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Unified optimal projection equations for simultaneous reduced-order, robust modelling, estimation and control

WASSIM M. HADDAD† and DENNIS S. BERNSTEIN‡

An optimal design problem which unifies reduced-order modelling, estimation and control problems is stated. Necessary conditions for optimality are obtained in the form of a coupled system of modified Riccati and Lyapunov equations. The results permit treatment of several new problems, such as reduced-order dynamic compensation with partially known disturbances and unified reduced-order control and estimation. Upon appropriate specialization, results obtained previously for the individual problems of reduced-order modelling, estimation and control are recovered. An additional feature is the inclusion of parameter uncertainty bounds so that the necessary conditions for an auxiliary minimization problem serve as sufficient conditions for simultaneous robust, reduced-order modelling, estimation and control.

Notation and definitions

Note. All matrices have real entries.

$\mathbf{R}$; $\mathbf{R}^{r\times s}$; $\mathbf{R}^r$	real numbers; $r \times s$ real matrices; $\mathbf{R}^{r\times 1}$
I_r, $(\ )^{\mathrm{T}}$	$r \times r$ identity matrix, transpose
$\oplus$; $\otimes$	Kronecker sum; Kronecker product (Brewer 1978)
$\hat{\mathbf{S}}^r$	$r \times r$ symmetric matrices
$\hat{\mathbf{N}}^r$	$r \times r$ symmetric non-negative-definite matrices
$\hat{\mathbf{P}}^r$	$r \times r$ symmetric positive-definite matrices
$Z_1 \leqslant Z_2$	$Z_2 - Z_1 \in \hat{\mathbf{N}}^r$, $\quad Z_1, Z_2 \in \hat{\mathbf{S}}^r$
$Z_1 < Z_2$	$Z_2 - Z_1 \in \hat{\mathbf{P}}^r$, $\quad Z_1, Z_2 \in \hat{\mathbf{S}}^r$
asymptotically stable matrix	matrix with eigenvalues in open left half-plane
$n, m, \hat{m}, l, \hat{l}, n_c, q, p$	positive integers
$\tilde{n}$	$n + n_c$
$x, u, y, \hat{y}, x_c, y_c, y_m, \tilde{x}$	$n, m, l, \hat{l}, n_c, q, \hat{l}, \tilde{n}$-dimensional vectors
$A, \Delta A$; $B, \Delta B$; $C, \Delta C$	$n \times n$ matrices; $n \times m$ matrices; $l \times n$ matrices
A_i, B_i, C_i	$n \times n$, $n \times m$, $l \times n$ matrices, $i = 1, \ldots, p$
δ_i, α_i	positive numbers, $i = 1, \ldots, p$
A_α	$A + \dfrac{1}{2}\sum_{i=1}^{p} \delta_i \alpha_i I_n$
$A_c, B_c, B_m, C_c, C_e, C_m$	$n_c \times n_c$, $n_c \times l$, $n_c \times \hat{m}$, $m \times n_c$, $q \times n_c$, $\hat{l} \times n_c$ matrices
$\hat{B}$, $\hat{C}$	$n \times \hat{m}$, $\hat{l} \times n$ matrices
$A_{c\alpha}$	$A_c + \frac{1}{2}\sum_{i=1}^{l} \delta_i \alpha_i I_{n_c}$
L	$q \times n$ matrix

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R estimation-error weighting in $\mathbf{P}^q$
$\hat{R}$ model-error weighting in $\mathbf{P}^{\ell}$
R_1 state weighting matrix in $\mathbf{N}^n$
R_2 control weighting matrix in $\mathbf{P}^m$
R_{12} $n \times m$ cross weighting matrix such that $R_1 - R_{12}R_2^{-1}R_{12}^{\mathrm{T}} \geqslant 0$
$w(\cdot)$ $\hat{m}$-dimensional white noise independent of $w_1(\cdot)$ and $w_2(\cdot)$
$w_1(\cdot)$ n-dimensional white noise
$w_2(\cdot)$ l-dimensional white noise
V_1 intensity of $w_1(\cdot)$ in $\mathbf{N}^n$
V_2 intensity of $w_2(\cdot)$ in $\mathbf{P}^l$
V_{12} $n \times l$ cross intensity of $w_1(\cdot)$, $w_2(\cdot)$
V intensity of $w(\cdot)$ in $\mathbf{P}^{\hat{m}}$
$\mathbf{E}$ expected value
$\tilde{A}, \Delta\tilde{A}$ $\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & 0 \end{bmatrix}$
$\tilde{R}$ $\begin{bmatrix} R_1 + L^{\mathrm{T}}RL + \hat{C}^{\mathrm{T}}\hat{R}\hat{C} & R_{12}C_c - L^{\mathrm{T}}RC_e - \hat{C}^{\mathrm{T}}\hat{R}C_m \\ C_c^{\mathrm{T}}R_{12}^{\mathrm{T}} - C_e^{\mathrm{T}}RL - C_m^{\mathrm{T}}\hat{R}\hat{C} & C_c^{\mathrm{T}}R_2C_c + C_e^{\mathrm{T}}RC_e + C_m^{\mathrm{T}}\hat{R}C_m \end{bmatrix}$
$\tilde{w}(t)$ $\begin{bmatrix} w_1(t) + \hat{B}w(t) \\ B_cw_2(t) + B_mw(t) \end{bmatrix}$
$\tilde{V}$ $\begin{bmatrix} V_1 + \hat{B}V\hat{B}^{\mathrm{T}} & V_{12}B_c^{\mathrm{T}} + \hat{B}VB_m^{\mathrm{T}} \\ B_cV_{12}^{\mathrm{T}} + B_mV\hat{B}^{\mathrm{T}} & B_cV_2B_c^{\mathrm{T}} + B_mVB_m^{\mathrm{T}} \end{bmatrix}$

1. Introduction

The problems of quadratically optimal reduced-order modelling, estimation and control have been treated in a common framework by Hyland and Bernstein (1985), Bernstein and Hyland (1985), and Hyland and Bernstein (1984), respectively. Specifically, by carrying out a judicious transformation of variables, it was shown that the necessary conditions for optimality could be cast as coupled systems of 2, 3 and 4 modified Lyapunov and Riccati equations, respectively. The coupling is via an oblique projection (i.e. idempotent matrix) which arises as a direct consequence of optimality and which determines the geometric structure of the reduced-order model, estimator, or compensator. When the order of the estimator or compensator is set equal to the order of the plant, the additional modified Lyapunov equations drop out and the remaining modified Riccati equations reduce to the standard steady-state Riccati equations of Kalman filter and LQG theory.

