}\left[\Delta_r^{-1} \,\mathrm{vec}\, \Omega_r\right]. \tag{10.13}$$

Then $(A_{e\alpha}, V_e^{1/2})$ and $(A_{r\alpha}, V_1^{1/2})$ are assignable with respect to $\mathscr{E}(q, r_1, r_2)$ if and only if the observer poles and regulator poles lie in $\mathscr{E}(q, r_1, r_2)$. In this case $\sigma(\tilde{A}) \subset \mathscr{E}(q, r_1, r_2)$.

Remark 10.1: Equations (10.6) and (10.7) are used to construct the Kalman gain B_c, while (10.8) and (10.9) are used to construct the regulator gain C_c. Note that (10.6) and (10.7) are decoupled from (10.8) and (10.9) in accordance with the enforced separation.

Remark 10.2: As mentioned in Remark 9.1, special choices of δ and γ yield the circular, vertical strip, and parabolic regions.

Remark 10.3: An interesting generalization of Theorem 10.1 is to use the enforced regulator/estimator separation to develop hybrid regional pole constraints. For example, one can constrain the regulator poles to lie within an elliptical region while constraining the observer poles to lie within a parabolic region. Since the spectrum of the resulting closed-loop system consists of the union of the regulator and observer poles, this approach yields a "hybrid" design. A potentially useful application of this idea involves placing the regulator and estimator poles within disjoint vertical strips. As shown in [32], separation of regulator and estimator poles reduces closed-loop sensitivity to parameter uncertainty. The robustness ramifications of this approach will be explored in a future paper.

Remark 10.4: As with Theorem 6.1, Theorem 10.1 provides constructive sufficient conditions that yield feedback gains for pole-constrained dynamic compensation. If the Auxiliary Minimization Problem has a solution, then, as shown in Section V, these conditions are also necessary.

XI. Dynamic Output Feedback for the Sector Region

In this section we develop a somewhat different approach that characterizes dynamic output feedback controllers for regional pole placement in the sector $\mathscr{S}(\mu, 0, \theta)$ (see Fig. 6). Specifically, we utilize the *analysis* results developed in [33], [34] for controller synthesis. It is shown in [33], [34] that if $Z \in \mathbb{R}^{n\times n}$ then $\sigma(Z) \subset \mathscr{S}(\mu, 0, \theta)$ if and only if the $2n \times 2n$ matrix $\hat{Z} \triangleq \begin{bmatrix} Z_\alpha \cos\phi & -Z_\alpha \sin\phi \\ Z_\alpha \sin\phi & Z_\alpha \cos\phi \end{bmatrix}$ is asymptotically stable, where $Z_\alpha \triangleq Z + \alpha I_n$.

Next, as in Section X we consider the nth-order stabilizable and detectable plant (6.1), (6.2) and determine an nth-order dynamic compensator of the form (6.3), (6.4). Furthermore, as in Section X, we enforce regulator/estimator separation so that the compensator dynamics satisfy (10.3)

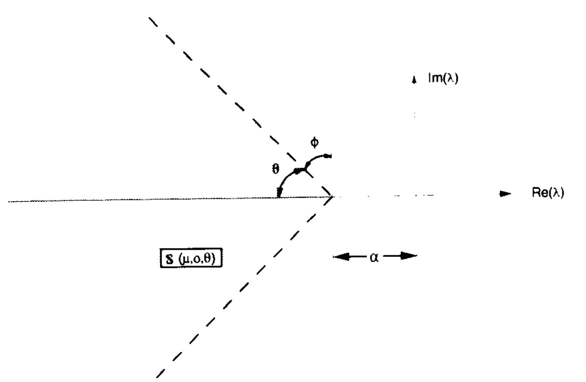

Fig. 6. Sector region $\mathscr{S}(\mu, 0, \theta)$.

and the error states satisfy (10.5). In order to exploit the analysis results of [33], [34] we reformulate the dynamic output feedback problem so as to form an augmented $2n \times 2n$ system.

