Global stabilization of the spinning top with mass imbalance

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Abstract. *We consider the stabilization of a top with known imbalance to the sleeping motion. We first define the sleeping motion and show that it is a solution of the equations of motion of a balanced top. In the general case where the top is unbalanced, we derive two families of control laws that globally asymptotically stabilize a top with known imbalance to the sleeping motion using torque actuators. The input torque is produced by two body-fixed torque actuators in one case, and is confined to the inertial* XY*-plane in the other. The control-design strategy is based on Hamilton–Jacobi–Bellman theory with zero dynamics. The result is global in the sense that the spinning top can be stabilized to the sleeping motion regardless of spin rate, and from an arbitrary initial motion that has a coning angle of up to 90°.*

1 Introduction

In the control of industrial rotating machinery, it is well known that one of the major causes of rotor vibrations is mass imbalance due to off-axis center of mass location, axis misalignment or both. While mechanical balancing of huge rotors, such as a turbine, is in itself a difficult task, the integration of the rotor with other subcomponents often introduces additional imbalance that becomes extremely difficult to eliminate. The control of the rotation of a rigid, dynamically unbalanced body amounts to spin stabilization about a non-principal axis of inertia. In this light, we shall investigate the motion control of a spinning, unbalanced top, which is in effect a rotor pivoted at one end. The present paper, which addresses the case in which the imbalance is known, is part of an effort to investigate the control of rotating bodies possessing unknown imbalance.

The motion of the spinning top, which is essentially a rigid body rotating about a fixed point and being subject to gravity, is characterized by the Euler–Poisson

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equations. Treatments of the general motion of the spinning top can be found in Crabtree (1914) and Macmillan (1936). A familiar type of top is Lagrange's top, that is one which possesses an axis of symmetry. One particular motion of Lagrange's top is the sleeping motion, that is, one in which the top spins about its symmetry axis, which itself remains vertical. Stability analysis of the sleeping motion of Lagrange's top is well developed using various methods (see Rumjancev, 1956, 1983; Chetayev, 1961; Leimanis, 1965; Ge & Wu, 1984; Bahar, 1992; Lewis *et al.*, 1992; Wang & Krishnaprasad, 1992). In Wan *et al.* (1994a) a family of control laws was obtained that globally asymptotically stabilize Lagrange's top to the sleeping motion using two force actuators, while in Wan *et al.* (1994b) torque actuators were used to achieve the same goal.

In this paper, we consider a top that is generally asymmetric and, in addition, possesses a mass imbalance so that the top axis, defined as the axis joining the center of mass of the top and its base, is not a principal axis of inertia. It is impossible for such a top to undergo the sleeping motion while spinning under the sole influence of gravity. In other words, the sleeping motion is not a solution of the equations of motion of a freely spinning, uncontrolled top with imbalance. Nevertheless, we show in this paper that, when the imbalance is known, the top can be put to sleep using external torque actuators. Two distinct actuation schemes are considered, namely, body-fixed and inertially fixed torques. In both cases and with two mutually orthogonal input torques, we obtain a family of control laws that asymptotically stabilize the sleeping motion of a top with imbalance. Moreover, this control objective is achieved regardless of the spin rate. In addition, the obtained results are global in the sense that the top can be stabilized from an initial motion that has a coning angle of up to 90°. These control laws are derived using Hamilton–Jacobi–Bellman theory with zero dynamics (see Bernstein, 1993; Wan & Bernstein, 1995). Some terminology in differential geometry is used; however, the stability analysis in this paper is done solely in the Lyapunov framework.

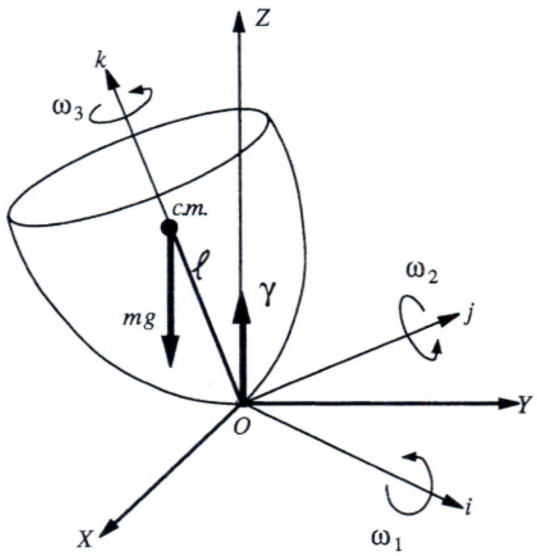
2 Equations of motion and problem statement

2.1 Dynamical equations of the freely spinning top

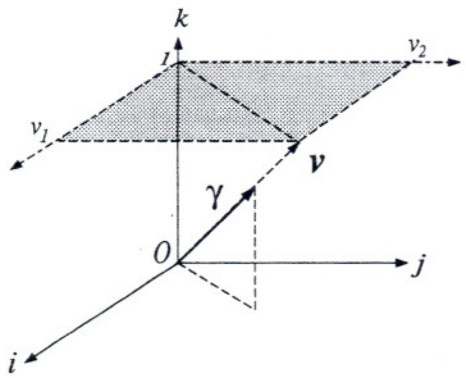
Figure 1(a) shows a rigid, fixed-base, freely spinning top under the influence of gravity, where the *ijk*-frame is the body frame attached to the top and rotating in the inertial *XYZ*-frame. The inertially stationary base of the top is chosen as the origin of both reference frames. The *k*-axis is chosen so that it passes through the center of gravity of the top. We shall call this axis the 'top axis'. We allow the top to be completely arbitrary with regard to its mass distribution. In other words, the top may or may not possess symmetry in mass distribution with respect to the top axis, while the *i*, *j*, *k*-axes may or may not be principal axes of inertia.

Let $\mathcal{J} \in \mathbb{R}^{3\times 3}$ be the inertia matrix of the top resolved in the *ijk*-frame. In general, $\mathcal{J}$ has non-zero (1,3)- and (2,3)-elements, in which case the top is said to be 'unbalanced'. The top is said to be 'balanced' with respect to the top axis if the top axis is a principal axis of inertia, that is, if $\mathcal{J}$ has the form

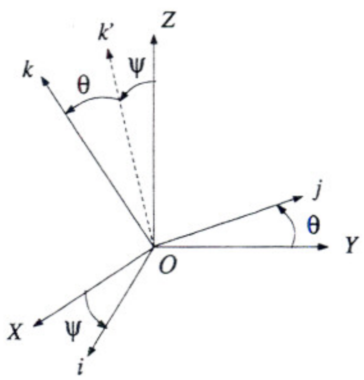
$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & 0 \\ \mathcal{J}_{12} & \mathcal{J}_{22} & 0 \\ 0 & 0 & \mathcal{J}_{a} \end{bmatrix}$$



(a) Fixed-Base Freely Spinning Top



(b) Definition of Projection Coordinates



(c) Definition of Euler Angles

Fig. 1. Spinning top.

In this case, $\mathcal{J}_a$ is called the 'axial' moment of inertia. If diagonalization of the i, j-coordinates yields two distinct eigenvalues, the top is balanced but 'asymmetric'. If however, diagonalization of the i, j-coordinates yields a repeated eigenvalue $\mathcal{J}_t$, then the top is said to be 'symmetric', and $\mathcal{J}_t$ is called the 'transverse' moment of inertia. Such a top is also known as Lagrange's top.

When the top is spinning in a manner such that the top axis is vertical, that is, parallel to the gravity direction, we say that it is 'sleeping'. In particular, when the spin is null, the sleeping position of the top corresponds to the (unstable) equilibrium of an inverted pendulum. However, due to mass imbalance, this position is no longer an equilibrium under the gyroscopic effect of the spin. As we shall see later, the particular case in which the sleeping motion remains an equilibrium with non-zero spin occurs if and only if the top is balanced.

The dynamics of the rigid, fixed-base, freely spinning top are completely described by the Euler–Poisson equations (see Hughes, 1986; Greenwood, 1988).

$$\mathcal{J}\dot{\omega} = -\omega \times \mathcal{J}\omega + mg\gamma \times l \tag{1}$$

$$\dot{\gamma} = \gamma \times \omega \tag{2}$$

where $\omega = [\omega_1\ \omega_2\ \omega_3]^{\mathrm{T}} \in \mathbb{R}^3$, $\gamma = [\gamma_1\ \gamma_2\ \gamma_3]^{\mathrm{T}} \in \mathbb{R}^3$, and $l = [0\ 0\ \ell]^{\mathrm{T}} \in \mathbb{R}^3$ are, respectively, the angular velocity of the top, the unit vector in the negative gravity direction and the position vector of the center of mass with ℓ being the distance from the origin to the center of mass, all of these vectors being resolved in body coordinates. Furthermore, g is the gravity constant, m is the mass of the top and $\mathcal{J}$ is the inertia matrix of the top resolved in the body frame. The vector equations (1) and (2) therefore comprise six scalar equations. However, the unit vector γ satisfies the constraint $\|\gamma\|^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, which means that (1) and (2) can be reduced to five independent ordinary differential equations as we shall now demonstrate.

Let us consider hereafter the case in which $\gamma_3 > 0$, that is, the top remains above the horizontal plane. Then, the redundant dimension in (1) and (2) can be removed by defining the 'projection vector' $v \triangleq [v_1\ v_2\ 1]^{\mathrm{T}}$, where v_1 and v_2 are defined by

$$v_1 \triangleq \frac{\gamma_1}{\gamma_3}, \qquad v_1 \triangleq \frac{\gamma_2}{\gamma_3} \tag{3}$$

As shown in Fig. 1(b), v is obtained by extending γ to the plane Π, which is parallel to the body *ij*-plane and which passes through the point (0, 0, 1). Then it can be shown using (2) and (3) that v_1 and v_2 satisfy

$$\begin{bmatrix} \dot{v}_1 \\ \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -\omega_2(1 + v_1^2) + \omega_1 v_1 v_2 + \omega_2 v_2 \\ \\ \omega_1(1 + v_2^2) - \omega_2 v_1 v_2 - \omega_3 v_1 \end{bmatrix} \tag{4}$$

Furthermore, the constraint $\|\gamma\|^2 = 1$ can be rewritten as

$$\frac{1}{\gamma_3} = (1 + v_1^2 + v_2^2)^{1/2} \tag{5}$$

Finally, replacing γ with (3) and (5) in equation (1) yields

$$\dot{\omega} = -\mathcal{J}^{-1}(\omega \times \mathcal{J}\omega) + \mathcal{J}^{-1}\left(\frac{mg}{\|v\|} v \times l\right) \tag{6}$$

The equations (4) and (6) completely describe the five-dimensional motion of the freely spinning top above the horizontal plane. By using the vector v, we have effectively mapped the precessional and nutational attitude of the top into the $\mathbb{R}^2$ space. Specifically, the sleeping position defined earlier is mapped on to itself, whereas the positions in which the top axis lies in the horizontal plane are mapped on to infinity. Although we consider only the case in which the top remains above the horizontal plane, it is not difficult to see that the same approach can be applied to the study of a hanging top instead of the conventional one by simply changing the sign of the gravity constant g.

The sleeping motion now corresponds to the case $\omega = \omega_s \equiv (0, 0, \Omega)$, where Ω is a non-zero constant, and $v_1 = v_2 = 0$. This motion is in general not a solution of (4) and (6). Replacing ω by ω_s and v by 0 in (4) and (6) yields

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\mathcal{J}^{-1}(\omega_s \times \mathcal{J}\omega_s) \end{bmatrix} \tag{7}$$

Therefore, the sleeping motion $\omega = \omega_s$, $v_1 = v_2 = 0$ is a solution of (4) and (6) if and only if the right-hand side of (7) is 0. Now suppose that the top axis, that is, the k-axis, is a principal axis of inertia. Then ω_s is an eigenvector of $\mathcal{J}$, and hence ω_s and $\mathcal{J}_s$ are colinear so that $\mathcal{J}^{-1}(\omega_s \times \mathcal{J}\omega_s)$ in (7) is zero. This implies that the sleeping motion is a solution of (4) and (6). Conversely, if the sleeping motion is a solution of (4) and (6), then the right-hand side of (7) is 0, which implies that $\omega_s \times \mathcal{J}\omega_s$ is 0. Since $\Omega \neq 0$, it follows that if $\omega_s \times \mathcal{J}\omega_s = 0$ then ω_s is an eigenvector of $\mathcal{J}$, that is, the k-axis is a principal axis of inertia. These observations are summarized by the following result.