An immediate benefit of this formulation of the necessary conditions is clarification of the relationship between the operations of model reduction and estimator or controller design. Specifically, although the additional pair of modified Lyapunov equations arising in the reduced-order estimation and control problems are analogous to the pair of modified Lyapunov equations characterizing the optimal reduced-order model, these equations are now inextricably coupled with the modified Riccati equations characterizing the estimator and controller design. Hence, because of the coupling, this approach is distinct from LQG controller-reduction techniques (see, for example, Liu and Anderson 1986, and Jonckheere and Silverman 1983). A comparison between the LQG reduction methods reviewed by Liu and Anderson (1986) and the optimal projection approach has been given by Greeley and Hyland (1988).

The goal of the present paper is to unify the results obtained previously for reduced-order modelling, estimation and control by deriving a *single* result which, upon appropriate specialization, yields the reduced-order modelling, estimation and control results as special cases. This is accomplished by defining a generalized performance functional which incorporates features of all three criteria. Thus the optimization problem involves determining a *single* reduced-order system which simultaneously serves as a reduced-order model, estimator and controller (or any two of these as desired). The necessary conditions now take the form of a coupled system of two modified Lyapunov equations and two modified Riccati equations which can be specialized to the necessary conditions obtained previously for the reduced-order modelling, estimation and control problems.

There are several motivations for developing a unified formulation encompassing all three results. For example, in the full-order case the certainty equivalence principle implies that the states of the optimal dynamic compensator are also optimal estimates of the plant states. This is definitely not the case for an optimal reduced-order controller in which the states may bear no resemblance to the plant states. The unified formulation of the present paper, however, expresses the desire that compensator states also provide estimates of selected plant states. Of course, except in the full-order case, such a compensator will generally be suboptimal from a strictly control point of view since the design also serves as an estimator. A similar formulation has been considered by Wilson and Kumar (1983).

Additional problems which can be handled in the unified setting involve reduced-order estimation and control in the presence of partially known plant disturbances. When measurements of disturbance components are available during real-time operation, such measurements can be used as inputs to the estimator or controller to improve performance. Note that this problem incorporates aspects of the model-reduction formulation in which the same white noise signal is injected into both the plant and the design system.

A practical motivation for the unified problem setting is convenience in developing numerical algorithms for treating different problems. In particular, a single algorithm for solving the unified optimal projection equations can readily be used for all special cases without reprogramming. (For discussions of numerical algorithms for the optimal projection equations, see Greeley and Hyland 1988, and Richter 1987.)

An additional feature of the results given herein is the treatment of parametric uncertainty in the plant matrices. By bounding the effects of parameter uncertainty on worst-case system performance, the necessary conditions for optimality effectively serve as sufficient conditions for robust stability and performance. A similar approach has been carried out by Bernstein and Haddad (1988), using structured stability radius bounds. In the present paper we use an alternative bound which corresponds to a right shift of the dynamics matrix (or equivalently, an exponential cost weighting) in conjunction with multiplicative white-noise type terms. The effect of multiplicative noise on the optimal projection equations has been developed by Bernstein and Hyland (1988). In the present paper such underlying interpretations will be suppressed since only the bound *per se* will be needed. Hence, although we use the phrase 'multiplicative white noise' for convenience in referring to the *type* of bound used, it should be stressed that our treatment of parameter uncertainty is wholly deterministic. (See Bernstein 1987 a, and Haddad 1987, for further background and discussion.)

2. Simultaneous reduced-order, robust modelling, estimation and control

In this section we state the 'robust performance problem' for simultaneous reduced-order modelling, estimation and control along with related notation for later use. Let $\mathbf{U} \subset \mathbf{R}^{n \times n} \times \mathbf{R}^{n \times m} \times \mathbf{R}^{l \times m}$ denote the set of uncertain perturbations $(\Delta A, \Delta B, \Delta C)$ of the nominal system matrices A, B and C.

Robust performance problem

For fixed $n_c \leqslant n$, determine $(A_c, B_c, B_m, C_c, C_e, C_m)$ such that, for the augmented system consisting of the nth-order controlled and disturbed plant

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) + \hat{B}w(t) + w_1(t), \quad t \in [0, \infty) \tag{2.1}$$

with noisy and non-noisy measurements

$$y(t) = (C + \Delta C)x(t) + w_2(t) \tag{2.2}$$

$$\hat{y}(t) = \hat{C}x(t) \tag{2.3}$$

and n_cth-order design system

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) + B_m w(t) \tag{2.4}$$

$$u(t) = C_c x_c(t) \tag{2.5}$$

$$y_e(t) = C_e x_c(t) \tag{2.6}$$

$$y_m(t) = C_m x_c(t) \tag{2.7}$$

the performance criterion

$$J(A_c, B_c, B_m, C_c, C_e, C_m) \triangleq J_c + J_e + J_m \tag{2.8}$$

is minimized, where

$$J_c \triangleq \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \to \infty} \mathbf{E}[x^{\mathrm{T}}(t)R_1 x(t) + 2x^{\mathrm{T}}(t)R_{12}u(t) + u^{\mathrm{T}}(t)R_2 u(t)] \tag{2.9}$$

$$J_e \triangleq \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \to \infty} \mathbf{E}[Lx(t) - y_e(t)]^{\mathrm{T}} R[Lx(t) - y_e(t)] \tag{2.10}$$

$$J_m \triangleq \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \to \infty} \mathbf{E}[\hat{y}(t) - y_m(t)]^{\mathrm{T}} \hat{R}[\hat{y}(t) - y_m(t)] \tag{2.11}$$

Remark 2.1

Suppose there are no uncertainties present, i.e. $\Delta A, \Delta B, \Delta C = 0$. By setting $L = 0$ and $\hat{C} = 0$, it follows that J_e and J_m play no role in the optimization problem when C_e and C_m are both taken to be zero. As will be seen in Theorem 6.1, this is indeed the optimal solution in this case. If, furthermore, $\hat{B} = 0$, then the reduced-order dynamic-compensation problem of Hyland and Bernstein (1984), is recovered. If, alternatively, $R_1 = 0$, $R_{12} = 0$, $B = 0$, $\hat{B} = 0$ and $\hat{C} = 0$ then the reduced-order state-estimation problem of Bernstein and Hyland (1985) is obtained. Finally, setting $R_1 = 0$, $R_{12} = 0$, $L = 0$, $V_1 = 0$, $B = 0$ and $C = 0$ yields the model-reduction problem considered by Hyland and Bernstein (1985).

