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A behavioral equation framework for time-domain transmissibilities[☆]

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ABSTRACT

Transmissibilities are widely used in engineering to express the relationship between sets of signals that are not necessarily inputs and outputs in the usual causal sense. Despite their usefulness, a rigorous intellectual framework for transmissibilities has been lacking. In this paper we demonstrate that behavioral equations provide a suitable framework for time-domain transmissibilities by choosing the latent variables in the behavioral equations to be the external signals that drive the system. This connection provides a theoretical foundation for time-domain transmissibilities and demonstrates the relevance of behavioral modeling to an important class of applications.

1. Introduction

Unlike traditional input–output modeling techniques, behaviors can be used to interconnect models without assigning the attributes of "input" or "output" (Markovsky, Willems, van Huffel, & de Moor, 2008; Polderman & Willems, 1998; Willems, 2007). Just as behaviors do not distinguish between inputs and outputs, the same can be said for transmissibilities. Transmissibilities are widely used in engineering to express the relationship between sets of signals that are not necessarily inputs and outputs in the usual causal sense (Chesné & Deraemaeker, 2013; Devriendt & Guillaume, 2008; Gajdatsy, Janssens, Desmet, & Van Der Auweraer, 2010; Hrovat, 1997; Johnson & Adams, 2002; Maia, Silva, & Ribeiro, 2001; Urgueira, Almeida, & Maia, 2011; Weijtjens, De Sitter, Devriendt, & Guillaume, 2014; Zhang, Pintelon, & Schoukens, 2013). A transmissibility operator is thus not a transfer function, and it does not have a state space realization with physically meaningful states. Despite their usefulness, a rigorous intellectual framework for transmissibilities has been lacking. In practice, transmissibilities are usually constructed in the frequency domain. The focus of the present paper is on time-domain transmissibilities.

Behavioral equations involve manifest and latent variables, where the latent variables can be eliminated to obtain a behavioral equation that involves only the manifest variables. On the other hand, the derivation of a transmissibility is predicated on the elimination of an external driving signal so that the resulting model involves only response variables (Aljanaideh & Bernstein, 2015). These observations suggest that there may be a connection between behavioral equations and transmissibilities.

The goal of this note is to demonstrate that behavioral equations provide a suitable framework for transmissibilities by choosing the latent variables in the behavioral equations to be the external signals that drive the system. This is done by applying Theorem 6.2.6 of Polderman and Willems (1998), and obtaining an explicit expression for the polynomial matrix that eliminates the latent variables. The main result shows that the cancellation of an external input within the context of transmissibility operators corresponds to the elimination of a latent variable in the behavioral setting. We illustrate this relationship by deriving transmissibilities for a spring–mass system that was previously analyzed in terms of behaviors.

2. Transmissibility Operators

Consider the multi-input, multi-output system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$y(t) = Cx(t) + Du(t) \in \mathbb{R}^p, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$. Multiplying (3) by $\det(\mathbf{p}I_n - A)$, where $\mathbf{p} \overset{\triangle}{=} \mathrm{d}/\mathrm{d}t$, yields the differential

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equation

$$\begin{aligned}\det(\mathbf{p}I_n - A)y(t) &= C\det(\mathbf{p}I_n - A)I_n x(t) + D\det(\mathbf{p}I_n - A)u(t)\\ &= C\mathrm{adj}(\mathbf{p}I_n - A)(\mathbf{p}I_n - A)x(t) + D\det(\mathbf{p}I_n - A)u(t)\\ &= C\mathrm{adj}(\mathbf{p}I_n - A)(\dot{x}(t) - Ax(t)) + D\det(\mathbf{p}I_n - A)u(t)\\ &= [C\mathrm{adj}(\mathbf{p}I_n - A)B + D\det(\mathbf{p}I_n - A)]u(t).\end{aligned} \tag{4}$$

Defining

$$\delta(\mathbf{p}) \triangleq \det(\mathbf{p}I - A) \in \mathbb{R}[\mathbf{p}], \tag{5}$$

$$\Gamma(\mathbf{p}) \triangleq C\mathrm{adj}(\mathbf{p}I - A)B + D\delta(\mathbf{p}) \in \mathbb{R}^{p\times m}[\mathbf{p}], \tag{6}$$