Proposition 1. The sleeping motion $\omega = \omega_s$, $v_1 = v_2 = 0$ is a solution of the equations of motion (4) and (6) if and only if the top axis is a principal axis of inertia, that is, if and only if the top is balanced.

2.2 Dynamical equations and stabilization of the controlled top

The sleeping motion of Lagrange's top and the stability thereof have been widely discussed in the previous literature (see Rumjancev, 1956; Chetayev, 1961; Liemanis, 1965; Ge & Wu, 1984; Wan *et al.*, 1994a). In this section, we investigate the asymptotic stabilization of the sleeping motion in the more general case, that is, one in which the top axis is not necessarily a principal axis of inertia. Here, the Euler equation (1) is rewritten with a control torque τ on the right-hand side as

$$\mathcal{J}\dot{\omega} = \omega \times \mathcal{J}\omega + \frac{mg}{\|v\|} v \times l + \tau$$

or, equivalently,

$$\dot{\omega} = -\mathcal{J}^{-1}(\omega \times \mathcal{J}\omega) + \mathcal{J}^{-1}\left(\frac{mg}{\|v\|} v \times l\right) + \mathcal{J}^{-1}\tau \tag{8}$$

where $\tau = (\tau_1\ \tau_2, \tau_3) \in \mathbb{R}^3$. Our aim is to determine a feedback control law $\tau = \tau(\omega, v)$ that brings the top to sleep from an arbitrary initial position above the horizontal plane. Following the discussion leading to Proposition 1, it can be seen that in order to put an unbalanced top to sleep, the sleeping motion must be rendered a solution of the equations of motion by means of an offset control torque that equals, at steady state, $\tau_s = \omega_s \times \mathcal{J}\omega_s$ so as to render the right-hand side of (7) 0. To put the top to sleep from an arbitrary initial position, one might propose to offset the control torque τ with τ_s, such as $\tau = \tau_s + T$, and derive a control law for T using techniques that work for a balanced top, such as those presented in Wan *et al.* (1994a,b). However, it can be easily verified that τ_s does not cancel the effects of imbalance except when the top is sleeping. Moreover, the spin rate at sleep, that is Ω, cannot be deduced solely from the initial conditions except in the special case of control torques confined to the inertial XY-plane, which we shall discuss later. More precisely, Ω depends on the motion that the top undergoes while approaching the sleeping position. Thus, rather than applying control laws found in Wan *et al.* (1994a,b) by offestting with τ_s, we shall derive control laws for an unbalanced top by directly accounting for the presence of mass imbalance.

We now consider two cases of actuation.

Case 1. The input torque is produced by two body-fixed torque actuators along the *i*- and *j*-axes. In this case, the control torque τ in (8) takes the form

$$\tau = b_1 u, \qquad b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 2}, \qquad u \in \mathbb{R}^2 \tag{9}$$

Body-fixed torques can be implemented externally by two pairs of gas jets mounted on the top. In each of these pairs, the jets produce equal and opposing thrusts resulting in a perfect couple along the *i*- or *j*-axis. Such an actuation scheme is often used in spacecraft control. Various studies of spacecraft control using body-fixed torque can be found in Byrnes *et al.* (1988), Hughes (1986), Lebedev (1990) and Zhao and Posbergh (1993).

Case 2. The input torque is confined to the inertial *XY*-plane. In this case, τ is constrained to remain perpendicular to the unit vector γ, that is,

$$\gamma_1\tau_1 + \gamma_2\tau_2 + \gamma_3\tau_3 = 0 \tag{10}$$

Since $\tau_3 = -\tau_1 v_1 - \tau_2 v_2$, the input torque τ can be written as

$$\tau = b_2(v)u, \qquad b_2(v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -v_1 & -v_2 \end{bmatrix} \in \mathbb{R}^{3\times 2}, \qquad u \in \mathbb{R}^2 \tag{11}$$

Input torques confined to an inertial plane can be implemented by using magnetic moments. For instance, the top may be located in a uniform magnetic field whose field strength vector is parallel to the local vertical. Then, a moment lying in the horizontal plane can be created by electromagnets or coils embedded in the top. Such actuation schemes can be found in some spacecraft control applications, where the external magnetic field is in effect the local earth magnetic field (see Rodden, 1984).

Let $H_0 \triangleq \gamma^T \mathcal{J}\omega$, that is, the component of the angular momentum along the inertial *Z*-axis. Note that since the input torque is confined to the *XY*-plane, that is, perpendicular to the *Z*-axis, H_0 is therefore a constant of motion. Furthermore, for the sleeping motion, since $\gamma = (0, 0, 1)$ and $\omega = (0, 0, \Omega)$, it follows that $\Omega = H_0/\mathcal{J}_{33}$. In other words, the spin rate of the sleeping top is predetermined by the value H_0, which depends solely on the initial conditions. Note that this is only a special case; in other actuation schemes, including that of body-fixed actuators, H_0 is in general not a constant of motion and Ω cannot be determined solely from the initial conditions.

It is not difficult to see that Case 2 is equivalent to that of a pair of independent, inertially fixed torque actuators. Indeed, τ satisfying (10) or, equivalently, (11), can always be synthesized by two mutually orthogonal torque actuators lying in the *XY*-plane. An equivalent scheme, that is, stabilization using two inertially fixed force actuators, was applied to Lagrange's top in Wan *et al.* (1994a). In that paper, the equations of motion of the symmetric top were formulated in Euler angles, and the control torque was expressed in terms of its X, Y-components. In the present paper, we propose an alternative model of the dynamics, namely equations (4), (8) and (11). However, notice that v_1 and v_2 locate the local vertical in the body frame, but not the azimuth (X and Y) directions. Thus, the design of a controller with inertially fixed torque actuators needs to be carried out in two steps. First, in Section 4, the control law is derived using (4), (8) and (11). Next, to implement the control law using a pair of actuators fixed in the X- and Y-axes, the control torque $\tau = b_2(v)u(\omega, v)$ as defined by (11) needs to be resolved on to the X, Y-axes. This can easily be done by transforming from the variable ω and v to a chosen set of Euler angles, for example, the 2–1–3 Euler angles. The main advantage of this two-step design process is that it avoids the use of Euler angles, which are cumbersome for modeling the dynamics of the unbalanced top.

3 Hamilton–Jacobi–Bellman theory with zero dynamics

In this section, we briefly review Hamilton–Jacobi–Bellman theory with zero dynamics (see Bernstein, 1993; Wan *et al.*, 1994b) by considering the system

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0 \tag{12}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, and f: $\mathbb{R}^n \mapsto \mathbb{R}^n$ and g:$\mathbb{R}^m \mapsto \mathbb{R}^n$ satisfy $f(0) = 0$ and $g(0) = 0$. For system (12), consider the cost functional

$$\mathcal{J}(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t \tag{13}$$

where

$$L(x, u) \triangleq L_1(x) + L_2(x)u + u^T Ru \tag{14}$$

L_1: $\mathbb{R}^n \mapsto \mathbb{R}_1$, L_2: $\mathbb{R}^n \mapsto \mathbb{R}^{1\times m}$ satisfies $L_2(0) = 0$, and $R \in \mathbb{R}^{m\times m}$ is (symmetric) positive definite.

Consider next an output function for (12) of the form

$$y = h(x) \tag{15}$$

where $y \in \mathbb{R}^m$, and h: $\mathbb{R}^n \mapsto \mathbb{R}^m$ satisfies $h(0) = 0$. We recall (see Isidori, 1989) that the 'zero-dynamics' of the non-linear system (12) and (15) are the dynamics of the system subject to the constraint that the output $y(t)$ be identically zero. Then, the system (12) and (15) is 'minimum phase' if its zero dynamics are stable. Furthermore, the system (12) and (15) is said to have relative degree $\{r_1, r_2, \ldots, r_m\}$ at the origin if there exists a neighborhood D_0 of the origin such that

$$L_{g_i} L_f^k h_j(x) = 0, \qquad x \in D_0, \qquad 0 \le k \le r_j - 2, \qquad 1 \le i, j \le m \tag{16}$$

and the $m \times m$ matrix

$$\begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \ldots & L_{g_m} L_f^{r_1-1} \mathrm{h}_1(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x) & \ldots & L_{g_m} L_f^{r_m-1} \mathrm{h}_\mathrm{m}(x) \end{bmatrix} \tag{17}$$

is non-singular for all $x \in D_0$, where $L_f h(x) \triangleq h'(x)f(x)$ denotes the derivative of $h(\cdot)$

along $f(\cdot)$. In particular, the system (12) and (15) has relative degree $\{1, \ldots, 1\}$ if the matrix

$$L_g h(x) \triangleq \begin{bmatrix} L_{g_1} \mathrm{h}_m(x) & \ldots & L_{g_m} h_1(x) \\ \vdots & \vdots & \vdots \\ L_{g_1} \mathrm{h}_m(x) & \ldots & L_{g_m} h_m(x) \end{bmatrix} \tag{18}$$

is non-singular for all $x \in D_0$. Finally, a smooth vector field f defined on a manifold $\mathcal{M} \subset \mathbb{R}^n$ is complete if the flow of f is defined on the entire Cartesian product $\mathbb{R} \times \mathcal{M}$. The following lemma is given in Byrnes and Isidori (1989).

Lemma 1. Assume that the system (12) and (15) is minimum phase with relative degree $\{1, \ldots, 1\}$. If the vector field $g(L_g h)^{-1}$ is complete, then there exists a diffeomorphism Φ: $\mathbb{R}^n \mapsto \mathbb{R}^n$, a C^∞ function f_0: $\mathbb{R}^{n-m} \mapsto \mathbb{R}^{n-m}$, and a C^∞ function r: $\mathbb{R}^{n-m} \times \mathbb{R}^m \mapsto \mathbb{R}^{(n-m)\times m}$ such that, by the change of coordinates

$$\begin{bmatrix} y \\ z \end{bmatrix} \triangleq \Phi(x) \tag{19}$$

the differential equation (12) can be rewritten in the normal form

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} L_f h(x) \\ f_0(z) + r(z, y) y \end{bmatrix} + \begin{bmatrix} L_g h(x) \\ 0 \end{bmatrix} u \tag{20}$$

Remark 1. In (20) with y as the output, the zero dynamics is therefore the system $\dot{z} = f_0(z)$, which is asymptotically stable at the origin due to the minimum phase assumption.

Theorem 1. Assume that the nonlinear system (12) and (15) is minimum phase with relative degree $\{1, \ldots, 1\}$, and assume that the vector field $g(L_g h)^{-1}$ is complete so that equations (19) and (20) hold. Furthermore, let V_0: $\mathbb{R}^{n-m} \mapsto \mathbb{R}$ be a Lyapunov function for $\dot{z} = f_0(z)$, that is, V_0: $\mathbb{R}^{n-m} \mapsto \mathbb{R}$ is positive definite such that $L_{f_0} V_0(z)$ is negative definite, and let $P \in \mathbb{R}^{m \times m}$ *and* $R \in \mathbb{R}^{m\times m}$ be positive-definite matrices. Define

$$L_2^{\mathrm{T}}(x) \triangleq R[L_g h(x)]^{-1}[P^{-1} r^{\mathrm{T}}(z, y) V_0'^{\mathrm{T}}(z) + 2L_f h(x)] \tag{21}$$

$$V(y, z) \triangleq V_0(z) + y^{\mathrm{T}} P y \tag{22}$$

Then the control law

$$\phi(x) \triangleq -\frac{1}{2}[L_g h(x)]^{-1}[P^{-1} r^{\mathrm{T}}(z, y)(V_0'(z))^{\mathrm{T}} + 2L_f h(x)] - R^{-1}[L_g h(x)]^{\mathrm{T}} P h(x) \tag{23}$$

asymptotically stabilizes (12) and minimizes $\mathcal{J}(x_0, u(\cdot))$ in the sense that there exists a neighborhood $D_0 \subset \mathbb{R}^n$ of the origin such that

$$\mathcal{J}(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} \mathcal{J}(x_0, u(\cdot)) = V(x_0) \tag{24}$$

for all $x_0 \in D_0$, where $\mathcal{J}(x_0, u(\cdot))$ is defined as in (13) and (14) with

$$L_1(x) \triangleq \phi^{\mathrm{T}}(x) R \phi(x) - L_f V(y, z) \tag{25}$$

and $\mathcal{S}(x_0)$ is the set of controls $u = \phi(x)$ such that the solution of the closed-loop system (12) and (23) with $x(0) = x_0$ satisfies $\lim x(t) = 0$ as $t \to \infty$. Furthermore, $V(y, z)$ is a Lyapunov function for the resulting closed-loop system. If, in addition,

$L_g h(x)$ is non-singular for all $x \in \mathbb{R}^n$ and the diffeomorphism (19) is global, and if $V_0(z)$ is radially unbounded, then the control law (23) globally asymptotically stabilizes (12).