Remark 2.2

Suppose $L = 0$ and $\hat{C} = 0$ (so that with C_c and C_e both zero J_e and J_m are ineffective) but that $\hat{B} \neq 0$. In this case, a portion of the plant disturbance, which is assumed to be measurable during on-line operation, is being fed directly into the compensator. Hence this problem, which generalizes that of Hyland and Bernstein (1984), can be thought of as reduced-order dynamic compensation with partially known disturbances. Similarly, the case $R_1 = 0$, $R_{12} = 0$, $B = 0$ and $\hat{C} = 0$ but $\hat{B} \neq 0$ provides a generalization of Bernstein and Hyland (1985), which can be thought of as reduced-order state estimation with partially known disturbances.

For each variation $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, the augmented system (2.1)–(2.5) can be written as

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{w}(t), \quad t \in [0, \infty) \tag{2.12}$$

where

$$\tilde{x}(t) \triangleq [x^{\mathrm{T}}(t), x_c^{\mathrm{T}}(t)]^{\mathrm{T}} \tag{2.13}$$

and $\tilde{w}(t)$ is white noise with intensity $\tilde{V} \in \hat{\mathbf{N}}^{\tilde{n}}$.

For the 'robust performance problem' the cost can be expressed in terms of the second-moment matrix of $\tilde{x}(t)$. The following result is immediate.

Proposition 2.1

For given $(A_c, B_c, B_m, C_c, C_e, C_m)$ and $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ the second-moment matrix

$$\tilde{Q}_{\Delta\tilde{A}}(t) \triangleq \mathbf{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)], \quad t \in [0, \infty) \tag{2.14}$$

satisfies

$$\dot{\tilde{Q}}_{\Delta\tilde{A}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}}(t) + \tilde{Q}_{\Delta\tilde{A}}(t)(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \tilde{V}, \quad t \in [0, \infty) \tag{2.15}$$

Furthermore,

$$J(A_c, B_c, B_m, C_c, C_e, C_m) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \limsup_{t \to \infty} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}}(t)\tilde{R} \tag{2.16}$$

3. Sufficient conditions for robust stability and performance

In practice, steady-state performance is only of interest when the augmented system is stable over $\mathbf{U}$. The following result is immediate.

Lemma 3.1

Suppose the system (2.12) is stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. Then

$$J(A_c, B_c, B_m, C_c, C_e, C_m) = \sup_{(\Delta A, \Delta B, \Delta C) \in \mathbf{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}}\tilde{R} \tag{3.1}$$

where $\tilde{Q}_{\Delta\tilde{A}} \in \hat{\mathbf{N}}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A} + \Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \tilde{V} \tag{3.2}$$

Remark 3.1

When $\mathbf{U}$ is compact, 'sup' in (3.1) can be replaced by 'max'.

Since it is difficult to determine $J(A_c, B_c, B_m, C_c, C_e, C_m)$ explicitly, we shall seek upper bounds.

Theorem 3.1

Let $\Omega: \hat{\mathbf{N}}^{\tilde{n}} \times \mathbf{R}^{n_c \times l} \times \mathbf{R}^{m \times n_c} \to \hat{\mathbf{S}}^{\tilde{n}}$ be such that

$$\Delta \tilde{A} \mathbf{Q} + \mathbf{Q} \Delta \tilde{A}^{\mathrm{T}} \leqslant \Omega(\mathbf{Q}, B_c, C_c)$$
$$(\Delta A, \Delta B, \Delta C) \in \mathbf{U}, \quad (\mathbf{Q}, B_c, C_c) \in \hat{\mathbf{N}}^{\tilde{n}} \times \mathbf{R}^{n_c \times l} \times \mathbf{R}^{m \times n_c} \tag{3.3}$$

and, for given $(A_c, B_c, B_m, C_c, C_e, C_m)$, suppose there exists $\mathbf{Q} \in \hat{\mathbf{N}}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathbf{Q} + \mathbf{Q}\tilde{A}^{\mathrm{T}} + \Omega(\mathbf{Q}, B_c, C_c) + \tilde{V} \tag{3.4}$$

Furthermore, suppose the pair $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. Then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$,

$$\tilde{Q}_{\Delta\tilde{A}} \leqslant \mathbf{Q}, (\Delta A, \Delta B, \Delta C) \in \mathbf{U} \tag{3.5}$$

and

$$J(A_c, B_c, B_m, C_c, C_e, C_m) \leqslant \operatorname{tr} \mathbf{Q}\tilde{R} \tag{3.6}$$

Proof

For all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, (3.4) is equivalent to

$$0 = (\tilde{A} + \Delta\tilde{A})\mathbf{Q} + \mathbf{Q}(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A}) + \tilde{V} \tag{3.7}$$

where

$$\Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A}) \triangleq \Omega(\mathbf{Q}, B_c, C_c) - (\Delta\tilde{A}\mathbf{Q} + \mathbf{Q}\Delta\tilde{A}^{\mathrm{T}})$$

Note that by (3.3), $\Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A}) \geqslant 0$ for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. Since $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, it follows from Theorem 3.6 of Wonham (1979), that $((\tilde{V} + \Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A}))^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. Hence Lemma 12.2 of Wonham (1979), implies that $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$.

Next, subtracting (3.2) from (3.7) yields

$$0 = (\tilde{A} + \Delta\tilde{A})(\mathbf{Q} - \tilde{Q}_{\Delta\tilde{A}}) + (\mathbf{Q} - \tilde{Q}_{\Delta\tilde{A}})(\tilde{A} + \Delta\tilde{A})^{\mathrm{T}} + \Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A})$$

or, equivalently (since $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable),

$$\mathbf{Q} - \tilde{Q}_{\Delta\tilde{A}} = \int_0^\infty \exp{(\tilde{A} + \Delta\tilde{A})t}\, \Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A}) \exp{(\tilde{A} + \Delta\tilde{A})^T t}\, dt \geqslant 0$$

which implies (3.5). Finally, (3.5) and (3.1) yield (3.6). □

Remark 3.2

For the dynamic-compensation problem the result that $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ is equivalent to robust stability of the closed-loop system. For the state-estimation and model-reduction problems, however, $\tilde{A} + \Delta\tilde{A}$ is lower block triangular (since $B = 0$) and block diagonal (since $C = 0$), respectively. Thus robust stability is equivalent to A_c stable and $A + \Delta A$ stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$.

We also note a sufficient condition for the solution $\mathbf{Q}$ of (3.4) to be positive definite.

Proposition 3.1

Let Ω be as in Theorem 3.1, let $(A_c, B_c, B_m, C_c, C_e, C_m)$ be given, and suppose there exists $\mathbf{Q} \in \hat{\mathbf{N}}^{\tilde{n}}$ satisfying (3.4). If $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is observable for some $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, then $\mathbf{Q}$ is positive definite.