(4) can be written as

$$\delta(\mathbf{p})y(t) = \Gamma(\mathbf{p})u(t). \tag{7}$$

Next, define

$$y_{\mathrm{i}}(t) \triangleq C_{\mathrm{i}}x(t) + D_{\mathrm{i}}u(t) \in \mathbb{R}^m, \tag{8}$$

$$y_{\mathrm{o}}(t) \triangleq C_{\mathrm{o}}x(t) + D_{\mathrm{o}}u(t) \in \mathbb{R}^{p-m}, \tag{9}$$

where $C_{\mathrm{i}} \in \mathbb{R}^{m\times n}$, $C_{\mathrm{o}} \in \mathbb{R}^{(p-m)\times n}$, $D_{\mathrm{i}} \in \mathbb{R}^{m\times m}$, $D_{\mathrm{o}} \in \mathbb{R}^{(p-m)\times m}$, and

$$y = \begin{bmatrix} y_{\mathrm{i}} \\ y_{\mathrm{o}} \end{bmatrix}, \qquad C = \begin{bmatrix} C_{\mathrm{i}} \\ C_{\mathrm{o}} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{\mathrm{i}} \\ D_{\mathrm{o}} \end{bmatrix}. \tag{10}$$

Hence,

$$\Gamma = \begin{bmatrix} \Gamma_{\mathrm{i}} \\ \Gamma_{\mathrm{o}} \end{bmatrix}, \tag{11}$$

where

$$\Gamma_{\mathrm{i}}(\mathbf{p}) \triangleq C_{\mathrm{i}}\mathrm{adj}(\mathbf{p}I - A)B + D_{\mathrm{i}}\delta(\mathbf{p}) \in \mathbb{R}^{m\times m}[\mathbf{p}], \tag{12}$$

$$\Gamma_{\mathrm{o}}(\mathbf{p}) \triangleq C_{\mathrm{o}}\mathrm{adj}(\mathbf{p}I - A)B + D_{\mathrm{o}}\delta(\mathbf{p}) \in \mathbb{R}^{(p-m)\times m}[\mathbf{p}]. \tag{13}$$

Using (10) and (11), we can write (7) as

$$\delta(\mathbf{p})y_{\mathrm{i}} = \Gamma_{\mathrm{i}}(\mathbf{p})u, \tag{14}$$

$$\delta(\mathbf{p})y_{\mathrm{o}} = \Gamma_{\mathrm{o}}(\mathbf{p})u. \tag{15}$$

Multiplying (14) by adj $\Gamma_{\mathrm{i}}(\mathbf{p})$ yields

$$\delta(\mathbf{p})\,\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{i}} = [\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})]\,\Gamma_{\mathrm{i}}(\mathbf{p})u = \det\Gamma_{\mathrm{i}}(\mathbf{p})u. \tag{16}$$

Next, multiplying (15) by det $\Gamma_{\mathrm{i}}(\mathbf{p})$ yields

$$[\det\Gamma_{\mathrm{i}}(\mathbf{p})]\,\delta(\mathbf{p})y_{\mathrm{o}} = [\det\Gamma_{\mathrm{i}}(\mathbf{p})]\,\Gamma_{\mathrm{o}}(\mathbf{p})u. \tag{17}$$

Substituting the left hand side of (16) into (17) yields

$$\delta(\mathbf{p})\det\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o}} = \delta(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p})\,\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{i}}. \tag{18}$$

As shown in Aljanaideh and Bernstein (2015), the common factor $\delta(\mathbf{p})$ in (18) can be canceled without excluding any solutions of (18). Therefore, (18) can be written as

$$\det\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o}} = \Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{i}}. \tag{19}$$

Definition 1. Assume that $\Gamma_{\mathrm{i}}(\mathbf{p})$ is nonsingular. Then, the *transmissibility operator* from y_{i} to y_{o} is defined by

$$\mathcal{T}(\mathbf{p}) \triangleq \Gamma_{\mathrm{o}}(\mathbf{p})\Gamma_{\mathrm{i}}^{-1}(\mathbf{p}). \tag{20}$$