Let $x_1(t),\ x_2(t) \in \mathbb{R}^n$ and consider the $2n$th-order dynamic system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_\alpha \cos\phi & -A_\alpha \sin\phi \\ A_\alpha \sin\phi & A_\alpha \cos\phi \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B\cos\phi \\ B\sin\phi \end{bmatrix} u_1(t) + \begin{bmatrix} -B\sin\phi \\ B\cos\phi \end{bmatrix} u_2(t) + \hat{w}(t), \quad (11.1)$$

$$y_1(t) = \begin{bmatrix} I_n & 0_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$y_2(t) = \begin{bmatrix} 0_n & I_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad (11.2)$$

with control inputs

$$u_1(t) = C_c y_1(t), \qquad u_2(t) = C_c y_2(t) \quad (11.3)$$

where $\hat{w}(t) \in \mathbb{R}^{2n}$ is white noise with nonnegative-definite intensity $\hat{V} \in \mathbb{R}^{2n \times 2n}$.

We now seek C_c to minimize

$$\hat{J} \triangleq \lim_{t\to\infty} \mathbb{E}\left\{ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathscr{R}_0 & 0 \\ 0 & \mathscr{R}_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u_1^{\mathrm{T}}(t)\mathscr{R}_2 u_1(t) + u_2^{\mathrm{T}}(t)\mathscr{R}_2 u_2(t) \right\} \quad (11.4)$$

where $\mathscr{R}_0,\ \mathscr{R}_1 \in \mathbb{R}^{n\times n}$ are nonnegative definite and $\mathscr{R}_2 \in R^{m\times m}$ is positive definite. It follows from (11.1)–(11.3) that x_1, x_2 satisfy

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} (A_\alpha + BC_c)\cos\phi & -(A_\alpha + BC_c)\sin\phi \\ (A_\alpha + BC_c)\sin\phi & (A_\alpha + BC_c)\cos\phi \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \hat{w}(t). \quad (11.5)$$

It now follows that if the above system is asymptotically stable, then the regulator poles $A + BC_c$ lie in the sector $\mathscr{S}(\mu, 0, \theta)$. Similarly, using a dual approach we can define error states $e_1(t),\ e_2(t) \in \mathbb{R}^n$ such that

$$\begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \end{bmatrix} = \left\{ \begin{bmatrix} A_\alpha \cos\phi & -A_\alpha \sin\phi \\ A_\alpha \sin\phi & A_\alpha\cos\phi \end{bmatrix} - \begin{bmatrix} I_n \\ 0_n \end{bmatrix} B_c \begin{bmatrix} C\cos\phi & -C\cos\phi \end{bmatrix} - \begin{bmatrix} 0 \\ I_n \end{bmatrix} B_c \begin{bmatrix} C\sin\phi & C\cos\phi \end{bmatrix} \right\} \cdot \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} + \begin{bmatrix} \mathscr{D}_0 - B_c\mathscr{D}_2 \\ \mathscr{D}_1 - B_c\mathscr{D}_2 \end{bmatrix} w(t) \quad (11.6)$$

where $\mathscr{D}_0,\ \mathscr{D}_1 \in \mathbb{R}^{n\times d}$ and $\mathscr{D}_2 \in \mathbb{R}^{l\times d}$. Using similar arguments it follows that if the above system is asymptotically stable then the observer poles $A - B_cC$ lie in $\mathscr{S}(\mu, 0, \theta)$.

At this point we make the following observations. First, the cost (11.4) is not directly related to the original problem introduced in Section VI. Rather, $\hat{J}$ can be viewed as a device for constructing feedback gains. Second, note that the reformulation of the problem (11.2)–(11.4) has a decentralized output feedback structure with two channels having the same gain C_c. Of course, similar remarks apply to the dual problem.