Proof

If $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is observable for some $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$, then, by Theorem 3.6 of Wonham (1979), $((\tilde{V} + \Psi(\mathbf{Q}, B_c, C_c, \Delta\tilde{A}))^{1/2}, \tilde{A} + \Delta\tilde{A})$ is also observable for the same $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$. It thus follows from (3.7) and Lemma 12.2 of Wonham (1979), that $\mathbf{Q}$ is positive definite. □

Remark 3.3

If $\tilde{V}$ is positive definite then the detectability and observability hypotheses of Theorem 3.1 and Proposition 3.1 are automatically satisfied.

Remark 3.4

Theorem 3.1 can be strengthened by noting that the detectability assumption is, in a sense, superfluous. To see this, first note that robust stability concerns only the undisturbed system while $\tilde{V}$ involves the disturbance noise. Hence robust stability is guaranteed by the existence of a solution $\mathbf{Q} \in \hat{\mathbf{N}}^{\tilde{n}}$ satisfying (3.4) with $\tilde{V}$ replaced by $\alpha I_{\tilde{n}}$ for some $\alpha > 0$. For this replacement detectability is automatic (see previous remark). For robust performance, however, $\mathbf{Q}$ in (3.5) must be obtained from (3.4).

4. Uncertainty structure and right shift multiplicative white noise bound

The uncertainty set $\mathbf{U}$ is assumed to be of the form

$$\mathbf{U} = \Big\{(\Delta A, \Delta B, \Delta C) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{l\times n} : \Delta A = \sum_{i=1}^{p} \sigma_i A_i,$$
$$\Delta B = \sum_{i=1}^{p} \sigma_i B_i, \quad \Delta C = \sum_{i=1}^{p} \sigma_i C_i, \quad |\sigma_i| \leq \delta_i, \quad i = 1, \ldots, p\Big\} \tag{4.1}$$

where, for $i = 1, \ldots, p$: $A_i \in \mathbf{R}^{n\times n}$, $B_i \in \mathbf{R}^{n\times m}$ and $C_i \in \mathbf{R}^{l\times n}$ are fixed matrices denoting the structure of the parametric uncertainty; δ_i is a given uncertainty bound; and σ_i is an uncertain real parameter. The closed-loop system thus has structured uncertainty of the form

$$\Delta\tilde{A} = \sum_{i=1}^{p} \sigma_i \tilde{A}_i$$

where

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i C_c \\ B_c C_i & 0 \end{bmatrix}, \quad i = 1, \ldots, p \tag{4.2}$$

To obtain an explicit gain expression for $(A_c, B_c, B_m, C_c, C_e, C_m)$ we require that

$$[B_i \neq 0 \Rightarrow C_i = 0], \quad i = 1, \ldots, p \tag{4.3}$$

That is, for each $i \in \{1, \ldots, p\}$ either B_i or C_i is zero. Of course, both $B_i = 0$ and $C_i = 0$ are possible for a given i, and there are no restrictions on A_i.

Given the structure of $\mathbf{U}$ defined by (4.1) we can define the bound satisfying (3.3).

Proposition 4.1

Let $\alpha_1, \ldots, \alpha_p$ be arbitrary positive scalars. Then the function

$$\Omega(\mathbf{Q}, B_c, C_c) = \sum_{i=1}^{p} \delta_i(\alpha_i \mathbf{Q} + \alpha_i^{-1} \tilde{A}_i \mathbf{Q} \tilde{A}_i^{\mathrm{T}}) \tag{4.4}$$

satisfies (3.3) with $\mathbf{U}$ given by (4.1).

Proof

Note that

$$\begin{aligned} 0 &\leqslant [\sigma_i(\alpha_i/\delta_i)^{1/2} I_{\tilde{n}} - (\delta_i/\alpha_i)^{1/2} \tilde{A}_i] \mathbf{Q} [\sigma_i(\alpha_i/\delta_i)^{1/2} I_{\tilde{n}} - (\delta_i/\alpha_i)^{1/2} \tilde{A}_i]^{\mathrm{T}} \\ &= \sigma_i^2(\alpha_i/\delta_i)\mathbf{Q} + (\delta_i/\alpha_i)\tilde{A}_i \mathbf{Q} \tilde{A}_i^{\mathrm{T}} - \sigma_i(\tilde{A}_i \mathbf{Q} + \mathbf{Q}\tilde{A}_i^{\mathrm{T}}) \end{aligned}$$

which, since $\sigma_i^2 \leqslant \delta_i^2$, implies (3.3). □

5. Auxiliary minimization problem

Rather than minimize the actual cost (2.8), we shall consider the upper bound (3.6). This leads to the following problem.

Auxiliary minimization problem

Determine $(A_c, B_c, B_m, C_c, C_e, C_m)$ and $\mathbf{Q} \in \hat{\mathbf{N}}^{\tilde{n}}$ which minimize

$$\mathbf{J}(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}) \triangleq \operatorname{tr} \mathbf{Q}\tilde{R} \tag{5.1}$$

subject to

$$0 = \tilde{A}\mathbf{Q} + \mathbf{Q}\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p} \delta_i(\alpha_i \mathbf{Q} + \alpha_i^{-1} \tilde{A}_i \mathbf{Q} \tilde{A}_i^{\mathrm{T}}) + \tilde{V} \tag{5.2}$$

and

$$(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A}) \text{ is detectable, } (\Delta A, \Delta B, \Delta C) \in \mathbf{U} \tag{5.3}$$

Proposition 5.1

If $(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q})$ is admissible, i.e. $(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q})$ satisfies (5.2) and (5.3), then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ and

$$J(A_c, B_c, B_m, C_c, C_e, C_m) \leqslant \mathbf{J}(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}) \tag{5.4}$$

Proof

With Ω given by (4.4), Proposition 4.1 implies that (3.3) is satisfied. Furthermore, admissibility implies that (3.4) has a solution $\mathbf{Q} \in \hat{\mathbf{N}}^{\tilde{n}}$. Hence, with (5.3), the hypotheses of Theorem 3.1 are satisfied so that robust stability with the performance bound (3.6) is guaranteed. Note that with the definition (5.1), (5.4) is merely a restatement of (3.6). □

6. Necessary conditions for the auxiliary minimization problem

The derivation of the necessary conditions for the 'auxiliary minimization problem' is based upon the Fritz John form of the Lagrange multiplier theorem.