Using (20), (19) can be written as

$$y_{\mathrm{o}} = \mathcal{T}(\mathbf{p})y_{\mathrm{i}}. \tag{21}$$

If y_{i} and y_{o} are scalar, then (19) becomes

$$\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o}} = \Gamma_{\mathrm{o}}(\mathbf{p})y_{\mathrm{i}}, \tag{22}$$

and the transmissibility operator (20) is written as

$$\mathcal{T}(\mathbf{p}) = \frac{\Gamma_{\mathrm{o}}(\mathbf{p})}{\Gamma_{\mathrm{i}}(\mathbf{p})}. \tag{23}$$

3. Behaviors

Definition 2 (*Polderman & Willems, 1998, pp. 8,15*)**.** A linear dynamical system $\mathcal{S}$ is the triple $\mathcal{S} = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T} \subset \mathbb{R}$ is the time set, the vector space $\mathbb{W}$ is the signal space, and the *behavior* $\mathcal{B}$ is a subspace of $\mathbb{W}^{\mathbb{T}}$.

Behavioral equations may contain both manifest variables w and latent variables ℓ.

Definition 3 (*Polderman & Willems, 1998, p. 7*)**.** A mathematical model with latent variables of a dynamical system $\mathcal{S} = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ is defined as a triple $(\mathbb{W}_m, \mathbb{W}_\ell, \mathcal{B}_{\mathrm{f}})$ with $\mathbb{W}_m$ the vector space of manifest variables, $\mathbb{W}_\ell$ the vector space of latent variables, $\mathbb{W} = \mathbb{W}_m \times \mathbb{W}_\ell$, and $\mathcal{B}_{\mathrm{f}} \subseteq \mathbb{W}^{\mathbb{T}}$ is the full behavior. The manifest mathematical model $(\mathbb{W}_m, \mathcal{B})$ is defined by $\mathcal{B} \triangleq \{u : \mathbb{T} \to \mathbb{W}_m | \exists \ell : \mathbb{T} \to \mathbb{W}_\ell$ such that $(u, \ell) \in \mathcal{B}_{rmf}\}$; $\mathcal{B}$ is the behavior and $(\mathbb{W}_m, \mathbb{W}_\ell, \mathcal{B}_{\mathrm{f}})$ is a latent variable representation of $(\mathbb{W}_m, \mathcal{B})$.

The following theorem concerns the elimination of latent variables from behavioral equations (Polderman & Willems, 1998, pp. 206–207).

Let $\mathbb{R}^+$ denote the nonnegative real numbers.

Theorem 1. *Consider the dynamical system $\mathcal{S} = (\mathbb{R}^+, \mathbb{R}^q \times \mathbb{R}^d, \mathcal{B})$ with $\mathcal{B} = \{w : \mathbb{R}^+ \to \mathbb{R}^q | \exists \ell : \mathbb{R}^+ \to \mathbb{R}^d$ such that $R(\mathbf{p})w(t) = M(\mathbf{p})\ell(t)\}$, $R(\mathbf{p}) \in \mathbb{R}^{g\times q}[\mathbf{p}]$, and $M(\mathbf{p}) \in \mathbb{R}^{g\times d}[\mathbf{p}]$. Then, there exists a unimodular matrix $U(\mathbf{p}) \in \mathbb{R}^{g\times g}[\mathbf{p}]$ such that*

$$U(\mathbf{p})M(\mathbf{p}) = \begin{bmatrix} 0_{(g-n_P)\times d} \\ P(\mathbf{p}) \end{bmatrix}, \tag{24}$$

$$U(\mathbf{p})R(\mathbf{p}) = \begin{bmatrix} Q(\mathbf{p}) \\ S(\mathbf{p}) \end{bmatrix}, \tag{25}$$

where $P(\mathbf{p}) \in \mathbb{R}^{n_P\times d}$ has full row rank, $Q(\mathbf{p}) \in \mathbb{R}^{n_Q\times q}$, and $S(\mathbf{p}) \in \mathbb{R}^{(g-n_Q)\times q}$. Furthermore,

$$Q(\mathbf{p})w(t) = 0. \tag{26}$$

Note that the behavioral equation (26) involves only the manifest variables.