Next, defining the notation

$$\hat{x}(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad \hat{e}(t) \triangleq \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

$$\hat{A} \triangleq \begin{bmatrix} A_\alpha\cos\phi & -A_\alpha\sin\phi \\ A_\alpha \sin & A_\alpha\cos\phi \end{bmatrix}$$

$$\hat{B}_1 \triangleq \begin{bmatrix} B\cos\phi \\ B\sin\phi \end{bmatrix},$$

$$\hat{B}_2 \triangleq \begin{bmatrix} -B\sin\phi \\ B\cos\phi \end{bmatrix}, \qquad \hat{C}_1 \triangleq \begin{bmatrix} C\cos\phi & -C\sin\phi \end{bmatrix},$$

$$\hat{C}_2 \triangleq \begin{bmatrix} C\sin\phi & C\cos\phi \end{bmatrix}, \qquad \hat{M}_1 = \begin{bmatrix} I_n & 0_n \end{bmatrix},$$

$$\hat{M}_2 = \begin{bmatrix} 0_n & I_n \end{bmatrix},$$

$$\hat{D} \triangleq \begin{bmatrix} \mathscr{D}_0 - B_c\mathscr{D}_2 \\ \mathscr{D}_1 - B_c\mathscr{D}_2 \end{bmatrix},$$

$$\hat{R}_r \triangleq \begin{bmatrix} \mathscr{R}_0 + C_c^{\mathrm{T}}\mathscr{R}_2 C_c & 0 \\ 0 & \mathscr{R}_1 + C_c^{\mathrm{T}}\mathscr{R}_2 C_c \end{bmatrix}$$

it follows from (11.5) and (11.6) that

$$\dot{\hat{x}}(t) = (\hat{A} + \hat{B}_1 C_c \hat{M}_1 + \hat{B}_2 C_c \hat{M}_2)\hat{x}(t) + \hat{w}(t), \quad (11.7)$$

$$\dot{\hat{e}}(t) = (\hat{A} - \hat{M}_1^{\mathrm{T}} B_c \hat{C}_1 - \hat{M}_2^{\mathrm{T}} B_c \hat{C}_2)\hat{e}(t) + \hat{D}w(t) \quad (11.8)$$

where $\hat{D}w(t)$ has $2n \times 2n$ nonnegative-definite intensity $\hat{V}_e \triangleq \begin{bmatrix} \mathscr{V}_0 + B_c\mathscr{V}_2 B_c^{\mathrm{T}} & 0 \\ 0 & \mathscr{V}_1 + B_c\mathscr{V}_2 B_c^{\mathrm{T}} \end{bmatrix}$. In what follows, we assume $\mathscr{R}_2$ and $\mathscr{V}_2$ are positive definite and $\hat{N}$ is an arbitrary $2n \times 2n$ positive definite matrix.

Theorem 11.1: Suppose there exist $2n \times 2n$ nonnegative-definite matrices $\hat{Q}_e$, $\hat{P}_e$, $\hat{Q}_r$, $\hat{P}_r$ satisfying

$$0 = (\hat{A} - \hat{M}_1^{\mathrm{T}} B_c \hat{C}_1 - \hat{M}_2^{\mathrm{T}} B_c \hat{C}_2)\hat{Q}_e + \hat{Q}_e(\hat{A} - \hat{M}_1^{\mathrm{T}} B_c \hat{C}_1 - \hat{M}_2^{\mathrm{T}} B_c \hat{C}_2)^{\mathrm{T}} + \hat{V}_e, \quad (11.9)$$

$$0 = (\hat{A} - \hat{M}_1^{\mathrm{T}} B_c \hat{C}_1 - \hat{M}_2^{\mathrm{T}} B_c \hat{C}_2)^{\mathrm{T}}\hat{P}_e + \hat{P}_e(\hat{A} - \hat{M}_1^{\mathrm{T}} B_c \hat{C}_1 - \hat{M}_2^{\mathrm{T}} B_c \hat{C}_2) + \hat{N}, \quad (11.10)$$

$$0 = (\hat{A} + \hat{B}_1 C_c \hat{M}_1 + \hat{B}_2 C_c \hat{M}_2)\hat{Q}_r + \hat{Q}_r(\hat{A} + \hat{B}_1 C_c \hat{M}_1 + \hat{B}_2 C_c \hat{M}_2)^{\mathrm{T}} + \hat{V}, \quad (11.11)$$

