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REDUCED-ORDER STATE ESTIMATION FOR LINEAR TIME-VARYING SYSTEMS

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ABSTRACT

We consider reduced-order and subspace state estimators for linear discrete-time systems with possibly time-varying dynamics. The reduced-order and subspace estimators are obtained using a finite-horizon minimization approach, and thus do not require the solution of algebraic Lyapunov or Riccati equations.

Key Words: Reduced-order Kalman filter, reduced-order state estimation, linear time-varying systems.

I. INTRODUCTION

Because the classical Kalman filter provides optimal least-squares estimates of all of the states of a linear time-varying system, there is longstanding interest in obtaining simpler state estimators that estimate only a subset of the system states. This objective is of particular interest when the system order is extremely large, which occurs for systems arising from discretized partial differential equations [1–3].

One approach to this problem is to consider reduced-order Kalman filters, which provide state estimates that are suboptimal [4–6]. Variants of the classical Kalman filter have been developed for computationally demanding applications such as weather forecasting [7–9]. A comparison of various techniques is given in [10]. An alternative approach to reducing complexity is to restrict the data-injection subspace to obtain a spatially localized state estimator [11, 12].

In the present paper we revisit the approach of [4, 13], which considers the problem of fixed-order steady-state reduced-order state estimation. For a linear time-invariant system, the optimal steady-state fixed-order state estimator is characterized in [4, 13] by coupled Riccati and Lyapunov equations, whose solution requires iterative techniques.

The contribution of the present paper is to derive Kalman-like reduced-order state estimators that are applicable to time-varying systems, thus extending the results of [4, 13]. To do so, we adopt the finite-horizon optimization technique used in [11]. This technique also avoids the periodicity constraint associated with the multirate state estimator derived in [14]. Related techniques are used in [15].

Furthermore, we also present fixed-structure subspace observers constrained to estimate a specified collection of states of a linear time-varying system. This problem is considered in [5, 16] for linear time-invariant systems. The difference between the reduced-order state estimator and subspace observer is apparent in the distinct oblique projectors τ and μ, which characterize the reduced-order state estimator and the subspace observer gains, respectively. While the former estimates a given partition of the state vector, the latter focuses on a specific subspace of the state vector. Moreover, for unstable time-invariant systems, reduced-order state estimators may diverge since they may fail to adequately track the unstable modes, while subspace estimators

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circumvent this problem by including all of the unstable modes within the observed subspace [5].

The paper is structured as follows. Section II presents the one-step and two-step finite-horizon reduced-order state estimators, while the infinite-horizon reduced-order state estimator is revisited in Section III. The one-step and two-step finite-horizon subspace state estimators are derived in Section IV, while Section V revisits the infinite-horizon subspace state estimator. Two illustrative examples are investigated in Sections VI and VII. Finally, concluding remarks are given in Section VIII. A preliminary version of this paper appears as [17].

II. OPTIMAL FINITE-HORIZON REDUCED-ORDER STATE ESTIMATOR

Consider the system

$$x_{k+1} = A_k x_k + D_{1,k} w_k, \tag{1}$$

$$y_k = C_k x_k + D_{2,k} w_k, \tag{2}$$

where $x_k \in \mathbb{R}^{n_k}$ is the state vector, $y_k \in \mathbb{R}^{p_k}$ is the measured output vector, and $w_k \in \mathbb{R}^{d_k}$ is a white noise process with zero mean and unit covariance. Furthermore, assume that $A_k \in \mathbb{R}^{n_{k+1}\times n_k}$, $C_k \in \mathbb{R}^{p_k \times n_k}$, $D_{1,k} \in \mathbb{R}^{n_{k+1}\times d_k}$, and $D_{2,k} \in \mathbb{R}^{p_k \times d_k}$ are known. Note that A_k need not be square and may have time-varying size.

2.1 One-step state estimator

We consider a one-step reduced-order state estimator with dynamics

$$x_{e,k+1} = A_{e,k} x_{e,k} + B_{e,k} y_k, \tag{3}$$

where $x_{e,k} \in \mathbb{R}^{n_{e,k}}$ and $1 \le n_{e,k} \le n_k$. Define the augmented state vector

$$\tilde{x}_k \triangleq \begin{bmatrix} x_k \\ x_{e,k} \end{bmatrix}, \tag{4}$$

where $\tilde{n}_k \triangleq n_k + n_{e,k}$, and

$$\tilde{Q}_k \triangleq \mathcal{E}[\tilde{x}_k \tilde{x}_k^{\mathrm{T}}]. \tag{5}$$

Consider the cost function

$$J_k(A_{e,k}, B_{e,k}) \triangleq \mathcal{E}[(L_{k+1}x_{k+1} - x_{e,k+1})^{\mathrm{T}} \times (L_{k+1}x_{k+1} - x_{e,k+1})], \tag{6}$$

where $L_{k+1} \in \mathbb{R}^{n_{e,k+1}\times n_{k+1}}$. Throughout this paper, L determines components of the state x whose estimates are desired. We assume that L has full row rank. It follows from (5) and (4) that J_k is given by

$$J_k(A_{e,k}, B_{e,k}) = \mathrm{tr}(\tilde{Q}_{k+1}\tilde{R}_{k+1}), \tag{7}$$

where $\tilde{R}_{k+1} \in \mathbb{R}^{(n_{k+1}+n_{e+1})\times(n_{k+1}+n_{e,k+1})}$ is defined by

$$\tilde{R}_{k+1} \triangleq \begin{bmatrix} L_{k+1}^{\mathrm{T}} L_{k+1} & -L_{k+1}^{\mathrm{T}} \\ -L_{k+1} & I_{n_{e,k+1}} \end{bmatrix}. \tag{8}$$

Note that (1) and (3) imply that

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{D}_{1,k} w_k, \tag{9}$$

where

$$\tilde{A}_k \triangleq \begin{bmatrix} A_k & 0_{n_{k+1}\times n_{e,k}} \\ B_{e,k} C_k & A_{e,k} \end{bmatrix}, \quad \tilde{D}_{1,k} \triangleq \begin{bmatrix} D_{1,k} \\ B_{e,k} D_{2,k} \end{bmatrix}. \tag{10}$$

Therefore,

$$\tilde{Q}_{k+1} = \tilde{A}_k \tilde{Q}_k \tilde{A}_k^{\mathrm{T}} + \tilde{V}_{1,k}, \tag{11}$$

where

$$\tilde{V}_{1,k} \triangleq \begin{bmatrix} V_{1,k} & V_{12,k} B_e^{\mathrm{T}} \\ B_e V_{12,k}^{\mathrm{T}} & B_{e,k} V_{2,k} B_{e,k}^{\mathrm{T}} \end{bmatrix}, \tag{12}$$

and

$$V_{1,k} \triangleq D_{1,k} D_{1,k}^{\mathrm{T}}, \quad V_{12,k} \triangleq D_{1,k} D_{2,k}^{\mathrm{T}}, \quad V_{2,k} \triangleq D_{2,k} D_{2,k}^{\mathrm{T}}. \tag{13}$$

Partitioning $\tilde{Q}_k$ as

$$\tilde{Q}_k = \begin{bmatrix} \tilde{Q}_{1,k} & \tilde{Q}_{12,k} \\ \tilde{Q}_{12,k}^{\mathrm{T}} & \tilde{Q}_{2,k} \end{bmatrix}, \tag{14}$$

it follows from (11) that

$$\tilde{Q}_{1,k+1} = A_k \tilde{Q}_{1,k} A_k^{\mathrm{T}} + V_{1,k}, \tag{15}$$

$$\tilde{Q}_{12,k+1} = A_k \tilde{Q}_{1,k} C_k^{\mathrm{T}} B_{e,k}^{\mathrm{T}} + A_k \tilde{Q}_{12,k} A_{e,k}^{\mathrm{T}} + V_{12,k} B_e^{\mathrm{T}}, \tag{16}$$

$$\tilde{Q}_{2,k+1} = B_{e,k}(C_k \tilde{Q}_{1,k} C_k^{\mathrm{T}} + V_{2,k}) B_{e,k}^{\mathrm{T}} + A_{e,k} \tilde{Q}_{12,k}^{\mathrm{T}} C_k^{\mathrm{T}} B_{e,k}^{\mathrm{T}} + B_{e,k} C_k \tilde{Q}_{12,k} A_{e,k}^{\mathrm{T}} + A_{e,k} \tilde{Q}_{2,k} A_{e,k}. \tag{17}$$

Therefore, (7) and (8) imply that J_k can be expressed as

$$\begin{aligned} J_k(A_{e,k}, B_{e,k}) = {} & \mathrm{tr}[L_{k+1}(A_k\tilde{Q}_{1,k}A_k^{\mathrm{T}} + V_{1,k})L_{k+1}^{\mathrm{T}}] \\ & -2\mathrm{tr}[B_{e,k}(C_k\tilde{Q}_{1,k}A_k^{\mathrm{T}} + V_{12,k}^{\mathrm{T}})L_{k+1}^{\mathrm{T}}] \\ & -2\mathrm{tr}[A_{e,k}\tilde{Q}_{12,k}^{\mathrm{T}}A_k^{\mathrm{T}}L_{k+1}^{\mathrm{T}}] \\ & +\mathrm{tr}[B_{e,k}(C_k\tilde{Q}_{1,k}C_k^{\mathrm{T}} + V_{2,k})B_{e,k}^{\mathrm{T}}] \\ & +\mathrm{tr}[A_{e,k}\tilde{Q}_{2,k}A_{e,k}^{\mathrm{T}}] \\ & +2\mathrm{tr}[A_{e,k}\tilde{Q}_{12,k}^{\mathrm{T}}C_k^{\mathrm{T}}B_{e,k}^{\mathrm{T}}]. \end{aligned} \tag{18}$$

Next, assuming that $\tilde{Q}_{2,k}$ is invertible, we define Q_k, $\hat{Q}_k \in \mathbb{R}^{n_k\times n_k}$, $\tilde{V}_{2,k} \in \mathbb{R}^{p_k\times p_k}$, and $G_k \in \mathbb{R}^{n_{e,k}\times n_k}$ by

$$\begin{aligned} Q_k &\triangleq \tilde{Q}_{1,k} - \tilde{Q}_{12,k}\tilde{Q}_{2,k}^{-1}\tilde{Q}_{12,k}^{\mathrm{T}}, \\ \hat{Q}_k &\triangleq \tilde{Q}_{12,k}\tilde{Q}_{2,k}^{-1}\tilde{Q}_{12,k}^{\mathrm{T}}, \end{aligned} \tag{19}$$

$$\tilde{V}_{2,k} \triangleq C_kQ_kC_k^{\mathrm{T}} + V_{2,k}, \tag{20}$$

$$G_k \triangleq \tilde{Q}_{2,k}^{-1}\tilde{Q}_{12,k}^{\mathrm{T}}. \tag{21}$$

We assume that $\tilde{V}_{2,k}$ is invertible.