4. Relationship between behavioral equations and transmissibility operators

The following corollary of Theorem 1 shows the equivalence between behavioral equations and transmissibility operators.

Corollary 4.1. *Consider the linear dynamical system $(\mathbb{R}^+, \mathbb{R}^p \times \mathbb{R}^m, \mathcal{B})$ with $\mathcal{B} = \{y : \mathbb{R}^+ \to \mathbb{R}^p \mid u : \mathbb{R}^+ \to \mathbb{R}^m$ such that (7) is satisfied$\}$, where Γ is given by (11)–(13) and Γ_{i} is nonsingular. Then,*

$$Q(\mathbf{p})y(t) = 0, \tag{27}$$

where

$$Q(\mathbf{p}) \triangleq \begin{bmatrix} -\delta(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p}) & \delta(\mathbf{p})\det\Gamma_{\mathrm{i}}(\mathbf{p})I_{p-m} \end{bmatrix}. \tag{28}$$

Proof. In Theorem 1, let $w = y$, $\ell = u$, $R(\mathbf{p}) = \delta(\mathbf{p})I_p$, $M(\mathbf{p}) = \Gamma(\mathbf{p})$, and

$$U(\mathbf{p}) = \begin{bmatrix} -\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p}) & \det \Gamma_{\mathrm{i}}(\mathbf{p})I_{p-m} \\ I_m & 0_{m\times(p-m)} \end{bmatrix}.$$

Then,

$$\begin{aligned} U(\mathbf{p})M(\mathbf{p}) &= U(\mathbf{p})\Gamma(\mathbf{p}) \\ &= \begin{bmatrix} -\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p}) & \det \Gamma_{\mathrm{i}}(\mathbf{p})I_{p-m} \\ I_m & 0_{m\times(p-m)} \end{bmatrix} \begin{bmatrix} \Gamma_{\mathrm{i}}(\mathbf{p}) \\ \Gamma_{\mathrm{o}}(\mathbf{p}) \end{bmatrix} \\ &= \begin{bmatrix} 0_{(p-m)\times m} \\ \Gamma_{\mathrm{i}}(\mathbf{p}) \end{bmatrix}, \end{aligned}$$

and thus, $P(\mathbf{p}) = \Gamma_{\mathrm{i}}(\mathbf{p})$ is nonsingular. Moreover,

$$\begin{aligned} U(\mathbf{p})R(\mathbf{p}) &= \delta(\mathbf{p})U(\mathbf{p}) \\ &= \begin{bmatrix} -\delta(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p}) & \delta(\mathbf{p})\det \Gamma_{\mathrm{i}}(\mathbf{p})I_{p-m} \\ \delta(\mathbf{p})I_m & 0_{m\times(p-m)} \end{bmatrix}, \end{aligned}$$

and thus (27) implies that Q is given by (28). □

Remark. Note that U constructed in the proof of Corollary 4.1 is not unimodular. Assume that $m = 1$ and $p = 2$ so that Γ_{i} and Γ_{o} are polynomials. Then, let $E_1, E_2 \in \mathbb{R}[\mathbf{p}]$ be such that U defined by

$$U(\mathbf{p}) = \begin{bmatrix} -\Gamma_{\mathrm{o}}(\mathbf{p}) & \Gamma_{\mathrm{i}}(\mathbf{p}) \\ E_1(\mathbf{p}) & E_2(\mathbf{p}) \end{bmatrix} \tag{29}$$