Rigorous application of this technique requires additional technical assumptions. Specifically, we further restrict $(A_c, B_c, B_m, C_c, C_e, C_m)$ to the set

$$\mathbf{S} \triangleq \{(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}) : \mathbf{Q} \in \hat{\mathbf{P}}^{\tilde{n}}, \tilde{\mathbf{A}} \text{ is asymptotically stable, and } (A_c, B_c, C_c) \text{ is minimal}\}$$

where

$$\tilde{\mathbf{A}} \triangleq \left(\tilde{A} + \frac{1}{2}\sum_{i=1}^{p} \delta_i \alpha_i I_{\tilde{n}}\right) \oplus \left(\tilde{A} + \frac{1}{2}\sum_{i=1}^{p} \delta_i \alpha_i I_{\tilde{n}}\right) + \sum_{i=1}^{p} \gamma_i \tilde{A}_i \otimes \tilde{A}_i$$

with, for convenience,

$$\gamma_i \triangleq \delta_i/\alpha_i$$

The following observation assures us that we can apply Lagrange multipliers over an open constraint set.

Proposition 6.1

The set $\mathbf{S}$ is open.

Proof

It need only be noted that $\mathbf{S}$ is the intersection of three open sets. □

Remark 6.1

The constraint $(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}) \in \mathbf{S}$ is not required for either robust stability or robust performance. As can be seen from the proof of Theorem 6.1 in the Appendix, the set $\mathbf{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the 'auxiliary minimization problem'. Specifically, asymptotic stability of $\tilde{\mathbf{A}}$ serves as a normality condition which further implies that the dual $\mathbf{P}$ of $\mathbf{Q}$ is non-negative definite. Furthermore, (A_c, B_c, C_c) minimal is a nondegeneracy condition which implies that the lower right $n_c \times n_c$ subblocks of $\mathbf{Q}$ and $\mathbf{P}$ are positive definite. It is extremely important to emphasize that Proposition 5.1 implies that it is *not* necessary for guaranteed robust stability and performance that an admissible quadruple, obtained by solving the necessary conditions, actually be shown to be an element of $\mathbf{S}$.

For arbitrary Q, P, $\hat{Q}$, $\hat{P} \in \mathbf{R}^{n \times n}$ define the following notation:

$$R_{2s} \triangleq R_2 + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}}(P + \hat{P})B_i, \quad V_{2s} \triangleq V_2 + \sum_{i=1}^{p} \gamma_i C_i(Q + \hat{Q})C_i^{\mathrm{T}}$$

$$Q_s \triangleq QC^{\mathrm{T}} + V_{12} + \sum_{i=1}^{p} \gamma_i A_i(Q + \hat{Q})C_i^{\mathrm{T}}, \quad P_s \triangleq B^{\mathrm{T}}P + R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}}(P + \hat{P})A_i$$

$$A_Q \triangleq A_\alpha - Q_s V_{2s}^{-1} C, \quad A_P \triangleq A_\alpha - BR_{2s}^{-1} P_s$$

The following factorization lemma is needed for the statement of the main result.

Lemma 6.1

If $\hat{Q}, \hat{P} \in \hat{\mathbf{N}}^n$ then $\hat{Q}\hat{P}$ is diagonalizable with non-negative eigenvalues. If, in addition, rank $\hat{Q}\hat{P} = n_c$, then there exist $n_c \times n$ G, Γ and $n_c \times n_c$ invertible M such that

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma \tag{6.1}$$

$$\Gamma G^{\mathrm{T}} = I_{n_c} \tag{6.2}$$

Furthermore, G, M and Γ are unique except for a change of basis in $\mathbf{R}^{n_c}$.

Proof

The result is an immediate consequence of Rao and Mitra (1971), Theorem 6.2.5, p. 123. □

A triple (G, M, Γ) satisfying (6.1) and (6.2) will be called a *projective factorization* of $\hat{Q}\hat{P}$. Since $\hat{Q}\hat{P}$ is diagonalizable it has a group generalized inverse $(\hat{Q}\hat{P})^{\#} = G^{\mathrm{T}} M^{-1} \Gamma$ and

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^{\mathrm{T}}\Gamma$$

is an oblique projection. Furthermore, define the complementary projection

$$\tau_{\perp} \triangleq I_n - \tau.$$

Theorem 6.1

Suppose $(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}) \in \mathbf{S}$ solves the 'auxiliary minimization problem'. Then there exist $Q, P, \hat{Q}, \hat{P} \in \hat{\mathbf{N}}^n$ such that, for some projective factorization (G, M, Γ) of $\hat{Q}\hat{P}$, A_c, B_c, B_m, C_c, C_e, C_m and $\mathbf{Q}$ are given by

$$A_c = \Gamma(A - BR_{2s}^{-1}P_s - Q_s V_{2s}^{-1} C)G^{\mathrm{T}} \tag{6.3}$$

$$B_c = \Gamma Q_s V_{2s}^{-1} \tag{6.4}$$

$$B_m = \Gamma \hat{B} \tag{6.5}$$

$$C_c = -R_{2s}^{-1} P_s G^{\mathrm{T}} \tag{6.6}$$

$$C_e = LG^{\mathrm{T}} \tag{6.7}$$

$$C_m = \hat{C} G^{\mathrm{T}} \tag{6.8}$$

$$\mathbf{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix} \tag{6.9}$$

and such that Q, P, $\hat{Q}$ and $\hat{P}$ satisfy

$$\begin{aligned} 0 = A_a Q + Q A_a^{\mathrm{T}} + V_1 + \sum_{i=1}^{p} \gamma_i [A_i Q A_i^{\mathrm{T}} + (A_i - B_i R_{2s}^{-1} P_s)\hat{Q}(A_i - B_i R_{2s}^{-1} P_s)^{\mathrm{T}}] \\ - Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} + \tau_{\perp}[Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} + \hat{B} V \hat{B}^{\mathrm{T}}]\tau_{\perp}^{\mathrm{T}} \end{aligned} \tag{6.10}$$

$$\begin{aligned} 0 = A_a^{\mathrm{T}} P + P A_a + R_1 + \sum_{i=1}^{p} \gamma_i [A_i^{\mathrm{T}} P A_i + (A_i - Q_s V_{2s}^{-1} C_i)^{\mathrm{T}} \hat{P} (A_i - Q_s V_{2s}^{-1} C_i) \\ - P_s^{\mathrm{T}} R_{2s}^{-1} P_s + \tau_{\perp}^{\mathrm{T}} [P_s^{\mathrm{T}} R_{2s}^{-1} P_s + L^{\mathrm{T}} R L + \hat{C}^{\mathrm{T}} \hat{R} \hat{C}]\tau_{\perp} \end{aligned} \tag{6.11}$$

$$0 = A_P \hat{Q} + \hat{Q} A_P^{\mathrm{T}} + Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} + \hat{B} V \hat{B}^{\mathrm{T}} - \tau_{\perp}[Q_s V_{2s}^{-1} Q_s^{\mathrm{T}} + \hat{B} V \hat{B}^{\mathrm{T}}]\tau_{\perp}^{\mathrm{T}} \tag{6.12}$$

$$0 = A_Q^{\mathrm{T}} \hat{P} + \hat{P} A_Q + P_s^{\mathrm{T}} R_{2s}^{-1} P_s + L^{\mathrm{T}} R L + \hat{C}^{\mathrm{T}} \hat{R} \hat{C} - \tau_{\perp}^{\mathrm{T}} [P_s^{\mathrm{T}} R_{2s}^{-1} P_s + L^{\mathrm{T}} R L + \hat{C}^{\mathrm{T}} \hat{R} \hat{C}]\tau_{\perp} \tag{6.13}$$

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{6.14}$$

Proof

For the proof see the Appendix.