The proof of Theorem 1 can be found in Wan and Bernstein (1995). We can specialize Theorem 1 by considering the case in which $V_0(z)$ is positive definite, but $L_{f_0}V_0(z)$ is negative semi-definite. Then, it can be shown similarly as in Wan and Bernstein (1995), that $\dot{V}(y, z) = 0$ if and only if $y = 0$ and $L_{f_0}V_0 = 0$. Thus, by applying the invariant set theorem, we have the following corollary.

Corollary 1. Assume that the nonlinear system (12) and (15) has relative degree $\{1, \ldots, 1\}$, and that there exists a diffeomorphism (19) so that (12) has the normal form (20). Furthermore, let V_0: $\mathbb{R}^{n-m} \mapsto \mathbb{R}$ be a positive-definite function such that $L_{f_0}V_0(z)$ is negative semi-definite, and let $V(y, z)$ be defined in (22). Then there exists a neighborhood $\mathcal{N}_0$ of the origin such that every solution of the closed-loop system obtained with the control law (23) and originating in $\mathcal{N}_0$ asymptotically approaches the set $\{(y, z) \in \mathcal{N}_0\colon\ y = 0,\ L_{f_0}V_0(z) = 0\}$. Furthermore, if $L_g h(x)$ is non-singular for all $x \in \mathbb{R}^n$ and the diffeomorphism (19) is global, and if $V(z)$ is radially unbounded, then the convergence is global, that is, $\mathcal{N}_0 = \mathbb{R}^n$.

Next we consider the case in which $V_0(\cdot)$ is only positive semi-definite and that $L_{f_0} V_0(\cdot)$ is only negative semi-definite. Under these weaker conditions, the following results show that the control law (23) asymptotically stabilizes the system (12) and (15) with respect to a subset of the state variables. The need to consider partial-state stability arises from the fact seen in Section 2.1 that the sleeping motion lies in the subspace $\{v_1 = v_2 = \omega_1 = \omega_2 = 0,\ \omega_3 \in \mathbb{R}\}$ instead of the origin. Hence, bringing the top to sleep requires that the four states ω_1, ω_2, v_1 and v_2 approach 0, while ω_3 approaches a constant value Ω. Related results can be found in Peiffer and Rouche (1969) and Rumjancev (1970). In the particular case of Lagrange's top, ω_3 is a constant of motion and can thus be omitted as a state variable, that is, the sleeping motion corresponds to the origin of a four-dimensional state space (see Wan *et al.*, 1994a).

Definition 1. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_2(x_1, x_2) \\ \\ f_2(x_1, x_2) \end{bmatrix}, \qquad x_1(0) = x_{01}, \qquad x_2(0) = x_{02} \tag{26}$$

where $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{n-p}$, $p \leq n$, and f_1 and f_2 are sufficiently smooth so that (26) has a unique solution for all $(x_{01}, x_{02}) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$. Assume that $f_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n-p}$. The system (26) is Lyapunov stable with respect to x_1 if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x_{01}\| < \delta$ implies $\|x_1(t)\| < \varepsilon$ for all $t \geq 0$. The system (26) is asymptotically stable with respect to x_1 if it is stable with respect to x_1 and if there exists $\delta > 0$ such that $\|x_{01}\| < \delta$ implies $x_1(t) \to 0$ as $t \to \infty$. Moreover, (26) is globally asymptotically stable with respect to x_1 if the latter holds with $\delta = \infty$.

Lemma 2. If there exists a C^1 positive-definite function V: $\mathbb{R}^p \mapsto \mathbb{R}$ such that $V'(x_1)f_1(x_1, x_2) \leq 0$, $(x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p}$, then the system (26) is stable with respect to x_1. If, in addition, there exists a continuous, strictly increasing function W: $(0, +\infty) \mapsto (0, +\infty)$, with $W(0) = 0$, such that

$$V'(x_1)f_1(x_1, x_2) \leq -W(\|x_1\|), \qquad (x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^{n-p} \tag{27}$$

then (26) is asymptotically stable with respect to x_1. If, furthermore, $V(\cdot)$ is radially unbounded, then the system (26) is globally asymptotically stable with respect to x_1.

Proof. Let $\varepsilon > 0$, and define $\mathscr{B}_\varepsilon \triangleq \{x_1 \in \mathbb{R}^p\colon \ \|x_1\| \leq \varepsilon\}$. Next, let $\beta \in \left(0, \min_{\|x_1\|=\varepsilon} V(x_1)\right)$, and define $\Omega_\beta \triangleq \{x_1 \in \mathscr{B}_\varepsilon\colon V(x_1) \leq \beta\}$. Since, along the trajectories of (26), $\dot{V}(x_1(t)) = V'(x_1)f_1(x_1, x_2) \leq 0$, it follows that $V(x_1(t))$ is a non-increasing function of time, and hence $\Omega_\beta \times \mathbb{R}^{n-p}$ is a positive-invariant set of (26). Since $V(\cdot)$ is continuous and $V(0) = 0$, there exists $\delta > 0$ such that $\mathscr{B}_\delta \subset \Omega_\beta$. Therefore, for all $(x_{01}, x_{02}) \in \mathscr{B}_\delta \times \mathbb{R}^{n-p}$, it follows that $(x_1(t), x_2(t)) \in \Omega_\beta \times \mathbb{R}^{n-p} \subset \mathscr{B}_\varepsilon \times \mathbb{R}^{n-p}$ for all $t \geq 0$, which proves that (26) is Lyapunov stable with respect to x_1.

Now, assume in addition that there exists a continuous, strictly increasing function $W\colon (0, +\infty) \mapsto (0, +\infty)$, with $W(0) = 0$, such that (27) holds. To prove asymptotic stability, we need to show that if $\|x_{01}\| \in \mathscr{B}_\delta$, then $x_1(t)$ tends towards 0. Since $V(x_1(t))$ is non-increasing in time, and is lower-bounded by zero, it admits a limit $c \geq 0$. If $c > 0$, then $\Omega_c \triangleq \{x_1 \in \mathscr{B}_\varepsilon\colon V(x_1) \leq c\}$ is non-empty and $V(x_1(t)) \geq c$ for all $t \geq 0$, and thus, $x_1(t)$ never enters Ω_c. Since $V(\cdot)$ ia continuous and $V(0) = 0$, there exists $d > 0$ such that $\mathscr{B}_d \subset \Omega_c$ and $W(d) > 0$. It then follows that for all $x_{01} \in \mathscr{B}_\delta$,

$$\begin{aligned} V(x_1(t)) &\leq V(x_{01}) + \int_0^t V'(x_1(v))f_1(x_1(v), x_2(v))\mathrm{d}v \\ &\leq \beta + \int_0^t W(\|x_1(v)\|)\mathrm{d}v \\ &< \beta - W(d)t \end{aligned}$$

which eventually becomes negative and contradicts the positive definiteness of $V(\cdot)$. Hence, $c = 0$, which proves that $V(x_1(t)) \to 0$ as $t \to 0$ for all $x_{01} \in \mathscr{B}_\delta$. Now, by continuity of $V(\cdot)$, and the earlier established fact that all $x_1(t)$ starting in $\mathscr{B}_\delta$ remains in the compact set Ω_β, it follows that $x_1(t) \to 0$ as $t \to \infty$, that is, (26) is asymptotically stable with respect to x_1.

Finally, assume that $V(\cdot)$ is radially unbounded. Let $x_{01} \in \mathbb{R}^p$, and define $b \triangleq V(x_{01})$ and $\Omega_b \triangleq \{x_1 \in \mathbb{R}^p\colon V(x_1) \leq b\}$. Then, radial unboundedness implies that there exists $r > 0$ such that $\Omega_b \subset \mathscr{B}_r$, and hence that Ω_b is compact. Since the solution $x_1(t)$ starting at x_{01} remains in Ω_b, we can reiterate the earlier argument to prove that for all $x_{01} \in \mathbb{R}^p$, $V(x_1(t)) \to 0$ as $t \to 0$. This completes the proof that (26) is globally asymptotically stable with respect to x_1.

Remark 2. By setting $p = n$, Lemma 2 specializes to the case of the autonomous system $\dot{x}_1 = f_1(x_1)$. In this case, (27) is equivalent to the assumption that $V'(x_1)f_1(x_1)$ is negative definite (see Vidyasagar, 1993, p. 149), so that Lemma 2 yields at the standard Layapunov stability theorem. There is a slight difference between Definition 1 and Lemma 2, and the definitions and stability theorems of partial stability given in Peiffer and Rouche (1969) and Rumjancev (1970), where V may be a function of both x_1 and x_2, positive definite and decrescent in x_1. For such a Lyapunov function candidate, the results of Peiffer and Rouche (1969) and Rumjancev (1970) requires that both x_{01} and x_{02} lie in a neighborhood of the origin, whereas in Lemma 2 x_{02} is arbitrary.

We now consider the problem of partial-state stabilization.

Definition 2. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix} u \tag{28}$$

The feedback control law $u \triangleq \phi(x_1, x_2)$ is asymptotically stabilizing with respect to x_1 if the resulting closed-loop system is asymptotically stable with respect to x_1. Furthermore, the feedback control law $u \triangleq \phi(x_1, x_2)$ is globally asymptotically stabilizing with respect to x_1 if the closed-loop system is globally asymptotically stable with respect to x_1.

The following result is a generalization of Theorem 1 to the case in which stabilization with respect to a subset of the state variables is desired and where the system is not assumed to be minimum phase.

Theorem 2. Assume that the system (12) and (15) has relative degree $\{1, \ldots, 1\}$, and assume that the diffeomorphism (19) exists so that (20) holds. Let $R \in \mathbb{R}^{m \times m}$ be a positive-definite matrix, and assume that there exists $\lambda_0 > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\lambda_{\min}(L_g h(x) R^{-1} L_g h(x)^{\mathrm{T}}) \geq \lambda_0 \tag{29}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. Furthermore, partition the partial state-vector $z \in \mathbb{R}^{n-m}$ as

$$z \triangleq \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad z_1 \in \mathbb{R}^p, \qquad z_2 \in \mathbb{R}^{n-m-p}$$

where $0 < p < n - m$, and its differential equation in the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f_{01}(z_1, z_2) + r_1(z_1, z_2, y)y \\ f_{02}(z_1, z_2) + r_2(z_1, z_2, y)y \end{bmatrix} \tag{30}$$

Assume moreover that there exists a C^1 positive-definite function $V_0 : \mathbb{R}^p \mapsto \mathbb{R}$, and a continuous, strictly increasing function $W_0 : (0, +\infty) \mapsto (0, +\infty)$, with $W_0(0) = 0$, such that

$$V_0'(z_1) f_{01}(z_1, z_2) \leq -W_0(\|z_1\|), \qquad (z_1, z_2) \in \mathbb{R}^p \times \mathbb{R}^{n-m-p} \tag{31}$$

Then the control law (23) is asymptotically stabilizing with respect to (y, z_1) with $V(y, z_1) \triangleq V_0(z_1) + y^{\mathrm{T}} P y$ as the Lyapunov function with respect to (y, z_1). Moreover, if $L_g h(x)$ is non-singular for all $x \in \mathbb{R}^n$, the diffeomorphism (19) is global, and $V_0(\cdot)$ is radially unbounded, then (23) is globally asymptotically stabilizing with respect to (y, z_1).