The following result characterizes $A_{e,k}$ and $B_{e,k}$ that minimize J_k.

Proposition II.1. Assume that $\tilde{Q}_{2,k}$ and $\tilde{V}_{2,k}$ are invertible and $A_{e,k}$ and $B_{e,k}$ minimize J_k. Then, $A_{e,k}$ and $B_{e,k}$ satisfy

$$A_{e,k} = L_{k+1}(A_k - Q_{s,k}\tilde{V}_{2,k}^{-1}C_k)G_k^{\mathrm{T}}, \tag{22}$$

$$B_{e,k} = L_{k+1}Q_{s,k}\tilde{V}_{2,k}^{-1}, \tag{23}$$

where

$$Q_{s,k} \triangleq A_kQ_kC_k^{\mathrm{T}} + V_{12,k}. \tag{24}$$

Proof. Setting $\frac{\partial J_k}{\partial A_{e,k}} = 0$, $\frac{\partial J_k}{\partial B_{e,k}} = 0$ and using (19)–(21) yield the result. □

Proposition II.2. Assume that $A_{e,k}$ and $B_{e,k}$ satisfy Proposition II.1. Then,

$$L_{k+1}\tilde{Q}_{12,k+1} = \tilde{Q}_{2,k+1}, \tag{25}$$

$$\tilde{Q}_{12,k+1} = \hat{Q}_{k+1}L_{k+1}^{\mathrm{T}}, \tag{26}$$

$$\tilde{Q}_{2,k+1} = L_{k+1}\hat{Q}_{k+1}L_{k+1}^{\mathrm{T}}. \tag{27}$$

Proof. Substituting (22) and (23) into (16) and (17) yields

$$\tilde{Q}_{12,k+1} = [A_k\hat{Q}_kA_k^{\mathrm{T}} + Q_{s,k}\tilde{V}_{2,k}^{-1}Q_{s,k}^{\mathrm{T}}]L_{k+1}^{\mathrm{T}}, \tag{28}$$

$$\tilde{Q}_{2,k+1} = L_{k+1}[A_k\hat{Q}_kA_k^{\mathrm{T}} + Q_{s,k}\tilde{V}_{2,k}^{-1}Q_{s,k}^{\mathrm{T}}]L_{k+1}^{\mathrm{T}}. \tag{29}$$

Pre-multiplying (28) by L_{k+1} yields $L_{k+1}\tilde{Q}_{12,k+1} = \tilde{Q}_{2,k+1}$. Using (19) and $L_{k+1}\tilde{Q}_{12,k+1} = \tilde{Q}_{2,k+1}$ yields $\tilde{Q}_{12,k+1} = \hat{Q}_{k+1}L_{k+1}^{\mathrm{T}}$ and $\tilde{Q}_{2,k+1} = L_{k+1}\hat{Q}_{k+1}L_{k+1}^{\mathrm{T}}$. □

Next, define $M_{k+1} \in \mathbb{R}^{n_{k+1}\times n_{k+1}}$ by

$$M_{k+1} \triangleq A_k\hat{Q}_kA_k^{\mathrm{T}} + Q_{s,k}\tilde{V}_{2,k}^{-1}Q_{s,k}^{\mathrm{T}}, \tag{30}$$

and define $\tau_{k+1}, \tau_{k+1\perp} \in \mathbb{R}^{n_{k+1}\times n_{k+1}}$ by

$$\tau_{k+1} \triangleq G_{k+1}^{\mathrm{T}}L_{k+1}, \quad \tau_{k+1\perp} \triangleq I_{n_{k+1}} - \tau_{k+1}. \tag{31}$$

Proposition II.3. Assume that $A_{e,k}$ and $B_{e,k}$ satisfy Proposition II.1. Then, $\tau_{k+1}^2 = \tau_{k+1}$, that is, τ_{k+1} is an oblique projector.

Proof. It follows from (30) that (28) and (29) can be expressed as

$$\begin{aligned} \tilde{Q}_{12,k+1} &= M_{k+1}L_{k+1}^{\mathrm{T}}, \\ \tilde{Q}_{2,k+1} &= L_{k+1}M_{k+1}L_{k+1}^{\mathrm{T}}. \end{aligned} \tag{32}$$

Hence, (31) implies that

$$\tau_{k+1} = M_{k+1}L_{k+1}^{\mathrm{T}}(L_{k+1}M_{k+1}L_{k+1}^{\mathrm{T}})^{-1}L_{k+1}. \tag{33}$$

Therefore, $\tau_{k+1}^2 = \tau_{k+1}$. □

Proposition II.4. Assume that $A_{e,k}$ and $B_{e,k}$ satisfy Proposition II.1. Then,

$$\tau_{k+1}\hat{Q}_{k+1} = \hat{Q}_{k+1}. \tag{34}$$

Proof. It follows from (19) that

$$\hat{Q}_{k+1} = \tilde{Q}_{12,k+1}\tilde{Q}_{2,k+1}^{-1}\tilde{Q}_{12,k+1}^{\mathrm{T}}. \tag{35}$$

Substituting (32) into (35) yields

$$\begin{aligned} \hat{Q}_{k+1} = {} & M_{k+1}L_{k+1}^{\mathrm{T}} \\ & \times(L_{k+1}M_{k+1}L_{k+1}^{\mathrm{T}})^{-1}L_{k+1}M_{k+1}. \end{aligned} \tag{36}$$

Hence, pre-multiplying (36) by τ_{k+1} and substituting (33) into the resulting expression yields (34). □

Proposition II.5. Assume that $A_{e,k}$ and $B_{e,k}$ satisfy Proposition II.1. Then,

$$Q_{k+1} = A_k Q_k A_k^{\mathrm{T}} - Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}} + V_{1,k} + \tau_{k+1\perp} \times [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] \tau_{k+1\perp}^{\mathrm{T}}, \quad (37)$$

$$\hat{Q}_{k+1} = \tau_{k+1} [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] \tau_{k+1}^{\mathrm{T}}, \quad (38)$$

$$\tau_{k+1} = M_{k+1} L_{k+1}^{\mathrm{T}} (L_{k+1} M_{k+1} L_{k+1}^{\mathrm{T}})^{-1} L_{k+1}. \quad (39)$$

Proof. It follows from (25) and (29) that

$$L_{k+1} \hat{Q}_{k+1} L_{k+1}^{\mathrm{T}} = L_{k+1} [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] L_{k+1}^{\mathrm{T}}. \quad (40)$$

Pre-multiplying and post-multiplying (40) by G_{k+1}^{T} and G_{k+1}, respectively, yields

$$\tau_{k+1} \hat{Q}_{k+1} \tau_{k+1}^{\mathrm{T}} = \tau_{k+1} [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] \tau_{k+1}^{\mathrm{T}}. \quad (41)$$

Hence, (38) follows from Proposition II.4.

Since $\tilde{Q}_{12,k+1} = \hat{Q}_{k+1} L_{k+1}$, (28) and (31) imply that

$$\tau_{k+1} \hat{Q}_{k+1} = \tau_{k+1} [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}]. \quad (42)$$

Therefore, (38) imply that

$$\tau_{k+1} [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] = \tau_{k+1} [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] \tau_{k+1}^{\mathrm{T}}. \quad (43)$$

Hence, $\hat{Q}_{k+1}$ can be expressed as

$$\hat{Q}_{k+1} = A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}} - \tau_{k+1\perp} \times [A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}] \tau_{k+1\perp}^{\mathrm{T}}. \quad (44)$$

Furthermore, it follows from (15) and (19) that

$$Q_{k+1} = A_k Q_k A_k^{\mathrm{T}} + V_{1,k} + A_k \hat{Q}_k A_k^{\mathrm{T}} - \hat{Q}_{k+1}. \quad (45)$$

Therefore, substituting (44) into (45) yields (37). □

Note that although $A_{e,k}$ and $B_{e,k}$ depend on $\tilde{Q}_{12,k}$ and $\tilde{Q}_{2,k}$, it follows from Proposition II.2 that $\tilde{Q}_{2,k}$ and $\tilde{Q}_{12,k}$ can be obtained from Q_k and $\hat{Q}_k$. Hence, it suffices to propagate Q_k and $\hat{Q}_k$ using (37) and (38), respectively.

Finally, we summarize the one-step reduced-order state estimator, whose state estimate update is given by

$$x_{e,k+1} = L_{k+1} (A_k - K_k C_k) G_k^{\mathrm{T}} x_{e,k} + L_{k+1} K_k y_k, \quad (46)$$

and whose covariance update is given by

$$Q_{k+1} = A_k Q_k A_k^{\mathrm{T}} + V_{1,k} - Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s}^{\mathrm{T}} + \tau_{k+1\perp} M_{k+1} \tau_{k+1\perp}^{\mathrm{T}}, \quad (47)$$

where

$$G_k = (L_k \hat{Q}_k L_k^{\mathrm{T}})^{-1} L_k \hat{Q}_k, \quad (48)$$

$$K_k = Q_{s,k} \tilde{V}_{2,k}^{-1}, \quad (49)$$

$$M_{k+1} = A_k \hat{Q}_k A_k^{\mathrm{T}} + Q_{s,k} \tilde{V}_{2,k}^{-1} Q_{s,k}^{\mathrm{T}}, \quad (50)$$

$$\tau_{k+1} = M_{k+1} L_{k+1}^{\mathrm{T}} (L_{k+1} M_{k+1} L_{k+1}^{\mathrm{T}})^{-1} L_{k+1}, \quad (51)$$

$$\hat{Q}_{k+1} = \tau_{k+1} M_{k+1} \tau_{k+1}^{\mathrm{T}}, \quad (52)$$

$\tilde{V}_{2,k}$ is given by (20), and $Q_{s,k}$ is given by (24).

Remark II.1. Note that, since $x_{e,k+1}$ in (46) does not use the current measurement y_{k+1}, (46)–(47) comprise predictor equations rather than filter equations. The differences between predictors and filters are discussed in [18].

Remark II.2. As is commonly done in the Kalman filtering literature, we can rewrite (46)–(47) as

$$x_{e,k+1} = L_{k+1} [A_k G_k^{\mathrm{T}} x_{e,k} + K_k (y_k - C_k G_k^{\mathrm{T}} x_{e,k})], \quad (53)$$

$$Q_{k+1} = A_k Q_k A_k^{\mathrm{T}} + V_{1,k} - K_k \tilde{V}_{2,k} K_k^{\mathrm{T}} + \tau_{k+1\perp} M_{k+1} \tau_{k+1\perp}^{\mathrm{T}}, \quad (54)$$

where the Kalman gain is given by (49). Note that, if $L_{k+1} = I_{n_{k+1}}$, then $\tau_{k+1} = I_{n_{k+1}}$, $\tau_{k+1\perp} = 0_{n_{k+1}}$, $G_{k+1} = I_{n_{k+1}}$, and $M_{k+1} = \hat{Q}_{k+1}$, and we thus recover the full-order Kalman predictor.