$$0 = (\hat{A} + \hat{B}_1 C_c \hat{M}_1 + \hat{B}_2 C_c \hat{M}_2)^{\mathrm{T}}\hat{P}_r + \hat{P}_r(\hat{A} + \hat{B}_1 C_c \hat{M}_1 + \hat{B}_2 C_c \hat{M}_2) + \hat{R}_r, \quad (11.12)$$

and let

$$A_c = A - B_c C + B C_c, \quad (11.13)$$

$$B_c = \left[\sum_{i=1}^{2} \hat{M}_i \hat{P}_e \hat{M}_i^{\mathrm{T}}\right]^{-1}\left[\sum_{i=1}^{2} \hat{M}_i \hat{P}_e \hat{Q}_e \hat{C}_i^{\mathrm{T}}\right] \mathscr{V}_2^{-1}, \quad (11.14)$$

$$C_c = -\mathscr{R}_2^{-1}\left[\sum_{i=1}^{2} \hat{B}_i^{\mathrm{T}} \hat{P}_r \hat{Q}_r \hat{M}_i^{\mathrm{T}}\right]\left[\sum_{i=1}^{2} \hat{M}_i \hat{Q}_r \hat{M}_i^{\mathrm{T}}\right]^{-1}. \quad (11.15)$$

Then $\sigma(\tilde{A}) \subset \mathscr{S}(\mu, 0, \theta)$.

Remark 11.1: By setting $\alpha = \phi = 0$ in the design equations (11.9)–(11.12), one recovers the standard LQG result. Specifically, for (11.11), (11.12) note that in this case

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

so that

$$\hat{A} + \hat{B}_1 C_c \hat{M}_1 + \hat{B}_2 C_c \hat{M}_2 = \begin{bmatrix} A + BC_c & 0 \\ 0 & A + BC_c \end{bmatrix}.$$

Now using (11.11) and (11.12) it follows that $\hat{Q}_r$ is superfluous and $\hat{P}_r = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$ where P satisfies the standard regulator Riccati equation. Furthermore, C_c given by (11.15) corresponds to the standard regulator gain $C_c = -\mathscr{R}_2^{-1} B^{\mathrm{T}} P$. Similar remarks apply to (11.9), (11.10) which yield the observer Riccati equation.

XII. Numerical Results for the Sector Region

In this section we present a numerical algorithm for solving the design equations (11.9)–(11.12) and consider an illustrative numerical example.

Algorithm 12.1: To solve (11.9)–(11.12), carry out the following steps:

Step 1: Set $\phi = 0$.

Step 2: Initialize $k = 1$, $B_c^{(1)}$ = filter gain, and $C_c^{(1)}$ = regulator gain.

Step 3: With $B_c = B_c^{(k)}$, $C_c = C_c^{(k)}$ and ϕ given, solve (11.9)–(11.12) for $\hat{Q}_e^{(k)} = \hat{Q}_e$, $\hat{P}_e^{(k)} = \hat{P}_e$, $\hat{Q}_r^{(k)} = \hat{Q}_r$, and $\hat{P}_r^{(k)} = \hat{P}_r$.

Step 4: If convergence of $\hat{Q}_e^{(k)}$, $P_e^{(k)}$, $\hat{Q}_r^{(k)}$, and $\hat{P}_r^{(k)}$ has been attained, then evaluate A_c, B_c, C_c using (11.13)–(11.15); increment ϕ if desired and return to Step 3 with $k = 1$, $B_c^{(1)} = B_c$ and $C_c^{(1)} = C_c$; else continue.

Step 5: Use $\hat{Q}_e = \hat{Q}_e^{(k)}$, $\hat{P}_e = \hat{P}_e^{(k)}$, $\hat{Q}_r = \hat{Q}_r^{(k)}$, and $\hat{P}_r = \hat{P}_r^{(k)}$ to evaluate $B_c^{(k+1)} = B_c$ and $C_c^{(k+1)} = C_c$ using (11.14) and (11.15).

Step 6: Replace k by $k + 1$ and go to Step 3.