is nonsingular. Then,

$$\begin{aligned} U(\mathbf{p})M(\mathbf{p}) &= U(\mathbf{p})\Gamma(\mathbf{p}) \\ &= \begin{bmatrix} -\Gamma_{\mathrm{o}}(\mathbf{p}) & \Gamma_{\mathrm{i}}(\mathbf{p}) \\ E_1(\mathbf{p}) & E_2(\mathbf{p}) \end{bmatrix} \begin{bmatrix} \Gamma_{\mathrm{i}}(\mathbf{p}) \\ \Gamma_{\mathrm{o}}(\mathbf{p}) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ E_1(\mathbf{p})\Gamma_{\mathrm{i}}(\mathbf{p}) + E_2(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p}) \end{bmatrix}, \end{aligned}$$

and thus, $P(\mathbf{p}) = E_1(\mathbf{p})\Gamma_{\mathrm{i}}(\mathbf{p}) + E_2(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p}) = -\det U$ is not zero. Moreover,

$$\begin{aligned} U(\mathbf{p})R(\mathbf{p}) &= \delta(\mathbf{p})U(\mathbf{p}) \\ &= \begin{bmatrix} -\delta(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p}) & \delta(\mathbf{p})\Gamma_{\mathrm{i}}(\mathbf{p}) \\ \delta(\mathbf{p})E_1(\mathbf{p}) & \delta(\mathbf{p})E_2(\mathbf{p}) \end{bmatrix}, \end{aligned} \tag{30}$$

and thus (25) implies that Q is given by

$$Q(\mathbf{p}) = \begin{bmatrix} -\delta(\mathbf{p})\Gamma_{\mathrm{o}}(\mathbf{p}) & \delta(\mathbf{p})\Gamma_{\mathrm{i}}(\mathbf{p}) \end{bmatrix}. \tag{31}$$

Therefore, U satisfies (24)–(26). Note that

$$\det U(\mathbf{p}) = -\Gamma_{\mathrm{o}}(\mathbf{p})E_2(\mathbf{p}) - \Gamma_{\mathrm{i}}(\mathbf{p})E_1(\mathbf{p}). \tag{32}$$

It follows from the Bezout identity that if Γ_{i} and Γ_{o} are coprime, then there exist $E_1, E_2 \in \mathbb{R}[\mathbf{p}]$ such that U defined by (29) is unimodular.

Define

$$T(\mathbf{p}) \triangleq \begin{bmatrix} -\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p}) & \det \Gamma_{\mathrm{i}}(\mathbf{p})I_{p-m} \end{bmatrix}. \tag{33}$$

Then

$$Q(\mathbf{p}) = \delta(\mathbf{p})T(\mathbf{p}), \tag{34}$$

and thus (27) implies that

$$Q(\mathbf{p})y(t) = \delta(\mathbf{p})T(\mathbf{p})y(t) = 0. \tag{35}$$

As shown in Aljanaideh and Bernstein (2015), $\delta(\mathbf{p})$ can be canceled in (35), which yields

$$T(\mathbf{p})y(t) = 0, \tag{36}$$

which is identical to (19). This shows that the factor δ in (28) can be removed.

Corollary 4.1 implies that a transmissibility equation is equivalent to a behavioral equation with the manifest variable set to $w = [y_{\mathrm{i}}\ y_{\mathrm{o}}]^{\mathrm{T}}$ and the latent variable set to $\ell = u$. Letting $Q = [Q_1\ Q_2]$, where $Q_1(\mathbf{p}) \triangleq -\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})$, and $Q_2(\mathbf{p}) \triangleq \det \Gamma_{\mathrm{i}}(\mathbf{p})I_{p-m}$, it follows that

$$\mathcal{T}(\mathbf{p}) = -Q_1(\mathbf{p})Q_2^{-1}(\mathbf{p}). \tag{37}$$

5. Example

We consider the spring–mass system in Fig. 1, whose dynamics are

$$M\ddot{q} + Kq = F, \tag{38}$$

where $q \triangleq [q_1\ q_2\ q_3]^{\mathrm{T}}$, $F \triangleq [f_1\ f_2\ f_3]^{\mathrm{T}}$, and