2.2 Two-step state estimator

We now consider a two-step state estimator. The data assimilation step is given by

$$x_{e,k}^{\mathrm{da}} = C_{e,k}^{\mathrm{da}} x_{e,k}^{\mathrm{f}} + D_{e,k}^{\mathrm{da}} y_k, \quad (55)$$

where $x_{\mathrm{e},k}^{\mathrm{da}} \in \mathbb{R}^{n_{\mathrm{e},k}}$ is the reduced-order data assimilation estimate of $L_k x_k$, and $x_{\mathrm{e},k}^{\mathrm{f}} \in \mathbb{R}^{n_{\mathrm{e},k}}$ is the reduced-order forecast estimate of $L_k x_k$. The forecast step or physics update of the estimator is given by

$$x_{\mathrm{e},k+1}^{\mathrm{f}} = A_{\mathrm{e},k}^{\mathrm{f}} x_{\mathrm{e},k}^{\mathrm{da}}. \tag{56}$$

Remark II.3. For large-scale applications, the processing time of $x_{\mathrm{e},k}^{\mathrm{da}}$ at time k using y_k in (55) may not be neglegible compared to the sample interval. We thus present the forecast estimate $x_{\mathrm{e},k+1}^{\mathrm{f}}$ as the final estimate of the two-step predictor (55)–(56).

Now, define the augmented forecast state vector $\tilde{x}_k^{\mathrm{f}} \in \mathbb{R}^{\tilde{n}_k}$ and augmented data-assimilation state vector $\tilde{x}_k^{\mathrm{da}} \in \mathbb{R}^{\tilde{n}_k}$, respectively, by

$$\tilde{x}_k^{\mathrm{f}} \triangleq \begin{bmatrix} x_k \\ x_{\mathrm{e},k}^{\mathrm{f}} \end{bmatrix}, \quad \tilde{x}_k^{\mathrm{da}} \triangleq \begin{bmatrix} x_k \\ x_{\mathrm{e},k}^{\mathrm{da}} \end{bmatrix}. \tag{57}$$

Also define,

$$\tilde{Q}_k^{\mathrm{f}} \triangleq \mathcal{E}[\tilde{x}_k^{\mathrm{f}}(\tilde{x}_k^{\mathrm{f}})^{\mathrm{T}}], \quad \tilde{Q}_k^{\mathrm{da}} \triangleq \mathcal{E}[\tilde{x}_k^{\mathrm{da}}(\tilde{x}_k^{\mathrm{da}})^{\mathrm{T}}]. \tag{58}$$

Defining the data assimilation cost

$$J_k^{\mathrm{da}}(C_{\mathrm{e},k}^{\mathrm{da}}, D_{\mathrm{e},k}^{\mathrm{da}}) \triangleq \mathcal{E}[(L_k x_k - x_{\mathrm{e},k}^{\mathrm{da}})^{\mathrm{T}}(L_k x_k - x_{\mathrm{e},k}^{\mathrm{da}})], \tag{59}$$

(58) implies that

$$J_k^{\mathrm{da}}(C_{\mathrm{e},k}^{\mathrm{da}}, D_{\mathrm{e},k}^{\mathrm{da}}) = \mathrm{tr}(\tilde{Q}_k^{\mathrm{da}} \tilde{R}_k), \tag{60}$$

where $\tilde{R}_k$ is defined by (8).

Next, it follows from (1), (55), and (57) that

$$\tilde{x}_k^{\mathrm{da}} = \tilde{A}_k^{\mathrm{da}} \tilde{x}_k^{\mathrm{f}} + \tilde{D}_{1,k}^{\mathrm{da}} w_k, \tag{61}$$

where $\tilde{A}_k^{\mathrm{da}} \in \mathbb{R}^{\tilde{n}_k \times \tilde{n}_k}$ and $\tilde{D}_{1,k}^{\mathrm{da}} \in \mathbb{R}^{\tilde{n}_k \times d_k}$ are defined by

$$\tilde{A}_k^{\mathrm{da}} \triangleq \begin{bmatrix} I_{n_k} & 0_{n_k \times n_{\mathrm{e},k}} \\ D_{\mathrm{e},k}^{\mathrm{da}} C_k & C_{\mathrm{e},k}^{\mathrm{da}} \end{bmatrix}, \quad \tilde{D}_{1,k}^{\mathrm{da}} \triangleq \begin{bmatrix} 0_{n_k \times d_k} \\ D_{\mathrm{e},k}^{\mathrm{da}} D_{2,k} \end{bmatrix}. \tag{62}$$

Therefore,

$$\tilde{Q}_k^{\mathrm{da}} = \tilde{A}_k^{\mathrm{da}} \tilde{Q}_k^{\mathrm{f}} (\tilde{A}_k^{\mathrm{da}})^{\mathrm{T}} + \tilde{D}_{1,k}^{\mathrm{da}} (\tilde{D}_{1,k}^{\mathrm{da}})^{\mathrm{T}}. \tag{63}$$

Hence, J_k^{da} can be expressed as

$$\begin{aligned} J_k^{\mathrm{da}}(C_{\mathrm{e},k}^{\mathrm{da}}, D_{\mathrm{e},k}^{\mathrm{da}}) = \mathrm{tr}[(&\tilde{A}_k^{\mathrm{da}} \tilde{Q}_k^{\mathrm{f}} (\tilde{A}_k^{\mathrm{da}})^{\mathrm{T}} \\ &+ \tilde{D}_{1,k}^{\mathrm{da}} (\tilde{D}_{1,k}^{\mathrm{da}})^{\mathrm{T}}) \tilde{R}_k]. \end{aligned} \tag{64}$$

Finally, partition $\tilde{Q}_k^{\mathrm{f}}$ as

$$\tilde{Q}_k^{\mathrm{f}} = \begin{bmatrix} \tilde{Q}_{1,k}^{\mathrm{f}} & \tilde{Q}_{12,k}^{\mathrm{f}} \\ (\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}} & \tilde{Q}_{2,k}^{\mathrm{f}} \end{bmatrix}, \tag{65}$$

so that substituting (62) into (64) yields

$$\begin{aligned} J_k^{\mathrm{da}}(C_{\mathrm{e},k}^{\mathrm{da}}, D_{\mathrm{e},k}^{\mathrm{da}}) = {} & \mathrm{tr}[L_k \tilde{Q}_{1,k}^{\mathrm{f}} L_k^{\mathrm{T}}] - 2\mathrm{tr}[D_{\mathrm{e},k}^{\mathrm{da}} C_k \tilde{Q}_{1,k}^{\mathrm{f}} L_k^{\mathrm{T}}] \\ & - 2\mathrm{tr}[L_k \tilde{Q}_{12,k}^{\mathrm{f}} (C_{\mathrm{e},k}^{\mathrm{da}})^{\mathrm{T}}] \\ & + \mathrm{tr}[C_{\mathrm{e},k}^{\mathrm{da}} \tilde{Q}_{2,k}^{\mathrm{f}} (C_{\mathrm{e},k}^{\mathrm{da}})^{\mathrm{T}}] \\ & + 2\mathrm{tr}[D_{\mathrm{e},k}^{\mathrm{da}} C_k \tilde{Q}_{12,k}^{\mathrm{f}} (C_{\mathrm{e},k}^{\mathrm{da}})^{\mathrm{T}}] \\ & + \mathrm{tr}[D_{\mathrm{e},k}^{\mathrm{da}} (C_k \tilde{Q}_{1,k}^{\mathrm{f}} C_k^{\mathrm{T}} + V_{2,k})(D_{\mathrm{e},k}^{\mathrm{da}})^{\mathrm{T}}] \end{aligned} . \tag{66}$$

Assuming that $\tilde{Q}_{2,k}^{\mathrm{f}}$ is invertible, define $Q_k^{\mathrm{f}}, \hat{Q}_k^{\mathrm{f}} \in \mathbb{R}^{n_k \times n_k}$ by

$$\begin{aligned} Q_k^{\mathrm{f}} &\triangleq \tilde{Q}_{1,k}^{\mathrm{f}} - \tilde{Q}_{12,k}^{\mathrm{f}} (\tilde{Q}_{2,k}^{\mathrm{f}})^{-1} (\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}, \\ \hat{Q}_k^{\mathrm{f}} &\triangleq \tilde{Q}_{12,k}^{\mathrm{f}} (\tilde{Q}_{2,k}^{\mathrm{f}})^{-1} (\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}} \end{aligned} \tag{67}$$

Finally, define $V_{2,k}^{\mathrm{da}} \in \mathbb{R}^{p_k \times p_k}$ by

$$V_{2,k}^{\mathrm{da}} \triangleq C_k Q_k^{\mathrm{f}} C_k^{\mathrm{T}} + V_{2,k}, \tag{68}$$

and $G_k^{\mathrm{da}} \in \mathbb{R}^{n_{\mathrm{e},k} \times n_k}$ by

$$G_k^{\mathrm{da}} \triangleq (\tilde{Q}_{2,k}^{\mathrm{f}})^{-1} (\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}. \tag{69}$$

We assume that $V_{2,k}^{\mathrm{da}}$ is invertible.

The following result characterizes $C_{\mathrm{e},k}^{\mathrm{da}}$ and $D_{\mathrm{e},k}^{\mathrm{da}}$ that minimize J_k^{da}.