Proof. Using (30), the closed-loop system consisting of the system (20) and the control law (23) can be written in the form

$$\begin{bmatrix} \dot{y} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} P^{-1} r^{\mathrm{T}}(z, y) [V_0'(z)]^{\mathrm{T}} - R^{-1} [L_g h(x)]^{\mathrm{T}} P h(x) \\ f_{01}(z_1, z_2) + r_1(z_1, z_2, y) y \\ f_{02}(z_1, z_2) + r_2(z_1, z_2, y) y \end{bmatrix} \tag{32}$$

Let the subsystem of (32) which comprises the variables y and z_1 be denoted by

$$\begin{bmatrix} \dot{y} \\ \dot{z}_1 \end{bmatrix} = f_1(y, z_1, z_2)$$

and let V: $\mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$ be the Lyapunov function candidate

$$V(y, z_1) = V_0(z_1) + y^{\mathrm{T}} P y$$

where $V_0'(z_1) f_{01}(z_1, z_2)$ satisfies (31), and $P \in \mathbb{R}^{m \times m}$ is a symmetric, positive-definite matrix. It is clear then that V is C^1 and positive definite. From (29), (31) and (32) it follows that

$$\begin{aligned} V'(y, z_1) f_1(y, z_1, z_2) &= V_0'(z_1)\dot{z}_1 + 2y^{\mathrm{T}} P \dot{y} \\ &= V_0'(z_1) f_{01}(z_1, z_2) - 2y^{\mathrm{T}} P [L_g h(x)] R^{-1} [L_g h(x)]^{\mathrm{T}} P y \\ &\leq -W_0(\|z_1\|) - 2\lambda_0 y^{\mathrm{T}} P^2 y \\ &\leq -W_0(\|z_1\|) - 2\lambda_0 \lambda_{\min}^2(P) \|y\|^2 \end{aligned}$$

with $\lambda_{\min}(P) > 0$ since P is positive definite. Let W:$(0, +\infty) \mapsto (0, +\infty)$ be defined by

$$W(r) \triangleq \min_{\|(y, z_1)\| = r} (W_0(\|z_1\|) + 2\lambda_0 \lambda_{\min}^2(P) \|y\|^2)$$

where $(y, z_1) \in \mathbb{R}^m \times \mathbb{R}^p$. It follows from Lemma A4 of the appendix that W is a continuous, strictly increasing function satisfying $W(0) = 0$. It then follows that $V'(y, z_1) f_1(y, z_1, z_2) \leq -W(\|(y, z_1)\|)$, $(y, z_1) \in \mathbb{R}^m \times \mathbb{R}^p$, that is, V satisfies all the conditions of Lemma 2. Therefore, we conclude that the control law (23) is asymptotically stabilizing with respect to (y, z_1).

Remark 3. In Theorem 2, the system (12) and (15) is not assumed to be minimum phase. More precisely, the state z_2 in the zero dynamics may not be asymptotically stable. Our goal, however, is not to stabilize z_2 but rather to achieve asymptotic stability with respect to the remaining state variables, namely y and z_1. Hence, in Theorem 2, we bypass the use of Lemma 1 by assuming that a diffeomorphism exists, and instead of minimum phase, we assume that there exists a Lyapunov function with respect to z_1.

4 Global partial-state stabilization of the spinning top

Recall that equations (4) and (8) describe the motion of a torque-controlled spinning top. For both of the actuation schemes discussed in Section 2.2, (4) and (8) can be rewritten in the form of (12) with $x \triangleq (\omega_1, \omega_2, \omega_3, v_1, v_2) \in \mathbb{R}^5$, $u \triangleq (u_1, u_2) \in \mathbb{R}^2$ and

$$f(x) \triangleq \begin{bmatrix} -\mathcal{J}^{-1}(\omega \times \mathcal{J}\omega) + \mathcal{J}^{-1}\left(\dfrac{mg}{\|v\|} v \times l\right) \\ -\omega_2(1 + v_1^2) + \omega_1 v_1 v_2 + \omega_3 v_2 \\ \omega_1(1 + v_2^2) - \omega_2 v_1 v_2 - \omega_3 v_1 \end{bmatrix}, \qquad g(x) \triangleq \begin{bmatrix} \mathcal{J}^{-1} b_i \\ 0_{2 \times 2} \end{bmatrix}, \qquad i \in \{1, 2\} \quad (33)$$

In the following subsections, we shall consider each actuation scheme separately.

4.1 Case 1: two body-fixed torque actuators

In this case, define the output

$$y = h(x) \triangleq \begin{bmatrix} \omega_1 + k_1 v_2 \\ \omega_2 - k_2 v_1 \end{bmatrix} \in \mathbb{R}^2 \quad (34)$$

where $k_1 > 0$ and $k_2 > 0$ are to be chosen. Next, let

$$z \triangleq \begin{bmatrix} v_1 \\ v_2 \\ \mathcal{J}_{13}\omega_1 + \mathcal{J}_{23}\omega_2 + \mathcal{J}_{33}\omega_3 \end{bmatrix} \in \mathbb{R}^3 \tag{35}$$

where $\mathcal{J}_{ij}$ denotes the (i, j)-element of the inertia matrix $\mathcal{J}$. Note that the state z_3 is the body component of the angular momentum along the k-axis, and that by (34) and (35) the sleeping motion is mapped on to the set

$$S \triangleq \{(y_1, y_2, z_1, z_2, z_3) \in \mathbb{R}^5 \colon y_1 = y_2 = z_1 = z_2 = 0, z_3 \in \mathbb{R}\}$$

Hence, finding a control law that globally asymptotically stabilizes the sleeping motion is equivalent to finding u in (20) that is globally asymptotically stabilizing with respect to (y_1, y_1, z_1, z_2). We then have the following proposition.

Proposition 2. Consider the system (12) and (15) where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and z are defined in (33), (34) and (35). Then, the system has relative degree $\{1, \ldots, 1\}$, there exist $\lambda_0 > 0$ and $R \in \mathbb{R}^{m \times m}$ such that $L_g h(x)$ satisfies (29), and the transformation Φ given by (19), (34) and (35) is a global diffeomorphism on $\mathbb{R}^5$ and transforms the system (12) and (15) into the form of (20). Furthermore, define V_0: $\mathbb{R}^2 \mapsto \mathbb{R}$ by

$$V_0(z_1, z_2) \triangleq \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 \tag{36}$$

Then the control law

$$\phi(x) = -R^{-1} L_g h^{\mathrm{T}} P \begin{bmatrix} \omega_1 + k_1 v_2 \\ \\ \omega_2 - k_2 v_1 \end{bmatrix} - L_g h^{-1} \left(\frac{\|v\|^2}{2} P^{-1} \begin{bmatrix} v_2 \\ \\ -v_1 \end{bmatrix} + L_f h(x) \right) \tag{37}$$

is globally asymetrically stabilizing with respect to (y_1, y_2, z_1, z_2).

Proof. From (34), it is clear that

$$h'(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & k_1 \\ & & & & \\ 0 & 1 & 0 & -k_2 & 0 \end{bmatrix} = \begin{bmatrix} b_1^{\mathrm{T}} & \begin{pmatrix} 0 & k_1 \\ & \\ -k_2 & 0 \end{pmatrix} \end{bmatrix} \tag{38}$$

where b_1 is the constant matrix given by (9). From (33) and (38), we have

$$L_g h(x) = h'(x) g(x) = b_1^{\mathrm{T}} \mathcal{J}^{-1} b_1 \tag{39}$$

Now, since $L_g h(x)$ is a constant, positive-definite matrix, the system has relative degree $\{1, \ldots, 1\}$, and it is easy to see that $L_g h(x)$ satisfies (29) for all positive-definite matrices $R \in \mathbb{R}^{m \times m}$. The transformation Φ with y and z defined in (34) and (35) is given by $\Phi(x) = Fx$, where

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & k_1 \\ 0 & 1 & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \mathcal{J}_{13} & \mathcal{J}_{23} & \mathcal{J}_{33} & 0 & 0 \end{bmatrix}$$

and clearly $\det(F) = \mathcal{J}_{33} \neq 0$. Hence, Φ is a global diffeomorphism. To see that Φ transforms (12) and (15) into the form of (20), we first note that z_1 and z_2 are simply v_1 and v_2. Moreover, since the input torque is perpendicular to the k-axis, it can be shown with (8) and (9) that

$$\dot{z}_3 = (\mathcal{J}_{11} - \mathcal{J}_{22})\omega_1\omega_2 + \mathcal{J}_{12}\omega_2^2 - \mathcal{J}\omega_1^2 + (\mathcal{J}_{13}\omega_2 - \mathcal{J}_{23}\omega_1)\omega_3 \tag{40}$$

Since the input u does not appear in (40) it can be seen that Φ transforms (12) and (15) into the form of (20). Using the transformation Φ and setting y_1 and y_2 to 0 in the derivatives of z, it follows that

$$f_0(z) = \begin{bmatrix} -k_2z_1(1+z_1^2) - k_1z_1z_2^2 + (z_3 + k_1\mathcal{J}_{13}z_2 - k_2\mathcal{J}_{23}z_1)z_2 \\ -k_1z_2(1+z_2^2) - k_2z_1^2z_2 - (z_3 + k_1\mathcal{J}_{13}z_2 - k_2\mathcal{J}_{23}z_1)z_1 \\ -k_1k_2(\mathcal{J}_{11} - \mathcal{J}_{22})z_1z_2 + k_2^2\mathcal{J}_{12}z_1^2 - k_1^2\mathcal{J}_{21}z_2^2 \\ \quad + \frac{1}{\mathcal{J}_{33}}(k_2\mathcal{J}_{13}z_1 + k_1\mathcal{J}_{23}z_2)(z_3 + k_1\mathcal{J}_{13}z_2 - k_2\mathcal{J}_{23}z_1) \end{bmatrix} \tag{41}$$

$$r(z, y) = \begin{bmatrix} z_1z_2 - \mathcal{J}_{13}z_2 & -(1+z_1^2) - \mathcal{J}_{23}z_2 \\ (1+z_2^2) + \mathcal{J}_{13}z_1 & -z_1z_2 + \mathcal{J}_{23}z_1 \\ r_{31}(z, y) & r_{32}(z, y) \end{bmatrix} \tag{42}$$

Then along the trajectories of $\dot{z} = f_0(z)$, we have

$$\dot{V}_0(z_1, z_2) = -(k_1z_2^2 + k_2z_1^2)(1 + z_1^2 + z_2^2) \tag{43}$$

for all $(z_1, z_2, z_3) \in \mathbb{R}^3$, which satisfies (31). With the above results, we can conclude by Theorem 2 that the control law (23) is globally asymptotically stabilizing with respect to (y_1, y_2, z_1, z_2).

We have thus obtained a family of control laws, parameterized by the positive real numbers k_1 and k_2 and the 2×2 positive-definite matrices P and R that globally asymptotically stabilize the sleeping motion of the top with a pair of body-fixed torque actuators. The result is 'global' in the sense that the sleeping motion has been stabilized for all initial motions of the top above the horizontal plane. Replacing $r(z, y)$ with (42), the control law (23) can be simplified in this case to the form of (37).

4.2 Case 2: torque confined to the inertial XY*-plane*

In this case, define

$$y = h(x) \triangleq \begin{bmatrix} \omega_1 + k_1v_2 - \omega_3v_1 \\ \\ \omega_2 - k_2v_1 - \omega_3v_2 \end{bmatrix} \in \mathbb{R}^2, \qquad z \triangleq \begin{bmatrix} v_1 \\ v_2 \\ v^{\mathrm{T}}\mathcal{J}\omega \end{bmatrix} \in \mathbb{R}^3 \tag{44}$$

Note that, as in Proposition 2, the sleeping motion is mapped on to the set $\mathcal{S}$. Note also that $z_3 = \frac{1}{\gamma_3}H_0$. Since H_0 is a constant of motion, as we have seen in Section 2.2, it follows that

$$\dot{z}_3 = -\frac{1}{\gamma_3^2}H_0\dot{\gamma}_3 = z_3(\omega_1v_2 - \omega_2v_1) \tag{45}$$

that is, z_3 is an uncontrolled state. We now have the following proposition.

Proposition 3. Consider the system (12) and (15) where $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ and z are defined in (33) and (44). Then, the system has relative degree $\{1, \ldots, 1\}$, and the transformation Φ given by (19) and (44) is a global diffeomorphism on $\mathbb{R}^5$ and transforms the system (12) and (15) into the form of (20), let V_0: $\mathbb{R}^3 \mapsto \mathbb{R}$ be the positive-definite function

$$V_0(z) \triangleq \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} z_3^2 \tag{46}$$

Then the control law

$$\phi(x) = -R^{-1}L_g h^{\mathrm{T}}(x) P \begin{bmatrix} \omega_1 + k_1 v_2 - \omega_3 v_1 \\ \\ \omega_2 - k_2 v_1 - \omega_3 v_1 \end{bmatrix} - L_g h^{-1}(x) \left(\|v\|^2 P^{-1} \begin{bmatrix} v_2 \\ \\ -v_1 \end{bmatrix} + L_f h(x) \right) \tag{47}$$

is globally asymptotically stabilizing with respect to (y_1, y_2, z_1, z_2).