The above algorithm is a straightforward iterative scheme in the spirit of [35] which is easy to implement. More sophisticated algorithms can be developed by utilizing homotopic continuation techniques as in [36]. The development of such numerical techniques and a proof of convergence remain areas for future research. For illustrative purposes consider a simply supported Euler–Bernoulli beam. The partial differential equation for the transverse deflection $w(x, t)$ is given by

$$m(x)\frac{\partial^2 w(x,t)}{\partial t^2} = \frac{-\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + f(x,t), \quad (12.1)$$

$$w(x,t)\big|_{x=0,L} = 0, \qquad EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\bigg|_{x=0,L} = 0, \quad (12.2)$$

where $m(x)$ is the mass per unit length and $EI(x)$ is the flexural rigidity with E denoting Young's modulus of elasticity and $I(x)$ denoting the cross-sectional area moment of inertia about an axis normal to the plane of vibration and passing through the center of the cross-sectional area. Finally, $f(x, t)$ is the force distribution due to a single control actuator. Assuming uniform beam properties, the modal decomposition of this system has the form

$$w(x,t) = \sum_{r=1}^{\infty} W_r(x) q_r(t), \quad (12.3)$$

$$\int_0^L m W_r^2(x)\,\mathrm{d}x = 1, \quad (12.4)$$

$$W_r(x) = \sqrt{\frac{2}{mL}} \sin\frac{r\pi x}{L} \quad (12.5)$$

where, assuming uniform proportional damping, the modal coordinates q_r satisfy

$$\ddot{q}_r(t) + 2\zeta w_r \dot{q}_r(t) + w_r^2 q_r(t) = \int_0^L f(x,t) W_r(x)\,\mathrm{d}x, r = 1, 2, \cdots. \quad (12.6)$$

For simplicity assume $L = \pi$ and $m = EI = 2/\pi$ so that $\sqrt{2/mL} = 1$. Furthermore, assume that $f(x, t)$ arises from a point force actuator located at $x = 0.55L$ and a position sensor at $x = 0.45L$. Finally, modeling the first five modes and defining the plant state as $x = [q_1, \dot{q}_1, \cdots, q_5, \dot{q}_5]^{\mathrm{T}}$, the

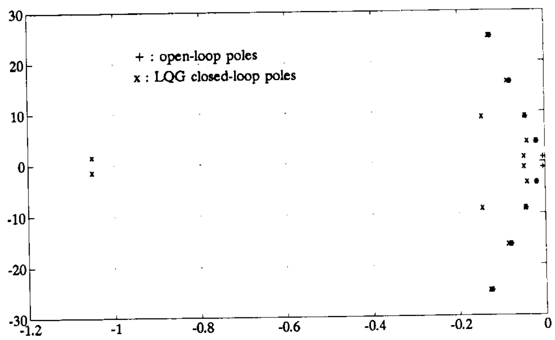

Fig. 7. Open-loop and LQG closed-loop poles.

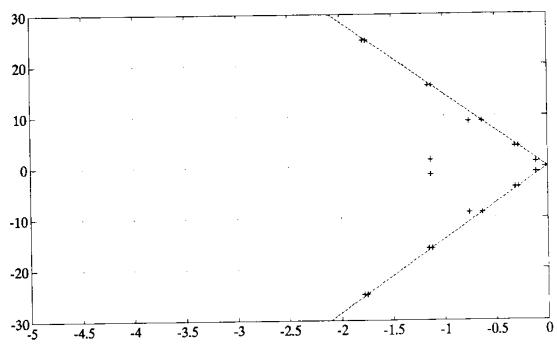

Fig. 8. Constrained closed-loop poles, $\zeta = 0.070$.