Proposition II.6. Assume that $C_{\mathrm{e},k}^{\mathrm{da}}$ and $D_{\mathrm{e},k}^{\mathrm{da}}$ minimize J_k^{da}, and assume that $\tilde{Q}_{2,k}^{\mathrm{f}}$ and $V_{2,k}^{\mathrm{da}}$ are invertible. Then,

$$C_{\mathrm{e},k}^{\mathrm{da}} = L_k (I_{n_k} - Q_k^{\mathrm{f}} C_k^{\mathrm{T}} (V_{2,k}^{\mathrm{da}})^{-1} C_k)(G_k^{\mathrm{da}})^{\mathrm{T}}, \tag{70}$$

$$D_{\mathrm{e},k}^{\mathrm{da}} = L_k Q_k^{\mathrm{f}} C_k^{\mathrm{T}} (V_{2,k}^{\mathrm{da}})^{-1}. \tag{71}$$

Proof. Setting $\frac{\partial J_k^{\mathrm{da}}}{\partial C_{\mathrm{e},k}^{\mathrm{da}}} = 0$, $\frac{\partial J_k^{\mathrm{da}}}{\partial D_{\mathrm{e},k}^{\mathrm{da}}} = 0$ and using (67)–(69) yields the result. □

Next, partition $\tilde{Q}_k^{\mathrm{da}}$ as

$$\tilde{Q}_k^{\mathrm{da}} = \begin{bmatrix} \tilde{Q}_{1,k}^{\mathrm{da}} & \tilde{Q}_{12,k}^{\mathrm{da}} \\ (\tilde{Q}_{12,k}^{\mathrm{da}})^{\mathrm{T}} & \tilde{Q}_{2,k}^{\mathrm{da}} \end{bmatrix}. \tag{72}$$

Proposition II.7. Assume that $x_{\mathrm{e},k}^{\mathrm{da}}$ is given by (55), and let $C_{\mathrm{e},k}^{\mathrm{da}}$ and $D_{\mathrm{e},k}^{\mathrm{da}}$ satisfy (70), (71). Then,

$$\tilde{Q}_{1,k}^{\mathrm{da}} = \tilde{Q}_{1,k}^{\mathrm{f}}, \tag{73}$$

$$\tilde{Q}^{\rm da}_{12,k} = (\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k) L^{\rm T}_k, \qquad (74)$$

$$\tilde{Q}^{\rm da}_{2,k} = L_k (\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k) L^{\rm T}_k. \qquad (75)$$

Proof. It follows from (63) that $\tilde{Q}^{\rm da}_{1,k} = \tilde{Q}^{\rm f}_{1,k}$ and

$$\tilde{Q}^{\rm da}_{12,k} = \tilde{Q}^{\rm f}_{12,k} (C^{\rm da}_{{\rm e},k})^{\rm T} + \tilde{Q}^{\rm f}_{1,k} C^{\rm T}_k (D^{\rm da}_{{\rm e},k})^{\rm T}. \qquad (76)$$

Substituting (70) and (71) into (76) yields (74). Similarly, it follows from (63) and (72) that

$$\begin{aligned}\tilde{Q}^{\rm da}_{2,k} = {} & C^{\rm da}_{{\rm e},k} \tilde{Q}^{\rm f}_{1,k} (C^{\rm da}_{{\rm e},k})^{\rm T} + C^{\rm da}_{{\rm e},k} (\tilde{Q}^{\rm f}_{12,k})^{\rm T} C^{\rm T}_k (D^{\rm da}_{{\rm e},k})^{\rm T} \\ & + D^{\rm da}_{{\rm e},k} C_k \tilde{Q}^{\rm f}_{12,k} (C^{\rm da}_{{\rm e},k})^{\rm T} \\ & + D^{\rm da}_{{\rm e},k} (C_k \tilde{Q}^{\rm f}_{1,k} C^{\rm T}_k + V_{2,k}) (D^{\rm da}_{{\rm e},k})^{\rm T}. \qquad (77)\end{aligned}$$

Finally, substituting (70) and (71) into (77) yields (75). □

Next, define $Q^{\rm da}_k \in \mathbb{R}^{n_k \times n_k}$ and $\hat{Q}^{\rm da}_k \in \mathbb{R}^{n_k \times n_k}$ by

$$\begin{aligned} Q^{\rm da}_k &\triangleq \tilde{Q}^{\rm da}_{1,k} - \tilde{Q}^{\rm da}_{12,k} (\tilde{Q}^{\rm da}_{2,k})^{-1} (\tilde{Q}^{\rm da}_{12,k})^{\rm T}, \\ \hat{Q}^{\rm da}_k &\triangleq \tilde{Q}^{\rm da}_{12,k} (\tilde{Q}^{\rm da}_{2,k})^{-1} (\tilde{Q}^{\rm da}_{12,k})^{\rm T}. \end{aligned} \qquad (78)$$

Corollary II.1. Assume that $C^{\rm da}_{{\rm e},k}$ and $D^{\rm da}_{{\rm e},k}$ satisfy Proposition II.6. Then,

$$\begin{aligned} L_k \tilde{Q}^{\rm da}_{12,k} &= \tilde{Q}^{\rm da}_{2,k}, \quad \tilde{Q}^{\rm da}_{12,k} = \hat{Q}^{\rm da}_k L^{\rm T}_k, \\ \tilde{Q}^{\rm da}_{2,k} &= L_k \hat{Q}^{\rm da}_k L^{\rm T}_k. \end{aligned} \qquad (79)$$

Next, define $G^{\rm f}_k \in \mathbb{R}^{n_{{\rm e},k} \times n_k}$ by

$$G^{\rm f}_k \triangleq (\tilde{Q}^{\rm da}_{2,k})^{-1} (\tilde{Q}^{\rm da}_{12,k})^{\rm T}. \qquad (80)$$

Also, define $M^{\rm da}_k \in \mathbb{R}^{n_k \times n_k}$ by

$$M^{\rm da}_k \triangleq \hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k, \qquad (81)$$

and $\tau^{\rm da}_k$, $\tau^{\rm da}_{k\perp} \in \mathbb{R}^{n_k \times n_k}$ by

$$\tau^{\rm da}_k \triangleq (G^{\rm f}_k)^{\rm T} L_k, \quad \tau^{\rm da}_{k\perp} \triangleq I_{n_k} - \tau^{\rm da}_k. \qquad (82)$$

Proposition II.8. Assume that $C^{\rm da}_{{\rm e},k}$ and $D^{\rm da}_{{\rm e},k}$ satisfy Proposition II.6. Then, $\tau^{\rm da}_k$ is an oblique projector.

Proof. The proof is similar to the proof of Proposition II.3. □

Proposition II.9. Assume that $C^{\rm da}_{{\rm e},k}$ and $D^{\rm da}_{{\rm e},k}$ satisfy Proposition II.6. Then,

$$\tau^{\rm da}_k \hat{Q}^{\rm da}_k = \hat{Q}^{\rm da}_k. \qquad (83)$$

Proof. The proof is similar to the proof of Proposition II.4. □

Proposition II.10. Assume that $x^{\rm da}_{{\rm e},k}$ is given by (55), and let $C^{\rm da}_{{\rm e},k}$ and $D^{\rm da}_{{\rm e},k}$ satisfy Proposition II.6. Then,

$$\hat{Q}^{\rm da}_k = \tau^{\rm da}_k (\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k)(\tau^{\rm da}_k)^{\rm T}, \qquad (84)$$

$$\begin{aligned} Q^{\rm da}_k = {} & Q^{\rm f}_k - Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k \\ & + \tau^{\rm da}_{k\perp} (\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k)(\tau^{\rm da}_{k\perp})^{\rm T}. \qquad (85)\end{aligned}$$

Proof. It follows from (75) and (79) that

$$L_k \hat{Q}^{\rm da}_k L^{\rm T}_k = L_k (\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k) L^{\rm T}_k. \qquad (86)$$

Pre-multiplying and post-multiplying (86) by $(G^{\rm f}_k)^{\rm T}$ and $G^{\rm f}_k$, respectively, yields (84).

Next, it follows from (74), (79), and (82) that

$$\tau^{\rm da}_k \hat{Q}^{\rm da}_k = \tau^{\rm da}_k (\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k). \qquad (87)$$

Therefore, Proposition II.9 and (84) imply that

$$\begin{aligned} & \tau^{\rm da}_k [\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k] \\ & = \tau^{\rm da}_k [\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k](\tau^{\rm da}_k)^{\rm T}. \qquad (88)\end{aligned}$$

Hence, $\hat{Q}^{\rm da}_k$ can be expressed as

$$\begin{aligned} \hat{Q}^{\rm da}_k = {} & \hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k - \tau^{\rm da}_{k\perp} \\ & \times [\hat{Q}^{\rm f}_k + Q^{\rm f}_k C^{\rm T}_k (V^{\rm da}_{2,k})^{-1} C_k Q^{\rm f}_k](\tau^{\rm da}_{k\perp})^{\rm T}. \qquad (89)\end{aligned}$$

Finally, note that (73) implies that $Q^{\rm da}_k = \tilde{Q}^{\rm f}_{1,k} - \hat{Q}^{\rm da}_k$. Hence, (89) yields (85). □

Next, we define the forecast cost $J^{\rm f}_k$ by

$$\begin{aligned} J^{\rm f}_k(A^{\rm f}_{{\rm e},k}) \triangleq {} & \mathcal{E}[(L_{k+1} x_{k+1} - x^{\rm f}_{{\rm e},k+1}) \\ & \times (L_{k+1} x_{k+1} - x^{\rm f}_{{\rm e},k+1})^{\rm T}]. \qquad (90)\end{aligned}$$

Hence, it follows from (58) that

$$J^{\rm f}_k(A^{\rm f}_{{\rm e},k}) = {\rm tr}(\tilde{Q}^{\rm f}_{k+1} \tilde{R}_{k+1}), \qquad (91)$$

where $\tilde{R}_{k+1}$ is given by (8). It follows from (1) and (56) that

$$\tilde{x}^{\rm f}_{k+1} = \tilde{A}^{\rm f}_k \tilde{x}^{\rm da}_k + \tilde{D}^{\rm f}_{1,k} w_k, \qquad (92)$$

where $\tilde{A}_k^{\mathrm{f}} \in \mathbb{R}^{\tilde{n}_{k+1}\times\tilde{n}_k}$ and $\tilde{D}_{1,k}^{\mathrm{f}} \in \mathbb{R}^{\tilde{n}_{k+1}\times d_k}$ are defined by

$$\tilde{A}_k^{\mathrm{f}} \triangleq \begin{bmatrix} A_k & 0_{n_{k+1}\times n_{\mathrm{e},k}} \\ 0_{n_{\mathrm{e},k+1}\times n_k} & A_{\mathrm{e},k}^{\mathrm{f}} \end{bmatrix},$$

$$\tilde{D}_{1,k}^{\mathrm{f}} \triangleq \begin{bmatrix} D_{1,k} \\ 0_{n_{\mathrm{e},k+1}\times d_k} \end{bmatrix}. \tag{93}$$

Therefore,

$$\tilde{Q}_{k+1}^{\mathrm{f}} = \tilde{A}_k^{\mathrm{f}} \tilde{Q}_k^{\mathrm{da}} (\tilde{A}_k^{\mathrm{f}})^{\mathrm{T}} + \tilde{D}_{1,k}^{\mathrm{f}} (\tilde{D}_{1,k}^{\mathrm{f}})^{\mathrm{T}}. \tag{94}$$

Proposition II.11. Assume that $A_{\mathrm{e},k}^{\mathrm{f}}$ minimizes J_k^{f}, and assume that $\tilde{Q}_{2,k}^{\mathrm{da}}$ is invertible. Then

$$A_{\mathrm{e},k}^{\mathrm{f}} = L_{k+1} A_k (G_k^{\mathrm{f}})^{\mathrm{T}}, \tag{95}$$

where G_k^{f} is given by (80).