Proof. From (44), we have

$$h'(x) = \begin{bmatrix} 1 & 0 & -v_1 & -\omega_3 & k_1 \\ \\ 0 & 1 & -v_2 & -k_2 & -\omega_3 \end{bmatrix} = \begin{bmatrix} b_2(v)^{\mathrm{T}} & \begin{pmatrix} -\omega_3 & k_1 \\ \\ -k_2 & -\omega_3 \end{pmatrix} \end{bmatrix} \tag{48}$$

where $b_2(v)$ is given by (11). Hence, we have

$$L_g h(x) = b_2^{\mathrm{T}}(v) \mathcal{J}^{-1} b_2(v) \tag{49}$$

which is positive definite for all $x \in \mathbb{R}^n$. This proves that the system has relative degree $\{1, \ldots, 1\}$. The transformation Φ with y and z defined in (44) has the Jacobian

$$\Phi'(x) = \begin{bmatrix} 1 & 0 & -v_1 & -\omega_3 & k_1 \\ 0 & 1 & -v_2 & -v_2 & -\omega_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ & & v^{\mathrm{T}}\mathcal{J} & \alpha_1(x) & \alpha_2(x) \end{bmatrix} \tag{50}$$

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are functions of x. By row combinations of (50), it can be shown that for all $x \in \mathbb{R}^5$,

$$\operatorname{rank}\left(\frac{\partial \Phi}{\partial x}(x)\right) = \operatorname{rank} \begin{bmatrix} 1 & 0 & -v_1 & -\omega_3 & k_1 \\ 0 & 1 & -v_2 & -v_2 & -\omega_3 \\ 0 & 0 & v^{\mathrm{T}}\mathcal{J}v & \beta_1(x) & \beta_2(x) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 5 \tag{51}$$

where $\beta_1(\cdot)$ and $\beta_2(\cdot)$ are functions of x. In (51), the second equality is obtained by noting that $\mathcal{J}$ is positive definite, which implies that the (3,3)-element of the second matrix is non-zero for all $(v_1, v_2) \in \mathbb{R}^2$. Equation (51) shows, by the inverse function theorem, that $\Phi(\cdot)$ is a global diffeomorphism on $\mathbb{R}^5$. Since $z_1 = v_1$, $z_2 = v_2$ and $\dot{z}_3$ is given by (45), it can be seen that $\Phi(\cdot)$ transforms the system (12) and (15) into the form of (20). $\Phi(\cdot)$ and setting y_1 and y_2 to 0 in the derivatives of z, it can be shown that

$$f_0(z) = \begin{bmatrix} -k_2 z_1(1 + z_1^2) - k_1 z_1 z_2^2 \\ -k_1 z_2(1 + z_2^2) - k_2 z_1^2 z_2 \\ -z_3(k_2 z_1^2 + k_1 z_2^2) \end{bmatrix} \tag{52}$$

$$r(z, y) = \begin{bmatrix} z_1 z_2 & -(1 + z_1^2) \\ (1 + z_2^2) & -z_1 z_2 \\ z_2 z_3 & -z_1 z_3 \end{bmatrix} \tag{53}$$

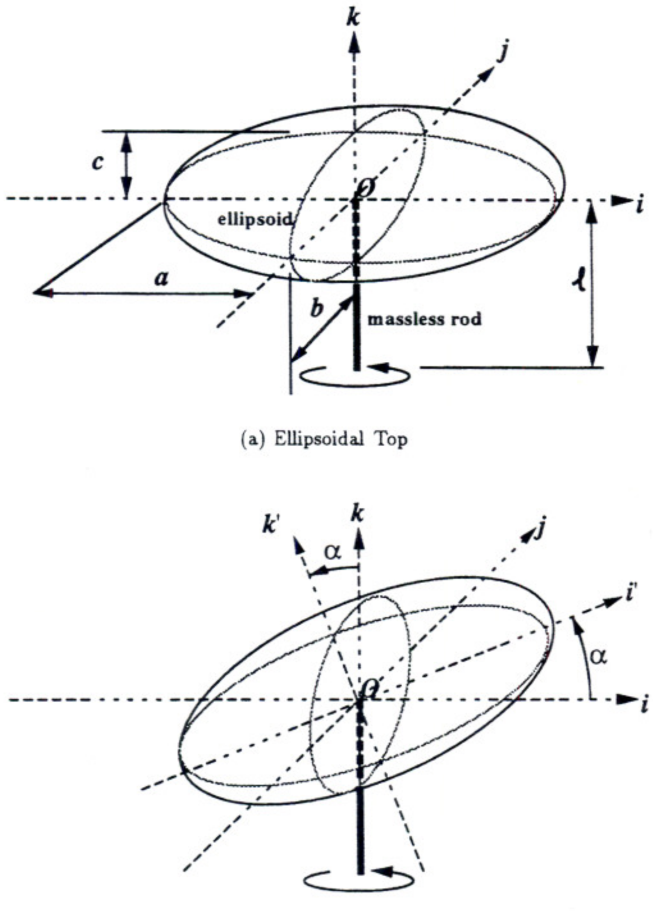


Fig. 2. Examples of asymmetric and unbalanced tops.

Then along the trajectories of $\dot{z} = f_0(z)$, we have

$$\dot{V}_0(z) = -(k_1 z_2^2 + k_2 z_1^2)(1 + z_1^2 + z_2^2 + z_3^2) \tag{54}$$

which is negative semi-definite, and is null on the set $\{z \in \mathbb{R}^3 : z_1 = z_2 = 0\}$. From Corollary 1, we hence conclude that all trajectories of the closed-loop system obtained with the control law (23) approach the set $\mathscr{S}$, that is, the control law (23) is globally asymptotically stabilizing with respect to (y_1, y_2, z_1, z_2).

Finally, substituting (53) into (23) and rescaling P yields the control law given by (47). Recall in this case that the input $u = \phi(x)$ is an element of $\mathbb{R}^2$, and is defined by (11) as the i- and j-components of the input torque τ. However, this case is different from Case 1 in that τ now has a k-component $\tau_3 = -u_1 v_1 - u_2 v_2$, and we have already seen that this will result in τ lying in the inertial XY-plane.

Remark 4. The steady-state control input for both actuation schemes can be obtained by setting $x = x_s \triangleq (0, 0, \Omega, 0, 0)$ in (37) and (47). Then, we have $\phi(x_s) = -L_g h^{-1}(x_s) L_f h(x_s)$. Now, it can be seen from (9) and (11) that $b_2(x_s) = b_1$ and hence, following (39) and (49), $L_g h(x_s) = b_1^{\mathrm{T}} \mathcal{J}^{-1} b_1$ for both actuation schemes.

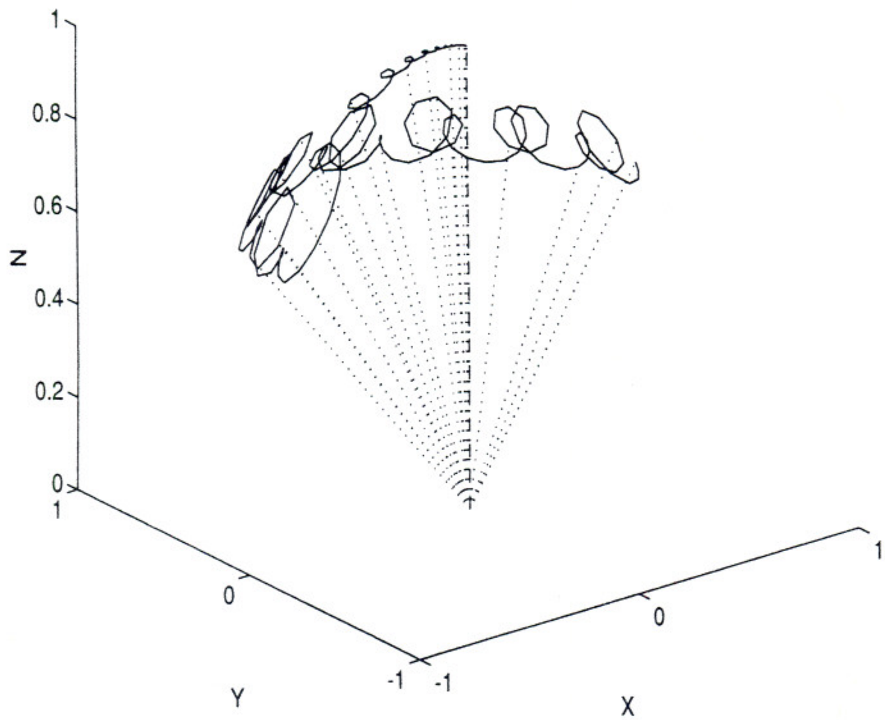


Fig. 3. Control of top using body torques: $k_1 = k_2 = 1$, $P = 0.1\ I_{2\times 2}$, $R = I_{2\times 2}$. Locus of center of mass (length dimensions are normalized by ℓ).

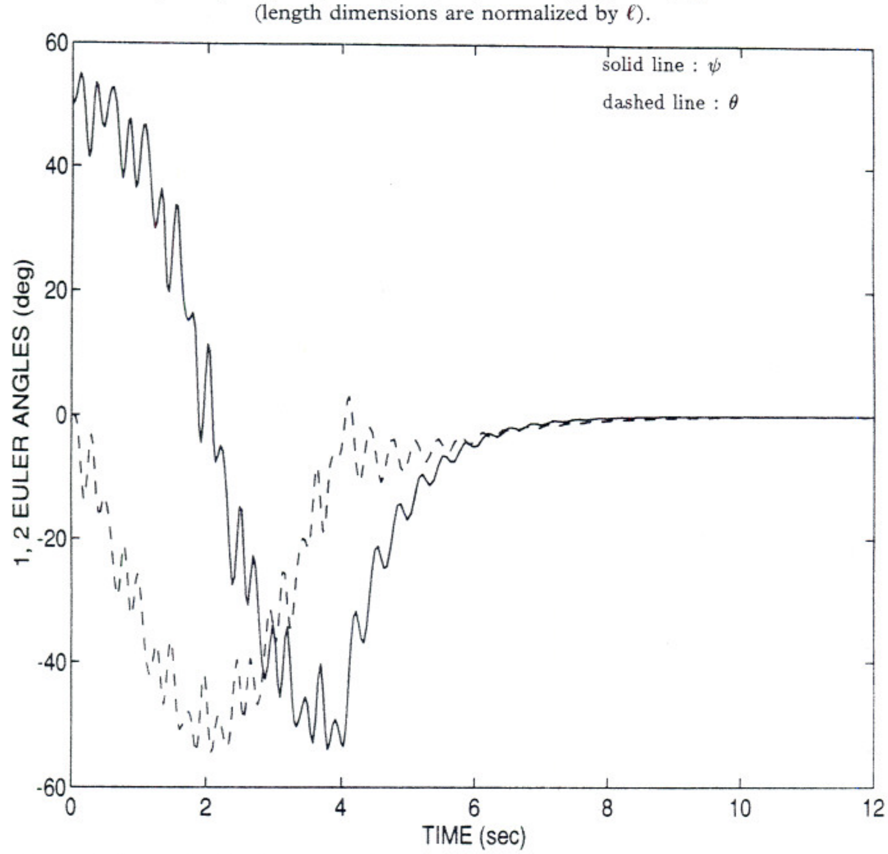


Fig. 4. Control of top using body torques: $k_1 = k_2 = 1$, $P = 0.1\ I_{2\times 2}$, $R = I_{2\times 2}$. Euler angles versus time.