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}. \tag{39}$$

5.1. Transmissibility operators

Consider the mass–spring system in Fig. 1 with $f_2 = f_3 = 0$, $y_{\mathrm{i}} = q_1$, $y_{\mathrm{o}} = q_3$, and $u = f_1$. Then (1) holds with

$$x \triangleq \begin{bmatrix} q_1 & q_2 & q_3 & \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{bmatrix}^{\mathrm{T}}, \qquad A \triangleq \begin{bmatrix} 0_{3\times3} & I_3 \\ \Omega & 0_{3\times3} \end{bmatrix}, \tag{40}$$

$$\Omega \triangleq -M^{-1}K = \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 \\ \dfrac{k_2}{m_2} & -\dfrac{k_2+k_3}{m_2} & \dfrac{k_3}{m_2} \\ 0 & \dfrac{k_3}{m_3} & -\dfrac{k_3+k_4}{m_3} \end{bmatrix}, \tag{41}$$

$$b \triangleq \begin{bmatrix} 0 & 0 & 0 & \frac{1}{m_1} & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \qquad u \triangleq f_1. \tag{42}$$

Moreover,

$$q_1 = y_{\mathrm{i}} = c_{\mathrm{i}}\mathrm{adj}(\mathbf{p}I - A)b, \tag{43}$$

$$q_3 = y_{\mathrm{o}} = c_{\mathrm{o}}\mathrm{adj}(\mathbf{p}I - A)b, \tag{44}$$

where

$$c_{\mathrm{i}} \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad c_{\mathrm{o}} \triangleq \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{45}$$

Therefore,

$$\Gamma_{\mathrm{i}}(\mathbf{p}) = c_{\mathrm{i}}\mathrm{adj}(\mathbf{p}I - A)b = \frac{m_2m_3\mathbf{p}^4 + a_1\mathbf{p}^2 + a_0}{m_1m_2m_3}, \tag{46}$$

$$\Gamma_{\mathrm{o}}(\mathbf{p}) = c_{\mathrm{o}}\mathrm{adj}(\mathbf{p}I - A)b = \frac{k_2k_3}{m_1m_2m_3}, \tag{47}$$

where

$$a_0 \triangleq k_2k_3 + k_2k_4 + k_3k_4, \tag{48}$$

$$a_1 \triangleq (k_3+k_4)m_2 + (k_2+k_3)m_3. \tag{49}$$

The corresponding transmissibility operator is

$$\mathcal{T}(\mathbf{p}) = \frac{\Gamma_{\mathrm{o}}(\mathbf{p})}{\Gamma_{\mathrm{i}}(\mathbf{p})} = \frac{k_2k_3}{m_2m_3\mathbf{p}^4 + a_1\mathbf{p}^2 + a_0}. \tag{50}$$

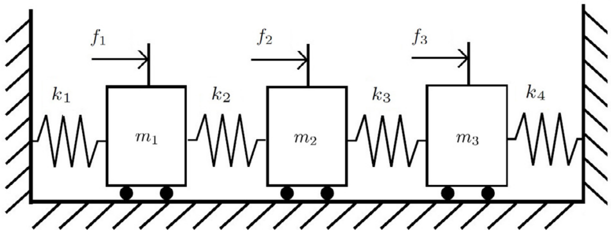

Fig. 1. Mass–spring system, where $q_1 q_2$, and q_3 are the displacements of m_1, m_2, and m_3, respectively, and f_1, f_2, and f_3 are external forces.

Using (22), (46), and (47), q_1 and q_3 satisfy

$$(m_2 m_3 \mathbf{p}^4 + ((k_3+k_4)m_2 + (k_2+k_3)m_3)\mathbf{p}^2 + k_2k_3 + k_2k_4 + k_3k_4)q_3 = k_2k_3q_1. \quad (51)$$

Next, consider the mass–spring system in Fig. 1 with $f_1 = f_3 = 0$, $y_\mathrm{i} = q_1$, $y_\mathrm{o} = q_3$, and $u = f_2$. Then (1) holds with (40), (41), $b = [0\ 0\ 0\ 0\ \frac{1}{m_2}\ 0]^\mathrm{T}$, and $u = f_2$. Therefore,

$$\Gamma_\mathrm{i}(\mathbf{p}) = c_\mathrm{i}\mathrm{adj}(\mathbf{p}I - A)b = \frac{k_2 m_3 \mathbf{p}^2 + k_2k_3 + k_2k_4}{m_1m_2m_3}, \quad (52)$$

$$\Gamma_\mathrm{o}(\mathbf{p}) = c_\mathrm{o}\mathrm{adj}(\mathbf{p}I - A)b = \frac{k_3 m_1 \mathbf{p}^2 + k_1k_3 + k_2k_3}{m_1m_2m_3}. \quad (53)$$