Proof. Setting $\frac{\partial J_k^{\mathrm{f}}}{\partial A_{\mathrm{e},k}^{\mathrm{f}}} = 0$ yields the result. □

Proposition II.12. Assume that $A_{\mathrm{e},k}^{\mathrm{f}}$ satisfies (95). Then,

$$L_{k+1} \tilde{Q}_{12,k+1}^{\mathrm{f}} = \tilde{Q}_{2,k+1}^{\mathrm{f}}, \tag{96}$$

$$\tilde{Q}_{12,k+1}^{\mathrm{f}} = \hat{Q}_{k+1}^{\mathrm{f}} L_{k+1}^{\mathrm{T}}, \tag{97}$$

$$\tilde{Q}_{2,k+1}^{\mathrm{f}} = L_{k+1} \hat{Q}_{k+1}^{\mathrm{f}} L_{k+1}^{\mathrm{T}}. \tag{98}$$

Proof. The proof is similar to the proof of Proposition II.2. □

Next, define $M_{k+1}^{\mathrm{f}} \in \mathbb{R}^{n_{k+1}\times n_{k+1}}$ by

$$M_{k+1}^{\mathrm{f}} \triangleq A_k \hat{Q}_k^{\mathrm{da}} A_k^{\mathrm{T}}, \tag{99}$$

and define $\tau_{k+1}^{\mathrm{f}}, \tau_{k+1\perp}^{\mathrm{f}} \in \mathbb{R}^{n_{k+1}\times n_{k+1}}$ by

$$\tau_{k+1}^{\mathrm{f}} \triangleq (G_{k+1}^{\mathrm{da}})^{\mathrm{T}} L_{k+1}, \quad \tau_{k+1\perp}^{\mathrm{f}} \triangleq I_{n_{k+1}} - \tau_{k+1}^{\mathrm{f}}. \tag{100}$$

Proposition II.13. Assume that $A_{\mathrm{e},k}^{\mathrm{f}}$ satisfies (95). Then, τ_{k+1}^{f} is an oblique projector.

Proof. The proof is similar to the proof of Proposition II.3. □

Proposition II.14. Assume that $A_{\mathrm{e},k}^{\mathrm{f}}$ satisfies (95). Then,

$$\tau_{k+1}^{\mathrm{f}} \hat{Q}_{k+1}^{\mathrm{f}} = \hat{Q}_{k+1}^{\mathrm{f}}. \tag{101}$$

Proof. The proof is similar to the proof of Proposition II.4. □

Proposition II.15. Assume that $A_{\mathrm{e},k}^{\mathrm{f}}$ satisfies (95). Then,

$$\hat{Q}_{k+1}^{\mathrm{f}} = \tau_{k+1}^{\mathrm{f}} A_k \hat{Q}_k^{\mathrm{da}} A_k^{\mathrm{T}} (\tau_{k+1}^{\mathrm{f}})^{\mathrm{T}}, \tag{102}$$

$$Q_{k+1}^{\mathrm{f}} = A_k Q_k^{\mathrm{da}} A_k^{\mathrm{T}} + V_{1,k} + \tau_{k+1\perp}^{\mathrm{f}} (A_k \hat{Q}_k^{\mathrm{da}} A_k^{\mathrm{T}}) (\tau_{k+1\perp}^{\mathrm{f}})^{\mathrm{T}}. \tag{103}$$

Proof. The proof is similar to the proof of Proposition II.5. □

Finally, we summarize the two-step reduced-order state estimator, whose data-assimilation step is given by

$$x_{\mathrm{e},k}^{\mathrm{da}} = L_k (I_{n_k} - K_k^{\mathrm{da}} C_k)(G_k^{\mathrm{da}})^{\mathrm{T}} x_{\mathrm{e},k}^{\mathrm{f}} + L_k K_k^{\mathrm{da}} y_k, \tag{104}$$

$$Q_k^{\mathrm{da}} = Q_k^{\mathrm{f}} - K_k^{\mathrm{da}} (V_{2,k}^{\mathrm{da}}) (K_k^{\mathrm{da}})^{\mathrm{T}} + \tau_{k\perp}^{\mathrm{da}} M_k^{\mathrm{da}} (\tau_{k\perp}^{\mathrm{da}})^{\mathrm{T}}, \tag{105}$$

where

$$G_k^{\mathrm{da}} = (L_k \hat{Q}_k^{\mathrm{f}} L_k^{\mathrm{T}})^{-1} L_k \hat{Q}_k^{\mathrm{f}}, \tag{106}$$

$$K_k^{\mathrm{da}} = Q_k^{\mathrm{f}} C_k^{\mathrm{T}} (V_{2,k}^{\mathrm{da}})^{-1}, \tag{107}$$

$$M_k^{\mathrm{da}} = \hat{Q}_k^{\mathrm{f}} + K_k^{\mathrm{da}} V_{2,k}^{\mathrm{da}} (K_k^{\mathrm{da}})^{\mathrm{T}}, \tag{108}$$

$$\tau_k^{\mathrm{da}} = M_k^{\mathrm{da}} L_k^{\mathrm{T}} (L_k M_k^{\mathrm{da}} L_k^{\mathrm{T}})^{-1} L_k, \tag{109}$$

$$\hat{Q}_k^{\mathrm{da}} = \tau_k^{\mathrm{da}} M_k^{\mathrm{da}} (\tau_k^{\mathrm{da}})^{\mathrm{T}}, \tag{110}$$

and $V_{2,k}^{\mathrm{da}}$ is given by (68), and whose forecast step is given by

$$x_{\mathrm{e},k+1}^{\mathrm{f}} = L_{k+1} A_k (G_k^{\mathrm{f}})^{\mathrm{T}} x_{\mathrm{e},k}^{\mathrm{da}}, \tag{111}$$

$$Q_{k+1}^{\mathrm{f}} = A_k Q_k^{\mathrm{da}} A_k^{\mathrm{T}} + V_{1,k} + \tau_{k+1\perp}^{\mathrm{f}} M_{k+1}^{\mathrm{f}} (\tau_{k+1\perp}^{\mathrm{f}})^{\mathrm{T}}, \tag{112}$$

where

$$G_k^{\mathrm{f}} = (L_k \hat{Q}_k^{\mathrm{da}} L_k^{\mathrm{T}})^{-1} L_k \hat{Q}_k^{\mathrm{da}}, \tag{113}$$

$$M_{k+1}^{\mathrm{f}} = A_k \hat{Q}_k^{\mathrm{da}} A_k^{\mathrm{T}}, \tag{114}$$

$$\tau_{k+1}^{\mathrm{f}} = M_{k+1}^{\mathrm{f}} L_{k+1}^{\mathrm{T}} (L_{k+1} M_{k+1}^{\mathrm{f}} L_{k+1}^{\mathrm{T}})^{-1} L_{k+1}, \tag{115}$$

$$\hat{Q}_{k+1}^{\mathrm{f}} = \tau_{k+1}^{\mathrm{f}} M_{k+1}^{\mathrm{f}} (\tau_{k+1}^{\mathrm{f}})^{\mathrm{T}}, \tag{116}$$

and $V_{1,k}$ is given by (13).

Remark II.4. Note that if we execute the forecast step (111)–(116) before the data-assimilation step (104)–(110), then we obtain the two-step reduced-order Kalman filter. As discussed in [18], the Kalman filter yields more precise estimates than the Kalman predictor.

III. OPTIMAL INFINITE-HORIZON REDUCED-ORDER STATE ESTIMATOR REVISITED

Consider the LTI system

$$x_{k+1} = Ax_k + D_1 w_k, \tag{117}$$

$$y_k = Cx_k + D_2 w_k, \tag{118}$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, and $w_k \in \mathbb{R}^d$ is a white noise process with zero mean and unit covariance. We consider a infinite-horizon reduced-order predictor

$$x_{\mathrm{e},k+1} = A_\mathrm{e} x_{\mathrm{e},k} + B_\mathrm{e} y_k, \tag{119}$$

where $x_{\mathrm{e},k} \in \mathbb{R}^{n_\mathrm{e}}$, and the cost

$$J(A_\mathrm{e}, B_\mathrm{e}) \triangleq \lim_{k\to\infty} \mathcal{E}[(Lx_k - x_{\mathrm{e},k})^\mathrm{T}(Lx_k - x_{\mathrm{e},k})]. \tag{120}$$

If $\tilde{A} \triangleq \begin{bmatrix} A_\mathrm{e} & 0_{n\times n_\mathrm{e}} \\ B_\mathrm{e}C & A_\mathrm{e} \end{bmatrix}$ is asymptotically stable, then

$$\tilde{Q} \triangleq \lim_{k\to\infty} \mathcal{E}[\tilde{x}_k \tilde{x}_k^\mathrm{T}] \tag{121}$$

exists, where $\tilde{x}_k \in \mathbb{R}^{\tilde{n}}$ is given by (4). Moreover, $\tilde{Q}$ and its nonnegative-definite dual $\tilde{P}$ are the unique solutions of the Lyapunov equations

$$\tilde{Q} = \tilde{A}\tilde{Q}\tilde{A}^\mathrm{T} + \tilde{V}, \tag{122}$$

$$\tilde{P} = \tilde{A}^\mathrm{T}\tilde{P}\tilde{A} + \tilde{R}, \tag{123}$$

where

$$\tilde{V} \triangleq \begin{bmatrix} V_1 & V_{12}B_\mathrm{e}^\mathrm{T} \\ B_\mathrm{e}V_{12}^\mathrm{T} & B_\mathrm{e}V_2B_\mathrm{e}^\mathrm{T} \end{bmatrix}, \quad \tilde{R} \triangleq \begin{bmatrix} L^\mathrm{T}L & -L^\mathrm{T} \\ -L & I_{n_\mathrm{e}} \end{bmatrix}, \tag{124}$$

and

$$V_1 \triangleq D_1 D_1^\mathrm{T}, \quad V_{12} \triangleq D_1 D_2^\mathrm{T}, \quad V_2 \triangleq D_2 D_2^\mathrm{T}. \tag{125}$$

Proposition III.1. Assume that A_e and B_e minimize $J(A_\mathrm{e}, B_\mathrm{e})$. Then, there exist nonnegative-definite matrices Q, $\hat{Q}$, $\hat{P} \in \mathbb{R}^{n\times n}$ such that A_e and B_e are given by

$$A_\mathrm{e} = \Gamma[A - KC]G^\mathrm{T}, \tag{126}$$

$$B_\mathrm{e} = \Gamma K, \tag{127}$$

and Q, $\hat{Q}$, $\hat{P}$ satisfy

$$Q = AQA^\mathrm{T} + V_1 - K\tilde{V}_2K^\mathrm{T} + \tau_\perp(A\hat{Q}A^\mathrm{T} + K\tilde{V}_2K^\mathrm{T})\tau_\perp^\mathrm{T}, \tag{128}$$

$$\hat{Q} = \tau(A\hat{Q}A^\mathrm{T} + K\tilde{V}_2K^\mathrm{T})\tau^\mathrm{T}, \tag{129}$$

$$\hat{P} = \tau^\mathrm{T}[(A - KC)^\mathrm{T}\hat{P}(A - KC) + L^\mathrm{T}L]\tau, \tag{130}$$

where

$$\operatorname{rank}(\hat{Q}) = \operatorname{rank}(\hat{P}) = \operatorname{rank}(\hat{Q}\hat{P}) = n_\mathrm{e}, \tag{131}$$

$$\tau \triangleq G^\mathrm{T}\Gamma = (\hat{Q}\hat{P})(\hat{Q}\hat{P})^\#, \tag{132}$$

$$\Gamma G^\mathrm{T} = I_{n_\mathrm{e}}, \tag{133}$$

$$\tau_\perp \triangleq I_n - \tau, \tag{134}$$

$$K \triangleq Q_\mathrm{s}\tilde{V}_2^{-1}, \tag{135}$$

$$Q_\mathrm{s} \triangleq AQC^\mathrm{T} + V_{12}, \tag{136}$$

$$\tilde{V}_2 \triangleq CQC^\mathrm{T} + V_2, \tag{137}$$

and $\tilde{V}_2$ is assumed to be invertible.