Remark 6.2

As in Remark 2.1 suppose $\Delta A, \Delta B, \Delta C = 0$. By setting $L = 0$, $\hat{B} = 0$ and $C = 0$, (6.10)–(6.13) specialize to the optimal projection equations (2.18)–(2.21) derived by Hyland and Bernstein (1984), with the added features of correlated plant/measurement noise (V_{12}) and cross weighting (R_{12}). If $R_1 = 0$, $R_{12} = 0$, $B = 0$, $\hat{B} = 0$ and $\hat{C} = 0$ then, since $P_s = 0$, (6.11) drops out and the remaining equations (6.10), (6.12) and (6.13) specialize to (2.10)–(2.12) of Bernstein and Hyland (1985). Finally, if $R_1 = 0$, $R_{12} = 0$, $L = 0$, $V_1 = 0$, $B = 0$ and $C = 0$, then, since $Q_s = P_s = 0$, (6.10) and (6.11) drop out and the remaining equations (6.12) and (6.13) specialize to (2.21) and (2.22) of Hyland and Bernstein (1985).

Remark 6.3

A more restrictive formulation for unified modelling, estimation and control is to require $C_c = C_e = C_m$ so that $u = y_e = y_m$. In this case the three outputs of the design system (2.4)–(2.7) are replaced by a single output. Again, the necessary conditions involve a system of four coupled matrix equations similar to (6.10)–(6.13) which specialize to previously known results. Since this formulation requires $m = q = \hat{l}$, it appears to be less useful than the three-output formulation.

7. Sufficient conditions for robust stability and performance

The main result guaranteeing robust stability and performance for the unified problem can now be stated.

Theorem 7.1

Suppose there exist $Q, P, \hat{Q}, \hat{P} \in \hat{\mathbf{N}}^n$ satisfying (6.10)–(6.14) and assume $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ with A_c, B_c, B_m, C_c, C_e, C_m given by (6.3)–(6.18). Then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathbf{U}$ and the closed-loop system satisfies the performance bound

$$J(A_c, B_c, B_m, C_c, C_e, C_m) \leqslant \operatorname{tr}[(Q + \hat{Q})R_1 + P_s^{\mathrm{T}} R_{2s}^{-1} R_2 R_{2s}^{-1} P_s \hat{Q} - 2R_{12} R_{2s}^{-1} P_s \hat{Q} + QL^{\mathrm{T}} RL + \hat{C}^{\mathrm{T}} \hat{R} \hat{C}(W_c - \hat{Q})] \tag{7.1}$$

where the controllability gramian W_c satisfies

$$0 = AW_c + W_c A^{\mathrm{T}} + \hat{B} V \hat{B}^{\mathrm{T}} \tag{7.2}$$

Proof

Theorem 7.1 implies $\mathbf{\Omega}$ given by (6.9) satisfies (5.2). With the detectability assumption the result follows from Proposition 5.1. □

8. Directions for further research

Several generalizations remain to be explored. These include:

(*a*) permit $w(\cdot)$ to be correlated with $w_1(\cdot)$ and $w_2(\cdot)$;

(*b*) replace (2.2) with

$$y(t) = (C + \Delta C)x(t) + (D + \Delta D)u(t) + w_2(t) \tag{8.1}$$

(*c*) replace (2.4) with

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) + B_m w(t) + w_3(t) \tag{8.2}$$

(*d*) replace (2.5) with

$$u(t) = C_c x_c(t) + D_c y(t) \tag{8.3}$$

The extension (8.3) has been studied by Bernstein (1987 b), for control and by Haddad and Bernstein (1987), for estimation.

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Appendix

Proof of Theorem 6.1

Partition $\tilde{n} \times \tilde{n}$ $\mathbf{Q}$, $\mathbf{P}$ into $n \times n$, $n \times n_c$, and $n_c \times n_c$ sub-blocks as

$$\mathbf{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}$$

and define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}}$$
$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}}$$

and the $n_c \times n$, $n_c \times n_c$, $n_c \times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2 P_2, \quad \Gamma \triangleq -P_2^{-1}P_{12}^{\mathrm{T}}$$

The existence of Q_2^{-1} and P_2^{-1} is shown below.

Clearly, Q, P, $\hat{Q}$ and $\hat{P}$ are symmetric and $\hat{Q}$ and $\hat{P}$ are non-negative definite. To show that Q and P are non-negative definite, note that Q is the upper left-hand block of the non-negative-definite matrix $Q^*\mathbf{Q}Q^{*\mathrm{T}}$, where

$$Q^* \triangleq \begin{bmatrix} I_n & -Q_{12}Q_2^{-1} \\ 0 & -I_{n_c} \end{bmatrix}$$

Similarly, P is non-negative definite.

To optimize (5.1) over the open set $\mathbf{S}'$, where

$$\mathbf{S}' \triangleq \{(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}) \in \mathbf{S}\text{: (5.3) is satisfied}\}$$

and subject to the constraint (5.2), form the lagrangian

$$L(A_c, B_c, B_m, C_c, C_e, C_m, \mathbf{Q}, \mathbf{P}, \lambda)$$
$$\triangleq \operatorname{tr}\left[\lambda \mathbf{Q}\tilde{R} + \left(\tilde{A}\mathbf{Q} + \mathbf{Q}\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p} \delta_i(\alpha_i \mathbf{Q} + \alpha_i^{-1}\tilde{A}_i \mathbf{Q}\tilde{A}_i^{\mathrm{T}}) + \tilde{V}\right)\mathbf{P}\right]$$

where the Lagrange multipliers $\lambda \geqslant 0$ and $\mathbf{P} \in \mathbf{P}^{\tilde{n}\times\tilde{n}}$ are not both zero. We thus obtain

$$\frac{\partial L}{\partial \mathbf{Q}} = \tilde{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\tilde{A} + \sum_{i=1}^{p} \delta_i(\alpha_i \mathbf{P} + \alpha_i^{-1}\tilde{A}_i^{\mathrm{T}}\mathbf{P}\tilde{A}_i) + \lambda\tilde{R}$$

Setting $\partial L/\partial \mathbf{Q} = 0$ yields

$$\mathbf{\AA}^{\mathrm{T}} \operatorname{vec} \mathbf{P} = -\lambda \operatorname{vec} \tilde{R}$$

where 'vec' is the column-stacking operation (see Brewer 1978). Since $\mathbf{\AA}$ is assumed to be stable and thus invertible, $\lambda = 0$ implies $\mathbf{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$.