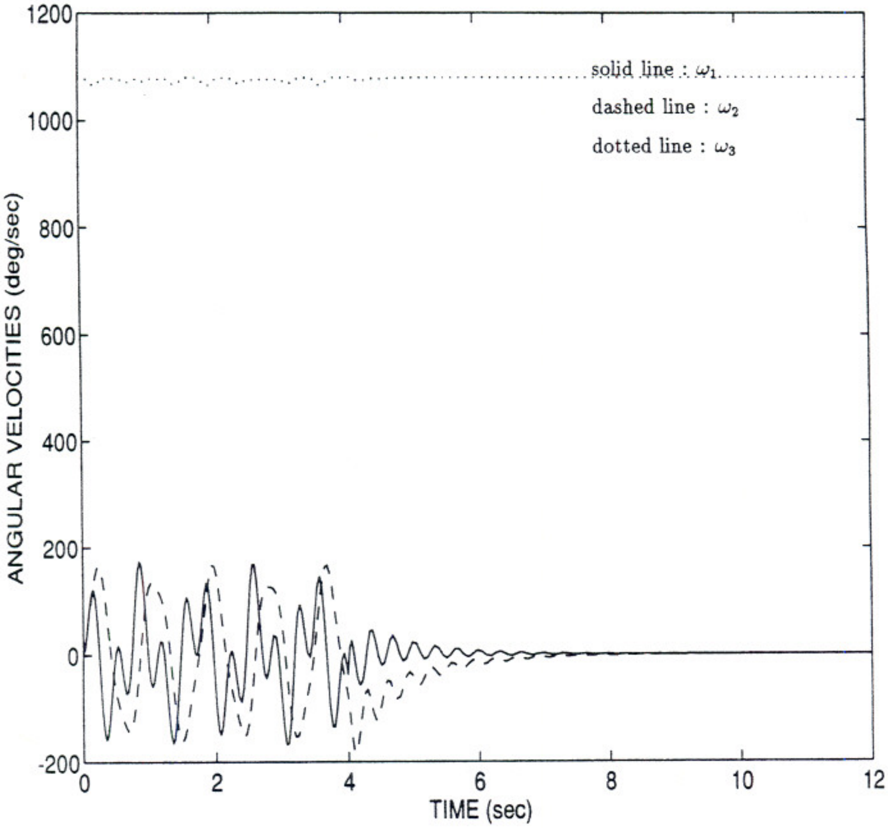


Fig. 5. Control of top using body torques: $k_1 = k_2 = 1$, $P = 0.1\, I_{2\times 2}$, $R = I_{2\times 2}$. Angular velocity versus time.

Moreover, using (33), (34) and (44), it can be verified that $L_f h(x_s) = -b_1^{\mathrm{T}} \mathcal{J}(\omega_s \times \mathcal{J}\omega_s)$ for both actuation schemes, where $\omega_s = (0, 0, \Omega)$ as introduced in Section 2.1. Hence, the steady-state control torques for both cases are identical and are given by

$$\tau(x_s) = b_1 \phi(x_s) = b_1 (b_1^{\mathrm{T}} \mathcal{J}^{-1} b_1)^{-1} b_1^{\mathrm{T}} \mathcal{J}^{-1} (\omega_s \times \mathcal{J}\omega_s) \tag{55}$$

Next, we observe that since $\omega_s \times \mathcal{J}\omega_s$ has a zero k-component. It then follows that $\omega_s \times \mathcal{J}\omega_s = b_1 b_1^{\mathrm{T}}(\omega_s \times \mathcal{J}\omega_s)$. Substituting this expression into (55) yields $\tau(x_s) = \tau_s$, where τ_s is the steady-state offset torque introduced in Section 2.2. Hence, the control torques resulting from both (37) and (47) converge to τ_s. This is not surprising since we have already seen in Section 2.2 that τ_s is the torque required to keep the top in the sleeping motion regardless of the actuation scheme.

5 Examples

Consider the top shown in Fig. 2(a), which consists of a uniform, flat ellipsoid mounted at its center of mass on a massless rod of length ℓ so that the i, j and k-axes are principal axes of inertia. In this case, we have a balanced, asymmetric top, and the inertia matrix $\mathcal{J}$ with respect to the pivot point O and resolved in body coordinates is diagonal with the diagonal terms

$$\mathcal{J}_{xx} = \tfrac{m}{5}(b^2 + c^2) + ml^2, \qquad \mathcal{J}_{yy} = \tfrac{m}{5}(c^2 + a^2) + ml^2, \qquad \mathcal{J}_{zz} = \tfrac{m}{5}(a^2 + b^2)$$

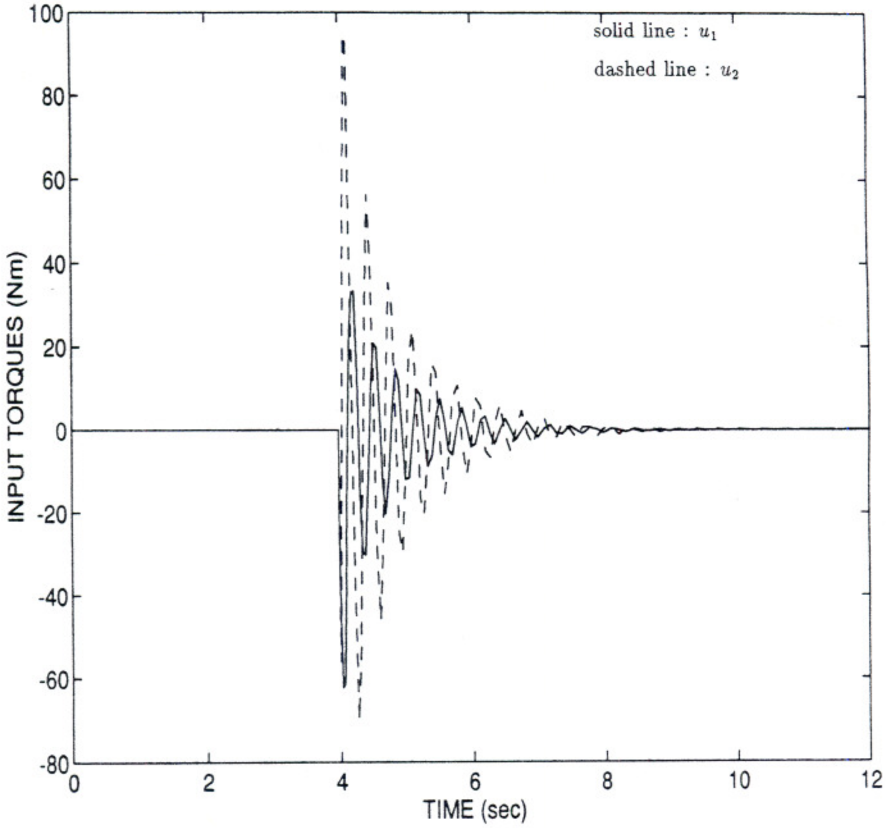


Fig. 6. Control of top using body torques: $k_1 = k_2 = 1$, $P = 0.1\ I_{2\times 2}$, $R = I_{2\times 2}$. Input body torques versus time.

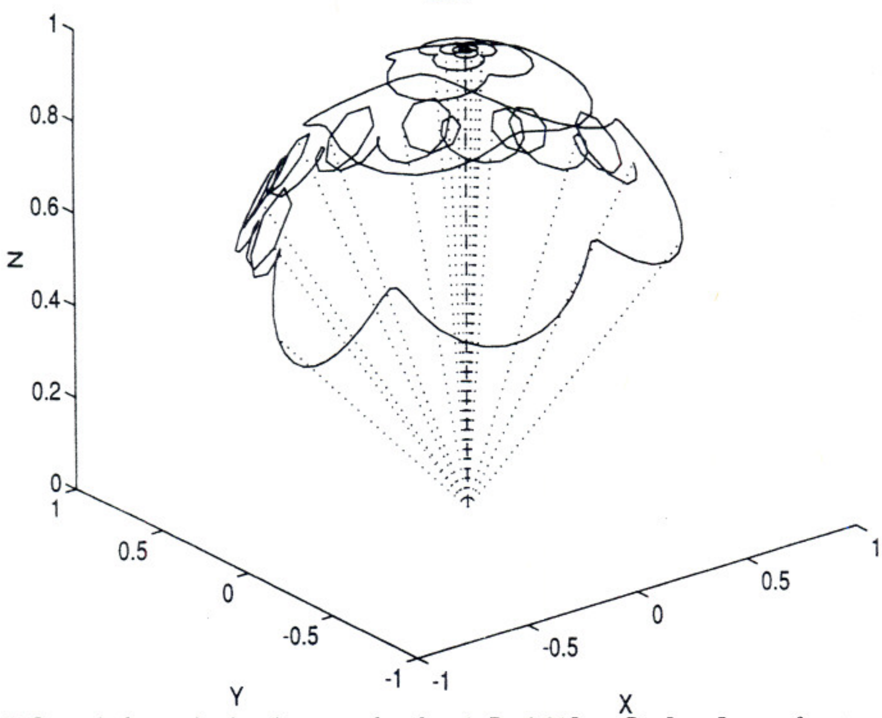


Fig. 7. Control of top using inertia torque: $k_1 = k_2 = 1$, $P = 0.01 I_{2\times 2}$, $R = I_{2\times 2}$. Locus of center of mass (length dimensions are normalized by ℓ).

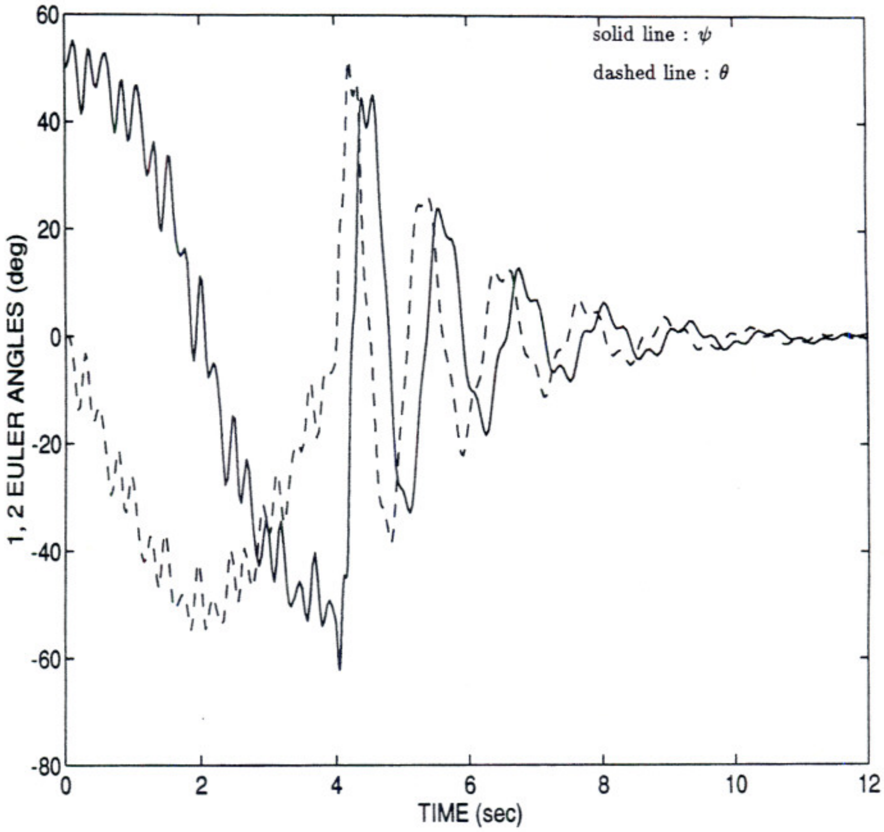


Fig. 8. Control of top using inertia torque: $k_1 = k_2 = 1$, $P = 0.01\ I_{2\times 2}$, $R = I_{2\times 2}$. Euler angles versus time.

For the purpose of illustration, let $m = 5$ kg, $a = 0.5$ m, $b = 0.3$ m, $c = 0.01$ m and $\ell = 0.1$ m, which give $\mathcal{J}_{xx} = 0.1401\ \mathrm{kg\,m^2}$, $\mathcal{J}_{yy} = 0.3001\ \mathrm{kg\,m^2}$, $\mathcal{J}_{zz} = 0.3400\ \mathrm{kg\,m^2}$. The top is initially released with a spin of 1080°/second (or 180 rpm) at a coning angle of 50°, and the control is enabled 4 seconds later.