Note that $\hat{P}$ and $\hat{Q}$ yield τ in (132). Also, from τ in (132) and from (133), we obtain G and Γ. Since $\Gamma G^\mathrm{T} = I_{n_\mathrm{e}}$, it follows that τ is an oblique projector. The notation $(\ )^\#$ indicates the group generalized inverse [19].

Remark III.1. Note that, unlike the finite-horizon case, the infinite-horizon state estimator uses constant gains; therefore, there is no advantage in recasting the estimator as a two-step algorithm.

IV. OPTIMAL FINITE-HORIZON SUBSPACE STATE ESTIMATOR

We now consider reduced-order state estimators that focus on a specific subspace of the state. Without loss of generality, we partition the system (1), (2) as

$$\begin{bmatrix} x_{\mathrm{r},k+1} \\ x_{\mathrm{s},k+1} \end{bmatrix} = \begin{bmatrix} A_{\mathrm{r},k} & A_{\mathrm{rs},k} \\ 0_{n_{\mathrm{s},k+1}\times n_{\mathrm{r},k}} & A_{\mathrm{s},k} \end{bmatrix} \begin{bmatrix} x_{\mathrm{r},k} \\ x_{\mathrm{s},k} \end{bmatrix} + \begin{bmatrix} D_{1\mathrm{r},k} \\ D_{1\mathrm{s},k} \end{bmatrix} w_k, \tag{138}$$

$$y_k = [C_{\mathrm{r},k} \;\; C_{\mathrm{s},k}] \begin{bmatrix} x_{\mathrm{r},k} \\ x_{\mathrm{s},k} \end{bmatrix} + D_{2,k} w_k. \tag{139}$$

In this formulation the plant state x_k is partitioned into subsystems for $x_{\mathrm{r},k} \in \mathbb{R}^{n_{\mathrm{r},k}}$ and $x_{\mathrm{s},k} \in \mathbb{R}^{n_{\mathrm{s},k}}$. The state $x_{\mathrm{r},k}$ may contain the components of x_k of interest.

4.1 One-step subspace state estimator

We seek a one-step reduced-order subspace state estimator of the form

$$x_{\mathrm{e},k+1} = A_{\mathrm{e},k} x_{\mathrm{e},k} + B_{\mathrm{e},k} y_k, \tag{140}$$

$$y_{\mathrm{e},k} = C_{\mathrm{e},k} x_{\mathrm{e},k}, \tag{141}$$

that minimizes

$$\begin{aligned} J_k(A_{\mathrm{e},k}, B_{\mathrm{e},k}, C_{\mathrm{e},k+1}) &\triangleq \mathcal{E}([L_{k+1}x_{k+1} - y_{\mathrm{e},k+1}]^{\mathrm{T}} \\ &\quad R_{k+1}[L_{k+1}x_{k+1} - y_{\mathrm{e},k+1}]), \end{aligned} \tag{142}$$

where $R_{k+1} \in \mathbb{R}^{q_{k+1}\times q_{k+1}}$ is a positive-definite weighting matrix. Furthermore, the state weighting matrix $L_k \in \mathbb{R}^{q_k\times n_k}$ is partitioned as $L_k \triangleq [L_{\mathrm{r},k} \quad L_{\mathrm{s},k}]$, where $L_{\mathrm{s},k} \in \mathbb{R}^{q_k\times n_{\mathrm{s},k}}$ and $L_{\mathrm{r},k} \in \mathbb{R}^{q_k\times n_{\mathrm{r},k}}$ is assumed to have full column rank. The order $n_{\mathrm{e},k}$ of the estimator state $x_{\mathrm{e},k}$ is chosen to be $n_{\mathrm{r},k}$.

We define the error state $z_k \triangleq x_{\mathrm{r},k} - x_{\mathrm{e},k}$, which satisfies

$$\begin{aligned} z_{k+1} = (A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k})x_{\mathrm{r},k} - A_{\mathrm{e},k}x_{\mathrm{e},k} \\ +(A_{\mathrm{us},k} - B_{\mathrm{e},k}C_{\mathrm{s},k})x_{\mathrm{s},k} \\ +(D_{1u,k} - B_{\mathrm{e},k}D_{2,k})w_k. \end{aligned} \tag{143}$$

By constraining

$$A_{\mathrm{e},k} = A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k}, \tag{144}$$

(143) becomes

$$\begin{aligned} z_{k+1} = (A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k})z_k + (A_{\mathrm{us},k} - B_{\mathrm{e},k}C_{\mathrm{s},k})x_{\mathrm{s},k} \\ +(D_{1u,k} - B_{\mathrm{e},k}D_{2,k})w_k. \end{aligned}$$

Furthermore, the estimation error in (142) becomes a function of z_k and $x_{\mathrm{s},k}$ by constraining

$$C_{\mathrm{e},k} = L_{\mathrm{r},k}. \tag{145}$$

Now, from (138)–(141) it follows that

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{D}_k w_k, \tag{146}$$

where

$$\begin{aligned} \tilde{x}_k &\triangleq \begin{bmatrix} z_k \\ x_{\mathrm{s},k} \end{bmatrix}, \\ \tilde{A}_k &\triangleq \begin{bmatrix} A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k} & A_{\mathrm{us},k} - B_{\mathrm{e},k}C_{\mathrm{s},k} \\ 0_{n_{\mathrm{s},k+1}\times n_{\mathrm{r},k}} & A_{\mathrm{s},k} \end{bmatrix}, \\ \tilde{D}_k &\triangleq \begin{bmatrix} D_{1\mathrm{r},k} - B_{\mathrm{e},k}D_{2,k} \\ D_{1\mathrm{s},k} \end{bmatrix}. \end{aligned} \tag{147}$$

Then, the problem can be restated as finding $B_{\mathrm{e},k}$ that minimizes

$$J_k(B_{\mathrm{e},k}) = \mathrm{tr}(Q_{k+1}\tilde{R}_{k+1}), \tag{148}$$

where $\tilde{R}_{k+1} \triangleq L_{k+1}^{\mathrm{T}} R_{k+1} L_{k+1}$ and $Q_k \triangleq \mathcal{E}[\tilde{x}_k \tilde{x}_k^{\mathrm{T}}] \in \mathbb{R}^{n_k\times n_k}$. The structure of the augmented state $\tilde{x}_k$ shows that the reduced-order subspace state estimator provides estimates of all of the states in the subspace corresponding to $x_{\mathrm{r},k}$.

Following the procedure in Section 2.1, we obtain the optimal finite-horizon reduced-order subspace state estimator given by

$$\begin{aligned} x_{\mathrm{e},k+1} = \Phi_{k+1}(A_k - K_kC_k)F_k^{\mathrm{T}} x_{\mathrm{e},k} \\ +\Phi_{k+1}K_k y_k, \end{aligned} \tag{149}$$

$$\begin{aligned} Q_{k+1} = A_k Q_k A^{\mathrm{T}} + V_{1,k} - K_k \hat{V}_k K_k^{\mathrm{T}} \\ +\mu_{k+1\perp} K_k \hat{V}_k K_k^{\mathrm{T}} \mu_{k+1\perp}^{\mathrm{T}}, \end{aligned} \tag{150}$$

where

$$\Phi_k \triangleq [I_{n_{\mathrm{r},k}} (L_{\mathrm{r},k}^{\mathrm{T}} R_k L_{\mathrm{r},k})^{-1} (L_{\mathrm{r},k}^{\mathrm{T}} R_k L_{\mathrm{s},k})], \tag{151}$$

$$\begin{aligned} \mu_k &\triangleq F_k^{\mathrm{T}} \Phi_k \\ &= \begin{bmatrix} I_{n_{\mathrm{r},k}} & (L_{\mathrm{r},k}^{\mathrm{T}} R_k L_{\mathrm{r},k})^{-1} (L_{\mathrm{r},k}^{\mathrm{T}} R_k L_{\mathrm{s},k}) \\ 0_{n_{\mathrm{s},k}\times n_{\mathrm{r},k}} & 0_{n_{\mathrm{s},k}} \end{bmatrix}, \end{aligned}$$

$$\mu_{k\perp} \triangleq I_{n_k} - \mu_k, \tag{152}$$

$$F_k \triangleq [I_{n_{\mathrm{r},k}} \quad 0_{n_{\mathrm{r},k}\times n_{\mathrm{s},k}}], \tag{153}$$

$$K_k \triangleq A_k Q_k C^{\mathrm{T}} \hat{V}_k^{-1}, \tag{154}$$

$$\hat{V}_k \triangleq C_k Q_k C_k^{\mathrm{T}} + V_{2,k}, \tag{155}$$

$V_{1,k}$, $V_{2,k}$ are given by (13), and $\hat{V}_k$ is assumed to be invertible. Note that Remark II.1 is also applicable to (149).