The corresponding transmissibility operator is

$$\mathcal{T}(\mathbf{p}) = \frac{\Gamma_\mathrm{o}(\mathbf{p})}{\Gamma_\mathrm{i}(\mathbf{p})} = \frac{k_3 m_1 \mathbf{p}^2 + k_1k_3 + k_2k_3}{k_2 m_3 \mathbf{p}^2 + k_2k_3 + k_2k_4}. \quad (54)$$

Using (22), (52), and (53), q_1 and q_3 satisfy

$$k_2(m_3\mathbf{p}^2 + k_3 + k_4)q_3 = k_3(m_1\mathbf{p}^2 + k_1 + k_2)q_1. \quad (55)$$

Next, consider the mass–spring system in Fig. 1 with $f_1 = f_2 = 0$, $y_\mathrm{i} = q_1$, $y_\mathrm{o} = q_3$, and $u = f_3$. Then (1) holds with (40), (41), $b = [0\ 0\ 0\ 0\ 0\ \frac{1}{m_3}]^\mathrm{T}$, and $u = f_3$. Therefore,

$$\Gamma_\mathrm{i}(\mathbf{p}) = c_\mathrm{i}\mathrm{adj}(\mathbf{p}I - A)b = \frac{k_2k_3}{m_1m_2m_3}, \quad (56)$$

$$\Gamma_\mathrm{o}(\mathbf{p}) = c_\mathrm{o}\mathrm{adj}(\mathbf{p}I - A)b = \frac{m_1m_2\mathbf{p}^4 + b_1\mathbf{p}^2 + b_0}{m_1m_2m_3}, \quad (57)$$

where

$$b_0 \triangleq k_1k_2 + k_1k_3 + k_2k_3, \quad (58)$$

$$b_1 \triangleq (k_2+k_3)m_1 + (k_1+k_2)m_2. \quad (59)$$

The corresponding transmissibility operator is

$$\mathcal{T}(\mathbf{p}) = \frac{\Gamma_\mathrm{o}(\mathbf{p})}{\Gamma_\mathrm{i}(\mathbf{p})} = \frac{m_1m_2\mathbf{p}^4 + b_1\mathbf{p}^2 + b_0}{k_2k_3}. \quad (60)$$

Using (22), (56) and (57), q_1 and q_3 satisfy

$$k_2k_3q_3 = (m_1m_2\mathbf{p}^4 + ((k_2+k_3)m_1 + (k_1+k_2)m_2)\mathbf{p}^2 + k_1k_2 + k_1k_3 + k_2k_3)q_1. \quad (61)$$

5.2. Behaviors

Suppose that $f_1 \neq 0, f_2 = f_3 = 0$, $w = [q_1\ q_3]^\mathrm{T}$ and $\ell = [f_2\ q_2]^\mathrm{T}$, and thus, q_1 and q_3 are the manifest variables and f_2 and q_2 are the latent variables. Moreover, define U by

$$U(\mathbf{p}) = \begin{bmatrix} -\Gamma_\mathrm{o}(\mathbf{p}) & \Gamma_\mathrm{i}(\mathbf{p}) \\ -\Gamma_\mathrm{o}(\mathbf{p}) & \Gamma_\mathrm{i}(\mathbf{p}) - \dfrac{1}{\Gamma_\mathrm{o}(\mathbf{p})} \end{bmatrix}, \quad (62)$$

where Γ_i and Γ_o are as in (46) and (47), respectively. Then, it follows from Corollary 4.1 that the behavioral equation is given by (36), where T is given by (33), that is,

$$T(\mathbf{p}) = \begin{bmatrix} -\Gamma_\mathrm{o}(\mathbf{p}) & \Gamma_\mathrm{i}(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_2k_3}{m_1m_2m_3} & \dfrac{m_2m_3\mathbf{p}^4 + a_1\mathbf{p}^2 + a_0}{m_1m_2m_3} \end{bmatrix}, \quad (63)$$