Figure 3 shows the locus of the top's center of mass in the inertial frame when the body-fixed torque control law (37) is applied. As indicated by the dotted lines joining points on the locus and the origin, the uncontrolled top precesses as well as nutates. However, when the control is enabled, the precession and nutation cease and the top is brought to sleep. This is also apparent in Fig. 4, where, for illustration, the Euler angles ψ and θ as defined in Fig. 1(c) are plotted against time. Here, it can be seen that the motion converges to the sleeping motion in about 4 seconds after the control is enabled. Figure 5 shows the body components of the angular velocity, and Fig. 6 shows the input body torques.

Figures 7–10 show the results for the same top as above, but here the inertially fixed torque control law (47) is applied. In Fig. 7, the top is brought to sleep in quite a different manner than in Fig. 3, and a comparison of Figs 5 and 9 reveals that with inertial torques, the change in ω_3 is more significant wherein the top de-spins while going to sleep. In fact, as shown in Section 2.2, ω_3 converges to $\Omega = H_0/\mathcal{J}_{33}$. In Fig. 10, U_1 and U_2 are the X- and Y-components of the input torque in the inertial frame, computed using (11) and (47).

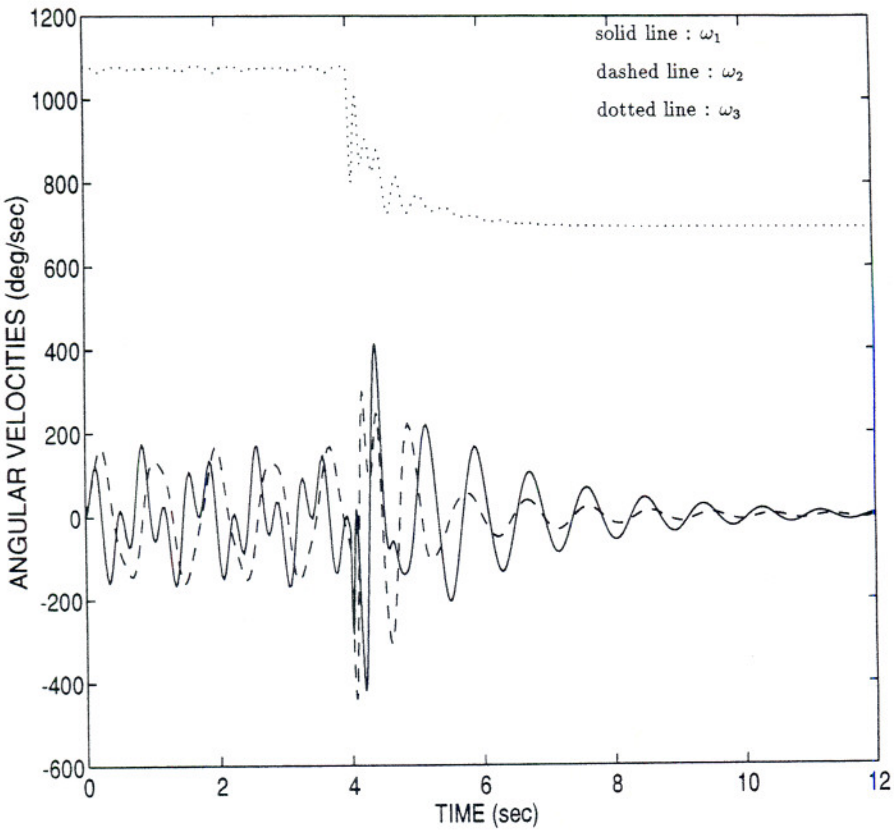


Fig. 9. Control of top using inertia torque: $k_1 = k_2 = 1$, $P = 0.01\ I_{2\times 2}$, $R = I_{2\times 2}$. Angular velocity versus time.

Next, consider the case in which the same ellipsoid is mounted on the rod at a skew angle α with respect to the i-axis. Consequently, the inertia matrix $\mathcal{J}$ becomes

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{xx}\cos^2\alpha + \mathcal{J}_{zz}\sin^2\alpha + ml^2 & 0 & (\mathcal{J}_{xx} - \mathcal{J}_{zz})\sin\alpha\ \cos\alpha \\ 0 & \mathcal{J}_{yy} + ml^2 & 0 \\ (\mathcal{J}_{xx} - \mathcal{J}_{zz})\sin\alpha\ \cos\alpha & 0 & \mathcal{J}_{xx}\sin^2\alpha + \mathcal{J}_{zz}\cos^2\alpha \end{bmatrix} \tag{56}$$

that is, an imbalance occurs as a result of the skewed top. In effect, (56) shows that for $\alpha \neq 0$ the top axis is not a principal axis of inertia. For $\alpha = 10°$,

$$\mathcal{J} = \begin{bmatrix} 0.1476 & 0 & -0.0427 \\ 0 & 0.3001 & 0 \\ -0.0427 & 0 & 0.3325 \end{bmatrix}$$

Figures 11–14 illustrate the results obtained with the body-fixed control law (37). Note in particular that in Fig. 14 the control torques do not go to zero as time progresses; instead, they approach constant offset values. This is because the

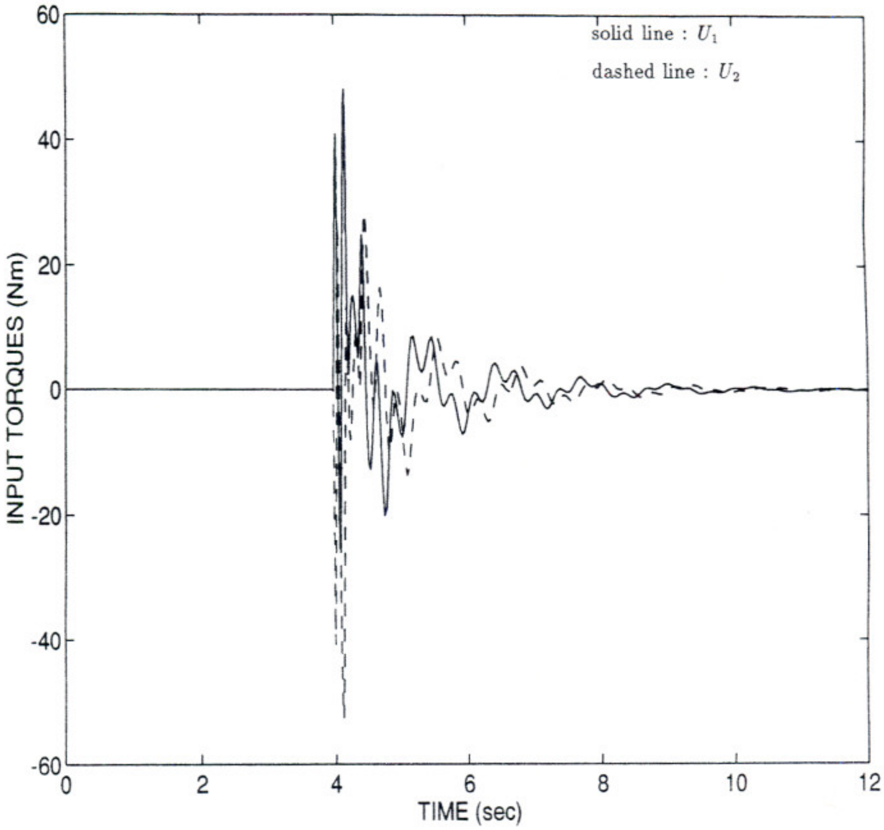


Fig. 10. Control of top using inertia torque: $k_1 = k_2 = 1$, $P = 0.01\ I_{2\times 2}$, $R = I_{2\times 2}$. Input inertia torque versus time.

sleeping motion, that is, spin about the top axis, is not a solution of the uncontrolled top; therefore, non-zero control effort is required to maintain the sleeping motion.

6 Conclusion

In this paper, we considered the stabilization of an unbalanced top to the sleeping motion. We saw that the sleeping motion is not a solution of the Euler–Poisson equations of motion of a top in general. However, we derived two families of control laws, (37) and (47), that globally asymptotically stabilize a top with known imbalance to the sleeping motion using torque actuators. In (37), the control torque is produced by two body-fixed torque actuators perpendicular to the top axis, whereas in (47), the control torque is confined to the inertial XY-plane. As we have seen earlier, the latter case is equivalent to having two torque actuators inertially fixed along the X- and Y-axes. The control-design strategy was based on Hamilton–Jacobi–Bellman theory with zero dynamics, and the result is global in the sense that the spinning top can be stabilized to the sleeping motion regardless of spin rate, and from an arbitrary initial motion having a coning angle of up to 90°. The behavior of the closed-loop systems were demonstrated in simulation.

The imbalance given in (56) is very particular and is only one of the many types

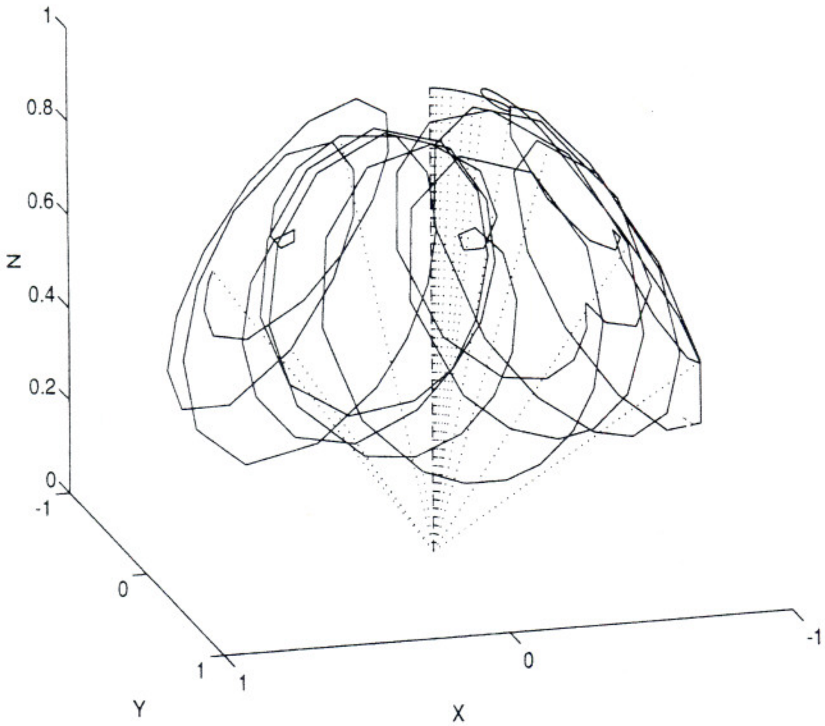


Fig. 11. Control of top with imbalance using body torques: $k_1 = k_2 = 1$, $P = I_{2\times 2}$, $R = I_{2\times 2}$. Locus of center of mass (length dimensions are normalized by ℓ).

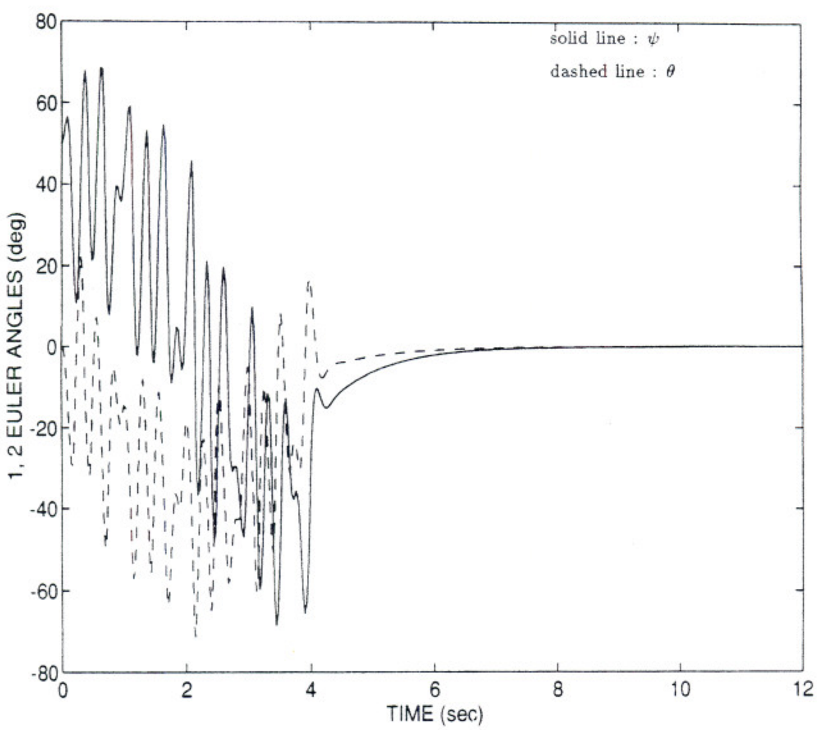


Fig. 12. Control of top with imbalance using body torques: $k_i = 1$, $P = 0.1\ I_{2\times 2}$, $R = I_{2\times 2}$. Euler angles versus time.