4.2 Two-step subspace state estimator

Next, we consider the two-step state estimator. The data-assimilation step is given by

$$x_{\mathrm{e},k}^{\mathrm{da}} = A_{\mathrm{e},k}^{\mathrm{da}} x_{\mathrm{e},k}^{\mathrm{f}} + B_{\mathrm{e},k}^{\mathrm{da}} y_k, \tag{156}$$

$$y_{\mathrm{e},k}^{\mathrm{da}} = C_{\mathrm{e},k}^{\mathrm{da}} x_{\mathrm{e},k}^{\mathrm{da}}, \tag{157}$$

where $x_{\mathrm{e},k}^{\mathrm{da}} \in \mathbb{R}^{n_{\mathrm{e}}}$ is the reduced-order data assimilation estimate of the subspace $x_{\mathrm{r},k}$, and $x_{\mathrm{e},k}^{\mathrm{f}} \in \mathbb{R}^{n_{\mathrm{e}}}$ is the reduced-order forecast estimate of subspace $x_{\mathrm{r},k}$, while the forecast step is given by

$$x_{\mathrm{e},k+1}^{\mathrm{f}} = A_{\mathrm{e},k}^{\mathrm{f}} x_{\mathrm{e},k}^{\mathrm{da}}, \tag{158}$$

$$y_{\mathrm{e},k+1}^{\mathrm{f}} = C_{\mathrm{e},k+1}^{\mathrm{f}} x_{\mathrm{e},k+1}^{\mathrm{f}}. \tag{159}$$

Defining the data-assimilation cost J_k^{da} and the forecast cost J_{k+1}^{f} as

$$\begin{aligned} J_k^{\mathrm{da}}(A_{\mathrm{e},k}^{\mathrm{da}}, B_{\mathrm{e},k}^{\mathrm{da}}, C_{\mathrm{e},k}^{\mathrm{da}}) &\triangleq \mathcal{E}([L_k x_k - y_{\mathrm{e},k}^{\mathrm{da}}]^{\mathrm{T}} \\ &\quad R_k[L_k x_k - y_{\mathrm{e},k}^{\mathrm{da}}]), \end{aligned} \tag{160}$$

$$\begin{aligned} J_{k+1}^{\mathrm{f}}(A_{\mathrm{e},k}^{\mathrm{f}}, C_{\mathrm{e},k+1}^{\mathrm{f}}) &\triangleq \mathcal{E}([L_{k+1} x_{k+1} - y_{\mathrm{e},k+1}^{\mathrm{f}}]^{\mathrm{T}} \\ &\quad R_{k+1}[L_{k+1} x_{k+1} - y_{\mathrm{e},k+1}^{\mathrm{f}}]), \end{aligned} \tag{161}$$

we obtain the two-step optimal finite-horizon subspace state estimator, whose data-assimilation step is given by

$$x_{\mathrm{e},k}^{\mathrm{da}} = \Phi_k (I_{n_k} - K_k^{\mathrm{da}} C_k) F_k^{\mathrm{T}} x_k^{\mathrm{da}} + \Phi_k K_k^{\mathrm{da}} y_k, \tag{162}$$

$$\begin{aligned} Q_k^{\mathrm{da}} &= Q_k^{\mathrm{f}} - K_k^{\mathrm{da}} \hat{V}_{2,k} (K_k^{\mathrm{da}})^{\mathrm{T}} \\ &\quad + \mu_{k\perp} K_k^{\mathrm{da}} \hat{V}_{2,k} (K_k^{\mathrm{da}})^{\mathrm{T}} \mu_{k\perp}^{\mathrm{T}}, \end{aligned} \tag{163}$$

and whose forecast step is given by

$$x_{\mathrm{e},k+1}^{\mathrm{f}} = \Phi_{k+1} A_k F_k^{\mathrm{T}} x_{\mathrm{e},k}^{\mathrm{da}}, \tag{164}$$

$$Q_{k+1}^{\mathrm{f}} = A_k Q_k^{\mathrm{da}} A_k^{\mathrm{T}} + V_{1,k}, \tag{165}$$

where

$$K_k^{\mathrm{da}} = Q_k^{\mathrm{f}} C_k^{\mathrm{T}} \hat{V}_{2,k}^{-1}, \tag{166}$$

$$\hat{V}_{2,k} = C_k Q_k^{\mathrm{f}} C_k^{\mathrm{T}} + V_{2,k}, \tag{167}$$

$$\mu_k = \Phi_k F_k^{\mathrm{T}}, \tag{168}$$

Φ_k is given by (151), F_k is given by (153), $V_{1,k}$, $V_{2,k}$ are given by (13), and $\hat{V}_{2,k}$ is assumed to be invertible. Note that Remark II.3 and Remark II.4 are also applicable to (162)–(165).

V. OPTIMAL INFINITE-HORIZON SUBSPACE STATE ESTIMATOR REVISITED

For the LTI system (117), (118), the optimal one-step infinite-horizon subspace state estimator can be obtained by reformulating the cost

$$J(B_{\mathrm{e}}) \triangleq \lim_{k\to\infty} \mathcal{E}([Lx_k - y_{\mathrm{e},k}]^{\mathrm{T}} R[Lx_k - y_{\mathrm{e},k}]), \tag{169}$$

where we constrain

$$A_{\mathrm{e}} \triangleq A_{\mathrm{r}} - B_{\mathrm{e}} C_{\mathrm{r}}, \tag{170}$$

$$C_{\mathrm{e}} \triangleq L_{\mathrm{r}}, \tag{171}$$

where A_{r} and C_{r} are the time-invariant counterparts of $A_{\mathrm{r},k}$ in (138) and $C_{\mathrm{r},k}$ in (139), respectively. If $\tilde{A} \triangleq \begin{bmatrix} A_{\mathrm{r}} - B_{\mathrm{e}} C_{\mathrm{r}} & A_{\mathrm{us}} - B_{\mathrm{e}} C_{\mathrm{s}} \\ 0_{n_{\mathrm{s}}\times n_{\mathrm{r}}} & A_{\mathrm{s}} \end{bmatrix}$ is asymptotically stable, then $Q \triangleq \lim_{k\to\infty} \mathcal{E}[\tilde{x}_k \tilde{x}_k^{\mathrm{T}}]$ exists.

Proposition V.1. Assume that B_{e} minimizes $J(B_{\mathrm{e}})$ with constraints (170) and (171). Then there exist nonnegative-definite matrices Q, $P \in \mathbb{R}^{n\times n}$ such that A_{e} and B_{e} are given by

$$A_{\mathrm{e}} = \Phi(A - KC)F^{\mathrm{T}}, \tag{172}$$

$$B_{\mathrm{e}} = \Phi K, \tag{173}$$

and Q and P satisfy

$$Q = AQA^{\mathrm{T}} + V_1 - K\hat{V}K^{\mathrm{T}} + \mu_\perp K\hat{V}K^{\mathrm{T}}\mu_\perp^{\mathrm{T}}, \tag{174}$$

$$\begin{aligned} P &= A^{\mathrm{T}} PA - Q_a \mu^{\mathrm{T}} PA - A^{\mathrm{T}} P\mu Q_a^{\mathrm{T}} \\ &\quad + Q_a \mu^{\mathrm{T}} P\mu Q_a^{\mathrm{T}} + L^{\mathrm{T}} RL, \end{aligned} \tag{175}$$

where

$$\Phi \triangleq [I_{n_r} \quad P_1^{-1}P_{12}], \tag{176}$$

$$\mu \triangleq F^{\mathrm{T}}\Phi = \begin{bmatrix} I_{n_r} & P_1^{-1}P_{12} \\ 0_{n_s\times n_r} & 0_{n_s\times n_s} \end{bmatrix}, \tag{177}$$

$$\mu_\perp \triangleq I_n - \mu, \tag{178}$$

$$F \triangleq [I_{n_r} \quad 0_{n_r\times n_s}], \tag{179}$$

$$K \triangleq AQC^{\mathrm{T}}\hat{V}^{-1}, \tag{180}$$

$$\hat{V} \triangleq CQC^{\mathrm{T}} + V_2, \tag{181}$$

$$Q_a \triangleq C^{\mathrm{T}}\hat{V}^{-1}C, \tag{182}$$

where $\begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix} \triangleq P$, $P_1 \in \mathbb{R}^{n_r\times n_r}$, $P_{12} \in \mathbb{R}^{n_r\times n_s}$, $P_2 \in \mathbb{R}^{n_s\times n_s}$, and $\hat{V}$ is assumed to be invertible.

The infinite-horizon subspace state-estimation problem with direct feedthrough in (141) is solved in [13, Theorem 2.2], while the continuous-time case is treated in [4].

VI. MASS-SPRING-DASHPOT SYSTEM

6.1 Asymptotically stable example

To illustrate the reduced-order state estimators of Section II and the subspace state estimators of Section IV, we consider a zero-order hold discretized model of the mass-spring-dashpot structure consisting of 10 masses shown in Fig. 1 for which $n = 20$. For $i = 1, \ldots, 10$, $m_i = 1.0\,\mathrm{kg}$, while, for $j = 1, \ldots, 11$, $k_j = 1.0\,\mathrm{N/m}$ and $c_j = 0.05\,\mathrm{N\text{-}s/m}$. We set the initial error covariance $Q_0 = 100I_n$, and we assume that $V_{1,k} = I_n$, $V_{2,k} = I_p$ for all $k \geqslant 0$. This example is also investigated in [11] using a spatially localized state estimator.

Let x_i denote the position of the ith mass so that

$$x \triangleq [x_1 \ \dot{x}_1 \ \cdots \ x_{10} \ \dot{x}_{10}]^{\mathrm{T}}.$$

We assume that measurements of position and velocities of $m_1, \ldots, m_4$ are available so that $C_k = [I_8 \ 0_{8\times 12}]$ for all $k \geqslant 0$. Next, we obtain state estimates from the reduced-order estimator with $n_e = 8$. Meanwhile, for the subspace estimator, we consider a change of basis so that the system has the block upper triangular structure shown in (138). The costs for the estimators are defined in (6) and (142) with $R_k = I_2$. The ratio of the cost J_k to the best achievable cost when a full-order Kalman predictor is used is shown in Fig. 2. As indicated by ratios greater than 1, the performance of the reduced-order state estimator is never better than the full-order state estimator.

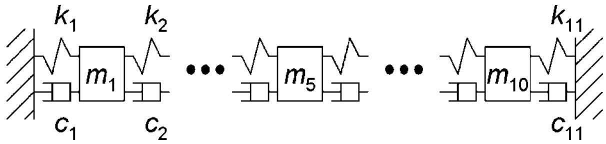

Fig. 1. Mass-spring-dashpot system.

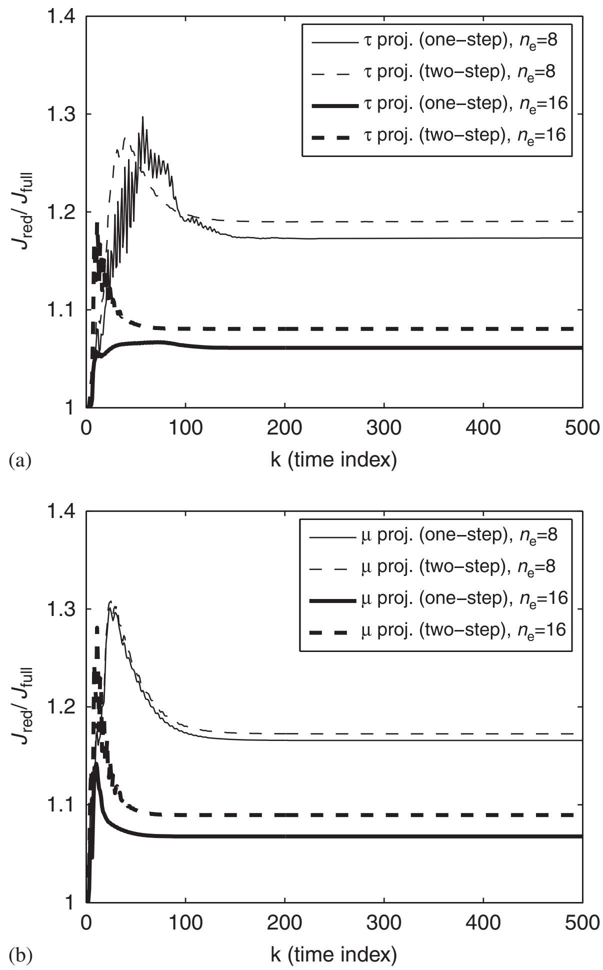

Fig. 2. Cost ratios for the (a) reduced-order state estimators and (b) subspace state estimators for the asymptotically stable mass-spring-dashpot system. J_{red} is the estimation cost for the reduced-order state estimator, and J_{full} is for the full-order system. The plots also demonstrate that the one-step and two-step estimators are not equivalent.