$\delta(\mathbf{p}) = \det(\mathbf{p}I - A)$, and A is given by (40). Therefore, using (36) with $y = w$, the behavioral equation of the behavior (q_1, q_3) is given by

$$(m_2 m_3 \mathbf{p}^4 + ((k_3+k_4)m_2 + (k_2+k_3)m_3)\mathbf{p}^2 + k_2k_3 + k_2k_4 + k_3k_4)q_3 = k_2k_3q_1. \quad (64)$$

Note from (34) that $Q(\mathbf{p}) = \delta(\mathbf{p})T(\mathbf{p})$. The latent variables f_1 and q_2 were thus eliminated to obtain the behavioral equation that corresponds to the behavior (q_1, q_3). Note that (64) is precisely (51).

Alternatively, suppose that $f_2 \neq 0$ and $f_1 = f_3 = 0$. Let $w = [q_1\ q_3]^\mathrm{T}$ and $\ell = [f_2\ q_2]^\mathrm{T}$, and thus, q_1 and q_3 are the manifest variables and f_2 and q_2 are the latent variables. Then following the same procedure above with U as in (62) and Γ_i and Γ_o as in (52) and (53), respectively, the behavioral equation of the behavior (q_1, q_3) is given by

$$k_2(m_3\mathbf{p}^2 + k_3 + k_4)q_3 = k_3(m_1\mathbf{p}^2 + k_1 + k_2)q_1. \quad (65)$$

Note from (34) that $Q(\mathbf{p}) = \delta(\mathbf{p})T(\mathbf{p})$. The latent variables f_2 and q_2 were thus eliminated to obtain the behavioral equation that corresponds to the behavior (q_1, q_3). Note that (65) is precisely (55).

Finally, suppose that $f_3 \neq 0$ and $f_1 = f_2 = 0$. Let $w = [q_1\ q_3]^\mathrm{T}$ and $\ell = [f_3\ q_2]^\mathrm{T}$, and thus, q_1 and q_3 are the manifest variables and f_3 and q_2 are the latent variables. Then following the same procedure above with U as in (62) and Γ_i and Γ_o as in (56) and (57), respectively, the behavioral equation of the behavior (q_1, q_3) is given by

$$k_2k_3q_3 = (m_1m_2\mathbf{p}^4 + ((k_2+k_3)m_1 + (k_1+k_2)m_2)\mathbf{p}^2 + k_1k_2 + k_1k_3 + k_2k_3)q_1. \quad (66)$$

Note from (34) that $Q(\mathbf{p}) = \delta(\mathbf{p})T(\mathbf{p})$. The latent variables f_3 and q_2 were thus eliminated to obtain the behavioral equation that corresponds to the behavior (q_1, q_3). Note that (66) is precisely (61).

6. Discussion

This paper showed that transmissibility operators arise from behaviors with the manifest variables chosen to be the output signals and the latent variables chosen to be the input signals. This observation has the following ramifications. First, it shows that time-domain transmissibility equations can be viewed as behavioral equations corresponding to specific behaviors, which deepens the theoretical foundation for time-domain transmissibilities and allows them to benefit from the rich literature on behaviors. For instance, this connection opens the door for the meaning of transmissibilities in linearized nonlinear systems, which is discussed in Polderman and Willems (1998) for behaviors. Moreover, this connection will help in understanding the roles of controllability and observability in constructing transmissibility operators, which is also discussed in Polderman and Willems (1998) for behaviors. At the same time, this paper shows that behavioral equations represent transmissibility operators that are valid in the presence of external inputs. These external inputs do not appear in the behavioral equation, which means that one response variable can be used to predict another response variable despite the presence of the unknown external excitation. The equivalence between behaviors and transmissibilities was illustrated on a mass–spring system.

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