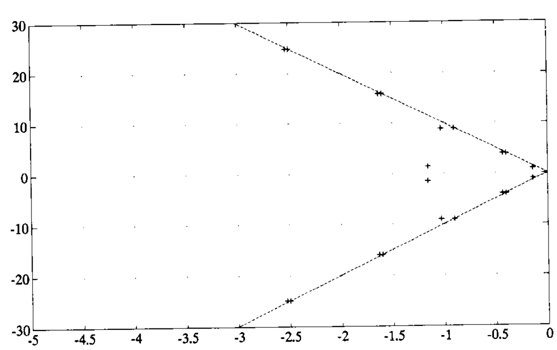

Fig. 9. Constrained closed-loop poles, $\zeta = 0.100$.

resulting state-space model and problem data are

$$A = \underset{i=1,\cdots,5}{\text{block}-\text{diag}} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \qquad w_i = i^2,$$

$$i = 1,\cdots,5, \qquad \zeta = 0.005,$$

$$B = \begin{bmatrix} 0 & 0.9877 & 0 & -0.3090 & 0 & -0.8910 & 0 & 0.5878 & 0 & 0.7071 \end{bmatrix}^{\mathrm{T}},$$

$$C = \begin{bmatrix} 0.9877 & 0 & 0.3090 & 0 & -0.8910 & 0 & -0.5878 & 0 & 0.7071 & 0 \end{bmatrix},$$

$$\mathscr{R}_0 = \mathscr{R}_1 = C^{\mathrm{T}}C, \qquad \mathscr{V}_0 = \mathscr{V}_1 = BB^{\mathrm{T}},$$

$$\mathscr{R}_2 = 100, \qquad \mathscr{V}_2 = 0.1,$$

$$\hat{N} = \hat{V} = I_{20}, \qquad \alpha = 0.$$

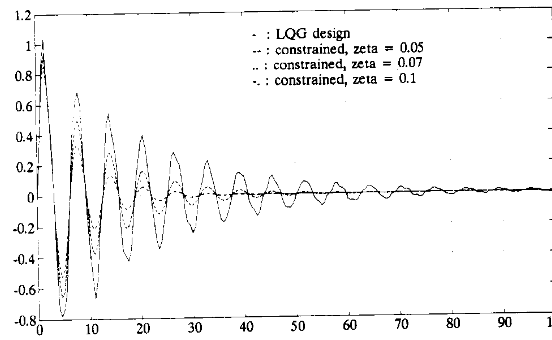

Fig. 10. Impulse response of LQG and constrained designs $\zeta = 0.050$, 0.070, 0.100.

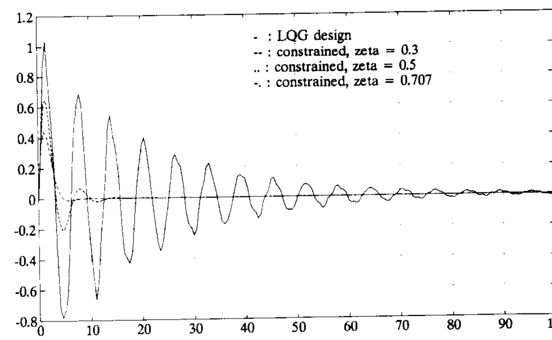

Fig. 11. Impulse response of LQG and constrained designs $\zeta = 0.300$, 0.500, 0.707.

The 10 poles of the open-loop system are $-0.005 \pm j1$, $-0.020 \pm j4$, $-0.045 \pm j9$, $-0.080 \pm j16$, $-0.125 \pm j25$. Note that the open-loop damping ratio is $\zeta = 0.005$ for all modes. Fig. 7 shows the 10 open-loop pole locations along with the 20 closed-loop pole locations for the 10th-order LQG design. Note that the worst-case ratio for the LQG design is no better than the open-loop system. By applying Algorithm 12.1, dynamic output-feedback compensators of order $n_c = 10$ were designed for the sector region $\mathscr{S}(0, 0, \theta)$ with θ chosen to correspond to $\zeta = 0.050$, 0.070, 0.100, 0.300, 0.500 and 0.707. The resulting closed-loop pole locations for $\zeta = 0.070$, and 0.100 are shown in Figs. 8 and 9. Finally, Figs. 10 and 11 show the corresponding closed-loop impulse responses for the LQG and each pole constrained design.

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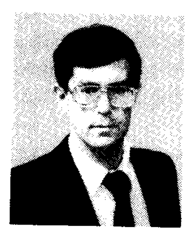

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