Next, we assume that measurements of positions and velocities of $m_1, \ldots, m_8$ are available so that $C_k = [I_{16} \ 0_{16\times 4}]$ for all $k \geq 0$. The performance of the reduced-order estimator with $n_e = 16$ is shown in Figure 2(a). The objective in both cases is to obtain estimates of Lx_k, where, for $i = 1, \ldots, n_e,\ j = 1, \ldots, n$, the (i, j) entry of $L \in \mathbb{R}^{n_e \times n}$ is given by

$$L_{(i,j)} \triangleq \begin{cases} 1, & \text{if } i = j, \\ 0.05, & \text{else.} \end{cases} \tag{183}$$

The plots also demonstrate that the one-step and two-step estimators are not equivalent.

6.2 Unstable example with rigid-body mode

We now consider a modification of the mass-spring-dashpot structure in Fig. 1. Specifically, we assume that both ends are free, that is, $k_1 = k_{11} = 0.0$ and $c_1 = c_{11} = 0.0$, and thus the structure has an unstable rigid-body mode. Let q_i denote the position of the ith mode in modal coordinates so that

$$x \triangleq [q_1 \ \dot{q}_1 \ \cdots \ q_{10} \ \dot{q}_{10}]^{\mathrm{T}}.$$

We consider only the subspace estimator with $x_r = [q_1 \ \dot{q}_1]^{\mathrm{T}}$. We assume that measurements of the position and velocity of m_1 are available and L is given by (183) in modal coordinates with $n_e = 4, 8$. The performance of the subspace estimator with $n_e = 4, 8$ is shown in Fig. 3. The plots show that the subspace estimator captures the unstable modes in the system.

$$V_r(x) = \sqrt{\frac{2}{ml}} \sin \frac{r\pi x}{l},$$

where the modal coordinates q_r satisfy

$$\begin{aligned} \ddot{q}_r(t) &= 2\zeta\omega_r \dot{q}_r(t) + \omega_r^2 q_r(t) \\ &= \int_0^l f(x, t) V_r(x) \mathrm{d}x, \quad r = 1, 2, \ldots. \end{aligned}$$

For simplicity we assume $l = \pi$ and $m = 2/\pi$ so that $\sqrt{\frac{2}{ml}} = 1$. We assume that displacement sensors located at $x = 0.55\pi$ and $x = 0.65\pi$ are sampled at 50 Hz and 30 Hz, respectively. Also, it is assumed that a white noise disturbance of unit intensity acts on the beam at $x = 0.45\pi$. For estimator design, we weight the performance of the beam displacement at $x = 0.65\pi$. Finally, retaining the first five modes and defining the plant states as

$$x \triangleq [q_1 \quad \dot{q}_1 \quad \ldots \quad q_5 \quad \dot{q}_5]^{\mathrm{T}},$$

the resulting sampled-data continuous-time state-space model is

$$\begin{aligned}
A &= \underset{i=1,\ldots,5}{\text{block}-\text{diag}} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \\
\omega_i &= i^2, \quad i = 1, \ldots, 5, \quad \zeta = 0.005, \\
C &= \begin{bmatrix} 0.9877 & 0 & -0.3090 & 0 & -0.8910 & 0 & 0.5878 & 0 & 0.7071 & 0 \\ 0.8910 & 0 & -0.8090 & 0 & -0.1564 & 0 & 0.9511 & 0 & -0.7071 & 0 \end{bmatrix}, \\
L &= [0.8910 \ 0 \ -0.8090 \ 0 \ -0.1564 \ 0 \ 0.9511 \ 0 \ -0.7071 \ 0], \\
D_1 &= [0 \ 0.9877 \ 0 \ 0.3090 \ 0 \ -0.8900 \ 0 \ -0.5878 \ 0 \ -0.7071], \\
V_2 &= \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},
\end{aligned}$$

VII. APPLICATION TO PERIODICALLY TIME-VARYING MULTIRATE ESTIMATION

Consider the transverse deflection $v(x, t)$ of a simply supported Euler–Bernoulli beam. The modal decomposition of $v(x, t)$ has the form

$$v(x, t) = \sum_{r=1}^{\infty} V_r(x) q_r(t), \qquad \int_0^l m V_r^2(x) dx = 1,$$

where $\mathrm{row}_1(C)$ accounts for sensor 1 sampled at 50 Hz, while $\mathrm{row}_2(C)$ accounts for sensor 2 sampled at 30 Hz. Then one period of the periodic sequence of sensor information $\mathcal{C}_k$ is given by

$$\mathcal{C}_k = \{s_1\}, \{s_2\}, \{s_1\}, \{s_1\}, \{s_1, s_2\}, \{s_1\}, \{s_1, s_2\},$$

where s_1 and s_2 denote the signals from sensor 1 and sensor 2, respectively, while one period of the

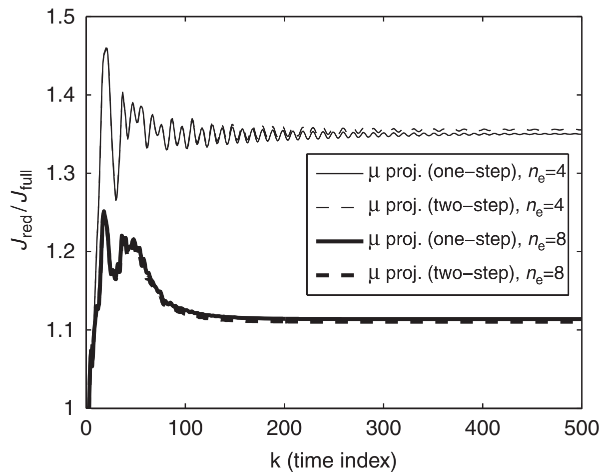

Fig. 3. Cost ratios of J for the subspace state estimator applied to the unstable mass-spring-dashpot system with a rigid body mode. The subspace estimator can handle the unstable modes in its filter structure.

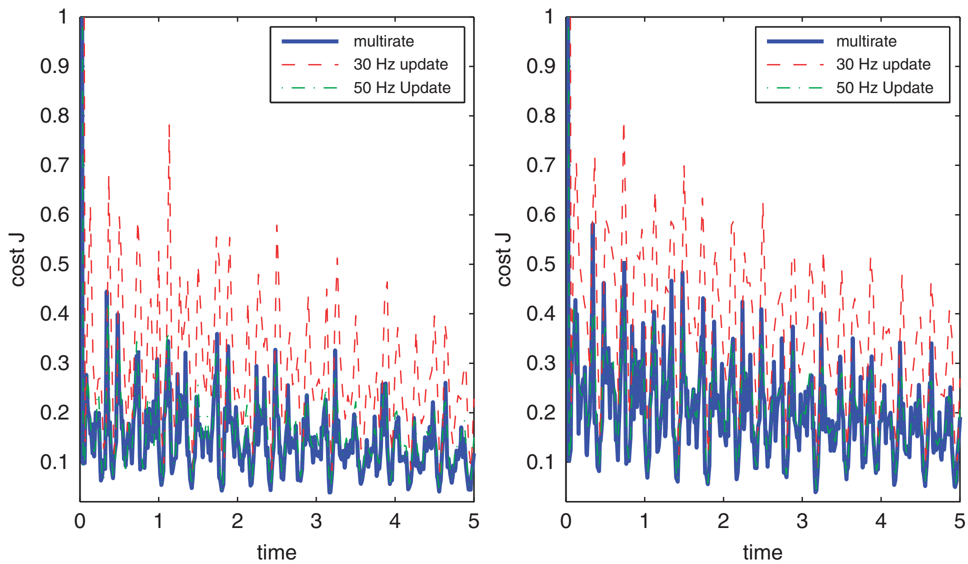

Fig. 4. Performance comparisons of reduced-order state estimators when applied to the periodically time-varying multirate sampling system and fixed sample-rate systems. (a) is for the one-step reduced-order state estimator, and (b) is for the two-step reduced-order state estimator.

periodically varying sample interval T_k is given by

$$T_k = 20,\ 40/3,\ 20/3,\ 20,\ 20/3,\ 40/3,\ 20,$$

where T_k is given in ms. This example is investigated in [14] with sampling rates 60 Hz and 30 Hz using a multirate state estimator.

The continuous-time model is discretized according to the given sample rates, which yields the time-varying system (1), (2), where A_k and C_k vary periodically as

$$A_k = e^{T_k A},$$

$$C_k = \begin{cases} \mathrm{row}_1(C), & \text{if } \mathcal{C}_k = \{s_1\}, \\ \mathrm{row}_2(C), & \text{if } \mathcal{C}_k = \{s_2\}, \\ C, & \text{if } \mathcal{C}_k = \{s_1, s_2\}. \end{cases}$$

Figure 4 shows the evolution of the costs of the one-step (Section 2.1) and two-step finite-horizon reduced-order state estimators (Section 2.2) with

$n = 10$, $n_e = 1$. The performance of the finite-horizon reduced-order state estimators for the multirate system is compatible with the performance of the same estimator applied to a single rate system where both signals are sampled at 50 Hz.

VIII. CONCLUSION

Using finite-horizon optimization, optimal reduced-order state estimators and optimal fixed-structure subspace state estimators were obtained in the form of recursive update equations for time-varying systems. These estimators are characterized by the oblique projectors τ and μ, respectively. Moreover, we derived one-step and two-step update equations for each class of state estimator. When the order of each estimator is equal to the order of the system, the oblique projectors become the identity and the estimators are equivalent to the classical optimal recursive full-order state estimator. We demonstrated the performance of the reduced-order and the subspace state estimators for lumped structures. Moreover, an application of the reduced-order state estimators to a multirate estimation problem was investigated.

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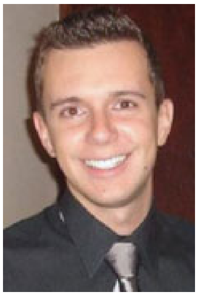

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