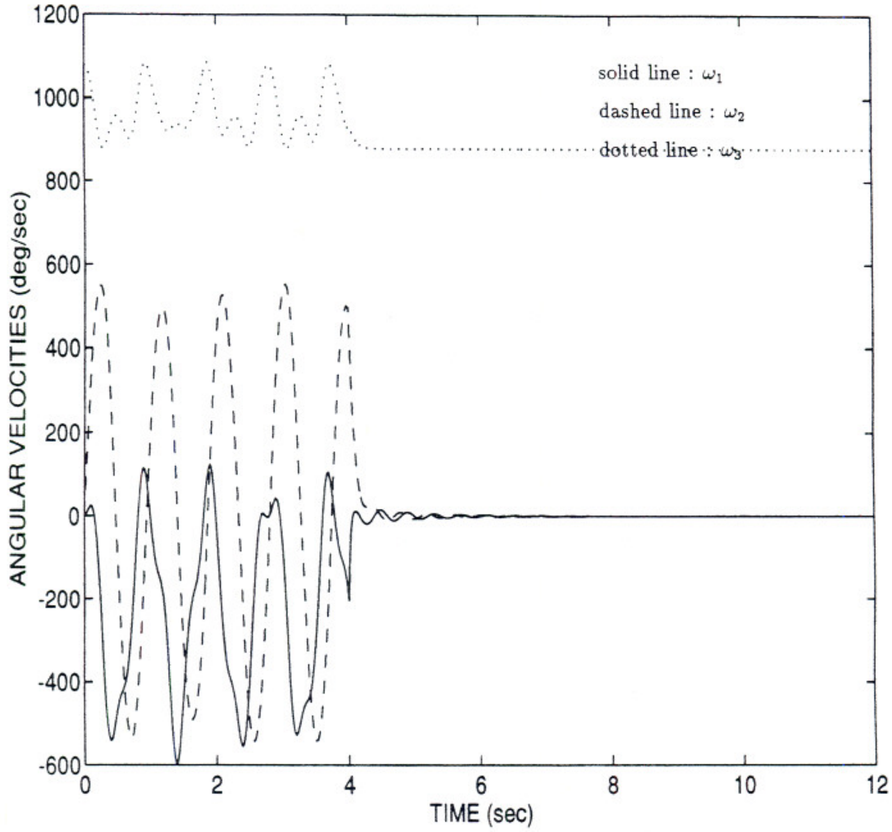


Fig. 13. Control of top with imbalance using body torques: $k_i = 1$, $P = 0.1\,I_{2\times 2}$, $R = I_{2\times 2}$. Angular velocity versus time.

of imbalance caused by manufacturing defects to the top. For instance, if the ellipsoid is tilted with respect to both the i- and j-axes, all of the off-diagonal terms of $\mathcal{J}$ will be non-zero. We can imagine other examples such as asymmetry due to actuators on non-principal axes of inertia, and imbalance due to an irregularly shaped top.

We note that the control laws obtained are feedback laws, where the feedback variables are the angular velocity vector ω measured in the body frame, and the projection vector v which locates the local vertical in body coordinates. In practical implementation, these variables can be measured using, for example, gyroscopes and accelerometers mounted on the top. Hence, the feedback control laws we obtained are physically realizable.

As with many non-linear control designs, the control laws derived in this paper rely on exact knowledge of the top model; in particular, knowledge of the inertia matrix is vital in order for the control laws to work. It has been verified in simulation that if the actual imbalance differs from the assumed imbalance model, then the control laws (37) and (47) will bring the top to some coning motion instead. The work presented in this paper is part of the ultimate objective of controlling rotating bodies with unknown imbalance.

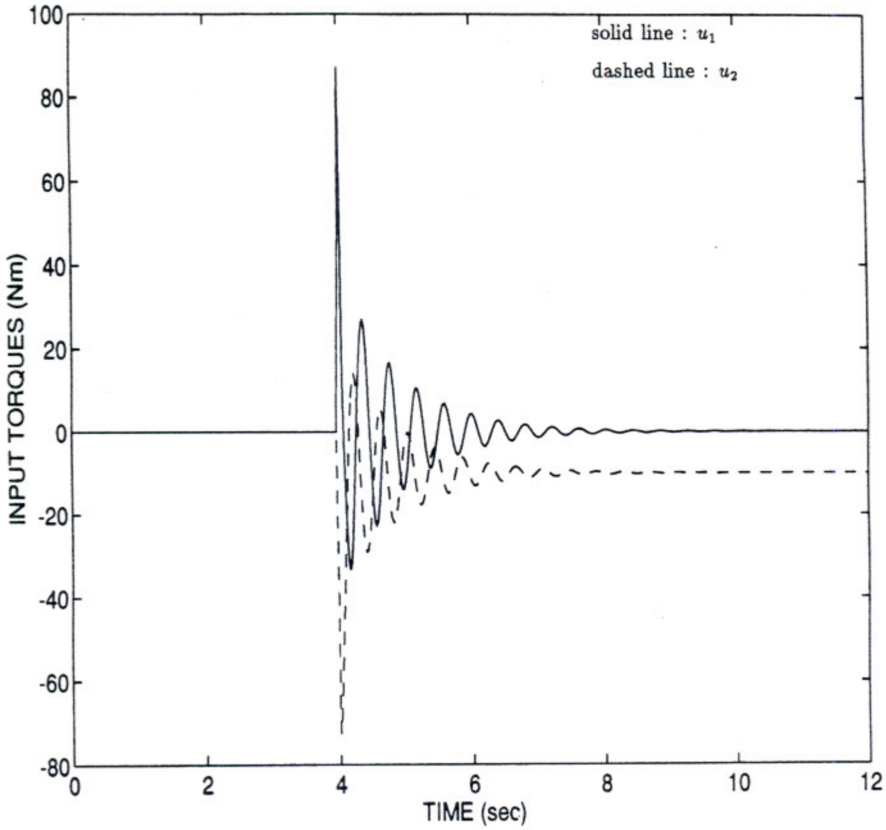


Fig. 14. Control of top with imbalance using body torques: $k_i = 1, P = 0.1\, I_{2\times 2}, R = I_{2\times 2}$. Input imbalance using body torques versus time.

Acknowledgements

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Appendix: complement to proof of Theorem 2

The following lemmas are used in the proof of Theorem 2.

Lemma A1. Let $W_1\colon \mathbb{R} \mapsto \mathbb{R}$ and $W_2\colon \mathbb{R} \mapsto \mathbb{R}$ be continuous, strictly increasing functions satisfying $W_1(0) = 0$ and $W_2(0) = 0$. Next, consider the partition $x = (x_1, x_2)$, where $x \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, so that $n_1 + n_2 = n$. Then, the function $\mathscr{W}\colon \mathbb{R}^n \mapsto \mathbb{R}$, $\mathscr{W}(x) \triangleq W_1(\|x_1\|) + W_2(\|x_2\|)$ is continuous on $\mathbb{R}^n$.

Proof. Let $x = (x_1, x_2) \in \mathbb{R}^n$, and $\varepsilon > 0$. Since for each $i \in \{1, 2\}$, W_i is continuous on $\mathbb{R}$, there exists $\delta_i > 0$ such that $|W_i(\|y_i\|) - W_i(\|x_i\|)| < \frac{1}{2}\varepsilon$ for all $y_i \in \mathbb{R}^{n_i}$, verifying $|\|y_i\| - \|x_i\|| < \delta_i$. Let $\delta \triangleq \min(\delta_1, \delta_2)$, and $y = (y_1, y_2) \in \mathbb{R}^n$ verifying $\|y - x\| < \delta$. Then, $|\|y_i\| - \|x_i\|| < \|y_i - x_i\| < \delta$, $i \in \{1, 2\}$, and it follows that

$$|\mathscr{W}(y) - \mathscr{W}(x)| \leq |W_1(\|y_1\|) - W_1(\|x_1\|)| + W_2(\|y_2\|) - W_2(\|x_2\|)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

This completes the proof that $\mathscr{W}$ is continuous on $\mathbb{R}^n$.

Lemma A2. Let $x \in \mathbb{R}^n$. Then, for all $\alpha < 1$, $\mathscr{W}(\alpha x) < \mathscr{W}(x)$.

Proof. Since W_1 and W_2 are strictly increasing, it follows that

$$\mathscr{W}(\alpha x) = W_1(\|\alpha x_1\|) + W_2(\|\alpha x_2\|) < W_1(\|x_1\|) + W_2(\|x_2\|) = \mathscr{W}(x)$$

which completes the proof for Lemma A2.

Since $\mathcal{W}$ as defined in Lemma A1 is continuous on $\mathbb{R}^n$, its minimum exists on any compact subset of $\mathbb{R}^n$. Hence, let the function W: $\mathbb{R}_+ \mapsto \mathbb{R}_+$ be defined by $W(r) \triangleq \min_{\|x\|=r} \mathcal{W}(x)$. Then, for all $r \in \mathbb{R}_+$, there exists $x_m \in \{\mathbb{R}^n: \|x\| = r\}$ such that $\mathcal{W}(x_m) = W(r)$.

Lemma A3. W: $\mathbb{R} \mapsto \mathbb{R}$ as defined above is a continuous, strictly increasing function verifying $W(0) = 0$.

Proof. The property $W(0) = 0$ is trivial. To show that W is strictly increasing, suppose that there exists $(r, r') \in \mathbb{R}^2_+$ such that $r > r' \geq 0$, and $W(r) \leq W(r')$. Let $x_m \in \{x \in \mathbb{R}^n: \|x\| = r\}$ be such that $\mathcal{W}(x_m) = W(r)$. Then,

$$\mathcal{W}(x_m) \leq W(r') = \min_{\|x\|=r'} \mathcal{W}(x) \leq \mathcal{W}\left(\frac{r'}{r} x_m\right)$$

where $r'/r < 1$, which contradicts Lemma A2. Hence, W is strictly increasing. Next, let $r_0 > 0$. Then, $\mathcal{W}$ is uniformly continuous on the compact set $\Omega \triangleq \{x \in \mathbb{R}^n: r_0/2 \leq \|x\| \leq 3r_0/2\}$. Let $\varepsilon < 0$. Then, there exists $\delta > 0$ such that $|\mathcal{W}(x) - \mathcal{W}(y)| < \varepsilon$ for all $(x, y) \in \Omega^2$ satisfying $\|x - y\| < \delta$. Now, let $(r_1, r_2) \in [r_0/2, 3r_0/2]$ such that $|r_1 - r_2| < \delta$, and assume without loss of generality that $r_1 < r_2$. Let $x \in \mathbb{R}^n, \|x\| = r_1$, be such that $\mathcal{W}(x) = W(r_1)$, and let $y = (r_2/r_1)x$. Then $(x, y) \in \Omega^2$, $\|x - y\| = r_2 - r_1 < \delta$, and it follows that

$$0 < W(r_2) - W(r_1) = \min_{\|z\|=r_2} \mathcal{W}(z) - \mathcal{W}(x) \leq \mathcal{W}(y) - \mathcal{W}(x) < \varepsilon$$

Therefore, W is uniformly continuous on $[r_0/2, 3r_0/2]$, and hence is continuous at r_0. Finally, continuity at 0 results from the fact that $0 \leq W(r) \leq W_1(r) + W_2(r) \to 0$ as $r \to 0$.

Finally, recognizing $W_2(r) = kr^2$, $k > 0$, as a particular continuous, strictly increasing function satisfying $W_2(0) = 0$, we thus have the following corollary which is required for the proof of Theorem 2.

Lemma A4. Let W_1: $\mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous, strictly increasing function such that $W_1(0) = 0$, and let $k > 0$. Then, W: $\mathbb{R}_+ \mapsto \mathbb{R}_+$ defined by $W(r) \triangleq \min_{\|x\|=r} (W_1(\|x_1\|) - k\|x_2\|^2)$ is also continuous, strictly increasing and verifies $W(0) = 0$.