Furthermore, the stability of $\mathbf{\AA}$ implies that $\mathbf{P}$ is non-negative definite. The stationarity conditions are given by

$$\frac{\partial L}{\partial \mathbf{P}} = \tilde{A}\mathbf{Q} + \mathbf{Q}\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p} [\delta_i \alpha_i \mathbf{Q} + \gamma_i \tilde{A}_i \mathbf{Q}\tilde{A}_i^{\mathrm{T}}] + \tilde{V} = 0 \tag{A 1}$$

$$\frac{\partial L}{\partial \mathbf{Q}} = \tilde{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p} [\delta_i \alpha_i \mathbf{P} + \gamma_i \tilde{A}_i^{\mathrm{T}}\mathbf{P}\tilde{A}_i] + \tilde{R} = 0 \tag{A 2}$$

$$\frac{\partial L}{\partial A_c} = P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2 = 0 \tag{A 3}$$

$$\frac{\partial L}{\partial B_c} = P_2 B_c V_{2s} + (P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}})C + P_{12}^{\mathrm{T}}\left(V_{12} + \sum_{i=1}^{p} \gamma_i A_i Q_1 C_i^{\mathrm{T}}\right) = 0 \tag{A 4}$$

$$\frac{\partial L}{\partial B_m} = P_{12}^{\mathrm{T}} \hat{B} V + P_2 B_m V = 0 \tag{A 5}$$

$$\frac{\partial L}{\partial C_c} = R_{2s} C_c Q_2 + B^{\mathrm{T}}(P_1 Q_{12} + P_{12} Q_2) + \left(R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}} P_1 A_i\right) Q_{12} = 0 \tag{A 6}$$

$$\frac{\partial L}{\partial C_e} = -RLQ_{12} + RC_e Q_{12} = 0 \tag{A 7}$$

$$\frac{\partial L}{\partial C_m} = -\hat{R}\hat{C}Q_{12} + \hat{R}C_m Q_2 = 0 \tag{A 8}$$

Expanding (A 1) and (A 2) yields

$$0 = A_\alpha Q_1 + BC_c Q_{12}^{\mathrm{T}} + Q_1 A_\alpha^{\mathrm{T}} + Q_{12} C_c^{\mathrm{T}} B^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i$$
$$\times [A_i Q_1 A_i^{\mathrm{T}} + B_i C_c Q_{12}^{\mathrm{T}} A_i^{\mathrm{T}} + A_i Q_{12} C_c^{\mathrm{T}} B_i^{\mathrm{T}} + B_i C_c Q_2 C_c^{\mathrm{T}} B_i^{\mathrm{T}}] + V_1 + \hat{B} V \hat{B}^{\mathrm{T}} \tag{A 9}$$

$$0 = A_\alpha Q_{12} + Q_{12} A_{c\alpha}^{\mathrm{T}} + BC_c Q_2 + Q_1 C^{\mathrm{T}} B_c^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i A_i Q_1 C_i^{\mathrm{T}} B_c^{\mathrm{T}}$$
$$+ V_{12} B_c^{\mathrm{T}} + \hat{B} V B_m^{\mathrm{T}} \tag{A 10}$$

$$0 = B_c C Q_{12} + A_{c\alpha} Q_2 + Q_{12}^{\mathrm{T}} C^{\mathrm{T}} B_c^{\mathrm{T}} + Q_2 A_{c\alpha}^{\mathrm{T}} + B_c V_{2s} B_c^{\mathrm{T}} + B_m V B_m^{\mathrm{T}} \tag{A 11}$$

$$0 = P_1 A_\alpha + P_{12} B_c C + A_\alpha^{\mathrm{T}} P_1 + C^{\mathrm{T}} B_c^{\mathrm{T}} P_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i \times [A_i^{\mathrm{T}} P_1 A_i + C_i^{\mathrm{T}} B_c^{\mathrm{T}} P_{12}^{\mathrm{T}} A_i + A_i^{\mathrm{T}} P_{12} B_c C_i + C_i^{\mathrm{T}} B_c^{\mathrm{T}} P_2 B_c C_i] + R_1 + L^{\mathrm{T}} R L + \hat{C}^{\mathrm{T}} \hat{R} \hat{C} \tag{A 12}$$

$$0 = P_1 B C_c + P_{12} A_{c\alpha} + A^{\mathrm{T}} P_{12} + C^{\mathrm{T}} B_c^{\mathrm{T}} P_2 + \sum_{i=1}^{p} \gamma_i A_i^{\mathrm{T}} P_1 B_i C_c + R_{12} C_c - L^{\mathrm{T}} R C_m - \hat{C}^{\mathrm{T}} R \hat{C}_m \tag{A 13}$$

$$0 = A_{c\alpha}^{\mathrm{T}} P_2 + P_2 A_{c\alpha} + C_c^{\mathrm{T}} B^{\mathrm{T}} P_{12} + P_{12}^{\mathrm{T}} B C_c + C_c^{\mathrm{T}} R_{2s} C_c + C_e^{\mathrm{T}} R C_e + C_m^{\mathrm{T}} \hat{R} C_m \tag{A 14}$$

Lemma A 1

Q_2 and P_2 are positive definite.

Proof

By a minor extension of the results from Albert (1969), (A 11) can be rewritten as

$$0 = (A_{c\alpha} + B_c C Q_{12} Q_2^{+}) Q_2 + Q_2 (A_{c\alpha} + B_c C Q_{12} Q_2^{+})^{\mathrm{T}} + \Lambda$$

where $\Lambda = B_c V_{2s} B_c^{\mathrm{T}} + B_m V B_m^{\mathrm{T}}$ and Q_2^{+} is the Moore–Penrose or Drazin generalized inverse of Q_2. Next note that since (A_c, B_c) is controllable then, by Theorem 3.6 of Wonham (1979), $(A_{c\alpha} + B_c C Q_{12} Q_2^{+}, \Lambda^{1/2})$ is also controllable. Now, since Q_2 and Λ are non-negative definite, it follows from Lemma 12.2 of Wonham (1979), that Q_2 is positive definite. Using (A 14), similar arguments show P_2 is positive definite. □

Since $\hat{R}$, R, R_{2s}, V, V_{2s}, Q_2 and P_2 are invertible, (A 3)–(A 8) can be written as

$$-P_2^{-1} P_{12}^{\mathrm{T}} Q_{12} Q_2^{-1} = I_{n_c} \tag{A 15}$$

$$B_c = -P_2^{-1} \left[(P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}}) C + P_{12}^{\mathrm{T}} \left(V_{12} + \sum_{i=1}^{p} \gamma_i A_i Q_1 C_i^{\mathrm{T}} \right) \right] V_{2s}^{-1} \tag{A 16}$$

$$B_m = -P_2^{-1} P_{12} \hat{B} \tag{A 17}$$

$$C_c = -R_{2s}^{-1} \left[B^{\mathrm{T}} (P_1 Q_{12} + P_{12} Q_2) + \left(R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} \gamma_i B_i^{\mathrm{T}} P_1 A_i \right) Q_{12} \right] Q_2^{-1} \tag{A 18}$$

$$C_e = L Q_{12} Q_2^{-1} \tag{A 19}$$

$$C_m = \hat{C} Q_{12} Q_2^{-1} \tag{A 20}$$

Note that because of (A 15), (6.1) and (6.2) hold. Since Q_2 and P_2 are positive definite and

$$Q_2 P_2 = P_2^{-1/2} (P_2^{1/2} Q_2 P_2^{1/2}) P_2^{1/2}$$

M is diagonalizable with positive eigenvalues. It is helpful to note the identities

$$\hat{Q} = Q_{12} G = G^{\mathrm{T}} Q_{12}^{\mathrm{T}} = G^{\mathrm{T}} Q_2 G, \quad \hat{P} = -P_{12} \Gamma = -\Gamma^{\mathrm{T}} P_{12}^{\mathrm{T}} = \Gamma^{\mathrm{T}} P_2 \Gamma \tag{A 21}$$

$$\tau G^{\mathrm{T}} = G^{\mathrm{T}}, \quad \Gamma \tau = \Gamma \tag{A 22}$$

$$\hat{Q} = \tau \hat{Q}, \quad \hat{P} = \hat{P} \tau \tag{A 23}$$

$$\hat{Q} \hat{P} = -Q_{12} P_{12}^{\mathrm{T}} \tag{A 24}$$

Using (6.2) and Silvester's inequality, it follows that rank $G = \text{rank } \Gamma = \text{rank } Q_{12} = \text{rank } P_{12} = n_c$ which in turn imply (6.14).

The components of Q and P can be written in terms of Q, P, $\hat{Q}$, $\hat{P}$, G and Γ as

$$Q_1 = Q + \hat{Q}, \quad P_1 = P + \hat{P} \tag{A 25}$$

$$Q_{12} = \hat{Q}\Gamma^T, \quad P_{12} = -\hat{P}G^T \tag{A 26}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^T, \quad P_2 = G\hat{P}G^T \tag{A 27}$$

The gain expressions (6.3)–(6.8) and (6.9) follow from (A 16)–(A 20) and the definition of $\mathbf{Q}$. Substituting (A 25)–(A 27) into (A 9)–(A 14) yields

$$0 = A_a Q + QA_a^T + V_1 + \hat{B}V\hat{B}^T + \sum_{i=1}^{p} \gamma_i \times [A_i Q A_i^T + (A_i - B_i R_{2s}^{-1} P_s)\hat{Q}(A_i - B_i R_{2s}^{-1} P_s)^T] + A_P\hat{Q} + \hat{Q}A_P^T \tag{A 28}$$

$$0 = [A_P\hat{Q} + \hat{Q}(\Gamma^T A_{ca}^T G + C^T V_{2s}^{-1} Q_s^T) + Q_s V_{2s}^{-1} Q_s^T + \hat{B}V\hat{B}^T]\Gamma^T \tag{A 29}$$

$$0 = [(G^T A_{ca}\Gamma + Q_s V_{2s}^{-1} C)\hat{Q} + \hat{Q}(G^T A_{ca}\Gamma + Q_s V_{2s}^{-1} C)^T + Q_s V_{2s}^{-1} Q_s^T + \hat{B}V\hat{B}^T]\Gamma^T \tag{A 30}$$

$$0 = A_a^T P + PA_a + R_1 + L^T RL + \hat{C}^T\hat{R}\hat{C} + \sum_{i=1}^{p} \gamma_i \times [A_i^T P A_i + (A_i - Q_s V_{2s}^{-1} C_i)^T \hat{P}(A_i - Q_s V_{2s}^{-1} C_i)] + A_Q^T\hat{P} + \hat{P}A_Q \tag{A 31}$$

$$0 = [A_Q^T\hat{P} + \hat{P}(G^T A_{ca}\Gamma + BR_{2s}^{-1} P_s) + P_s^T R_{2s}^{-1} P_s + L^T RL + \hat{C}^T\hat{R}\hat{C}]G^T \tag{A 32}$$

$$0 = G[(G^T A_{ca}\Gamma + BR_{2s}^{-1} P_s)^T\hat{P} + \hat{P}(G^T A_{ca}\Gamma + BR_{2s}^{-1} P_s) + P_s^T R_{2s}^{-1} P_s + L^T RL + \hat{C}^T\hat{R}\hat{C}]G^T \tag{A 33}$$

Next, computing either Γ(A 29) $-$ (A 30) or G(A 32) $-$ (A 33) yields (6.3). Substituting this expression for A_c into (A 28), (A 29), (A 31) and (A 32) and using

$$(\text{A } 28) + G^T\Gamma(\text{A } 29)G - (\text{A } 29)G - (\text{A } 29\ G)^T$$

and

$$G^T\Gamma(\text{A } 29)G - (\text{A } 29)G - (\text{A } 29\ G)^T$$

yields (6.10) and (6.12). Using

$$(\text{A } 31) + \Gamma^T G(\text{A } 32)\Gamma - (\text{A } 32)\Gamma - (\text{A } 32\ \Gamma)^T$$

and

$$\Gamma^T G(\text{A } 32)\Gamma - (\text{A } 32)\Gamma - (\text{A } 32\ \Gamma)^T$$

yields (6.11) and (6.13).

Finally, to show that the preceding development entails no loss of generality in the optimality condition we now use (6.3)–(6.14) to obtain (A 1)–(A 8). Let A_c, B_c, B_m, C_c, C_e, C_m, G, Γ, τ, Q, P, $\hat{Q}$, $\hat{P}$, $\mathbf{Q}$ be as in the statement of Theorem 6.1 and define Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2 by (A 25)–(A 27). Using (6.2) and (6.4)–(6.8) it is easy to verify (A 4)–(A 8). Finally, substitute the definitions for Q, P, $\hat{Q}$, $\hat{P}$, G and Γ into (6.10)–(6.13), reverse the steps taken earlier in the proof and use (6.3)–(6.8) to obtain (A 1) and (A 2), which completes the proof. □

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