results of [10]. This note, therefore, unveils a link between polynomial hyperbolicity and stability.

Finally, as pointed out in [3, Sec. 18.9], the applications of frequency response convexity in robust control have only been minimally explored. The explicit LMI formulation described in this note may motivate further research along these lines.

ACKNOWLEDGMENT

This research benefitted from discussions with B. Mourrain.

REFERENCES

[1] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*. Providence, RI: AMS, 1990.
[2] J. Ackermann, *Robust Control: The Parameter Space Approach*. New York: Springer-Verlag, 1993.
[3] B. R. Barmish, *New Tools for Robustness of Linear Systems*. New York: Macmillan, 1994.
[4] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*. New York: Springer-Verlag, 2003.
[5] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parametric Approach*. Englewood Cliffs, NJ: Prentice-Hall, 1995.
[6] D. Cox, J. Little, and D. O'Shea, *Ideals, Varietes, and Algorithms*. New York: Springer-Verlag, 1992.
[7] M. Elkadi and B. Mourrain, *Introduction à la résolution des systèmes polynomiaux*. New York: Springer-Verlag, 2007.
[8] K. Gu, "Comments on "Convexity of frequency response arcs associated with a stable polynomial," *IEEE Trans. Autom. Control*, vol. 39, no. 11, pp. 2262–2265, Nov. 1994.
[9] J. C. Hamann and B. R. Barmish, "Convexity of frequency response arcs associated with a stable polynomial," *IEEE Trans. Autom. Control*, vol. 38, no. 6, pp. 904–915, Jun. 1993.
[10] J. W. Helton and V. Vinnikov, "Linear matrix inequality representation of sets," *Commun. Pure Appl. Math.*, vol. 60, no. 5, pp. 654–674, 2007.
[11] J. Kogan, "Stability of a polynomial and convexity of a frequency response arc," in *Proc. IEEE Conf. Decis. Control*, Dec., 1993, pp. 519–520.
[12] N. Kravistky, "On the discriminant function of two commuting nonselfadjoint operators," *Integr. Equation Operator Theory*, vol. 3, no. 1, pp. 97–124, 1980.
[13] A. S. Lewis, P. A. Parrilo, and M. V. Ramana, "The Lax conjecture is true," *Proc. Amer. Math. Soc.*, vol. 133, no. 9, pp. 2495–2499, 2005.
[14] J. Renegar, "Hyperbolic programs and their derivative relaxations," *Found. Comput. Math.*, vol. 6, no. 1, pp. 59–79, 2006.
[15] B. Sturmfels, *Solving Systems of Polynomial Equations*. Providence, RI: AMS, 2002.

Internal Model Control in the Shift and Delta Domains

Jesse B. Hoagg, Mario A. Santillo, and Dennis S. Bernstein

***Abstract*—We construct multivariable internal model controllers in the shift and delta domains. To do so, we develop a linear algebraic approach to the multivariable command following and disturbance rejection problem that facilitates a unified treatment of the differential, shift, and delta domains.**

***Index Terms*—Command following, delta domain, disturbance rejection, internal model control.**

I. INTRODUCTION

Internal model control is essential for both command following and disturbance rejection, where the exogenous command and disturbance signals are outputs of an unforced linear system. In short, the internal model principle states that asymptotic command following and disturbance rejection can be achieved by incorporating copies of the exogenous dynamics in the feedback loop. The most familiar case of internal model control is integral control for following step commands or rejecting constant disturbances.

Internal model control in continuous time was developed in [1]–[3]. Specifically, [1] treats the disturbance rejection problem for a class of matched systems in which the range of the disturbance input matrix is contained in the range of the control input matrix. Multiinput multioutput (MIMO) internal model control for both command following and disturbance rejection problems is given in [4]–[7], including the synthesis of a servocompensator. An alternative geometric approach is given in [8]. The converse problem is addressed in [9], and necessary conditions for asymptotic regulation are developed. Continuous-time internal model control is considered in [10] and [11], while [12] constructs internal model controllers for disturbances with known characteristics. Optimal H_2 control is combined in [13] and [14] with the internal model principle to develop controllers for both command following and disturbance rejection.

The results of [1]–[9] on internal model control are confined to continuous-time systems. Although the principle is used in [12] for single-input single-output (SISO) discrete-time systems, results for MIMO discrete-time systems are not available in the literature. Furthermore, the results of [4]–[7] use analytic tools specific to continuous-time models, and thus, do not extend to discrete-time systems. In the present note, we develop an alternative approach to internal model control that is applicable to continuous-time systems as well as discrete-time systems in both the shift and delta domains [15]. The MIMO problem that we consider includes both the command following and disturbance rejection problems as special cases, as well as mixed problems that include aspects of both. In contrast to the analytical approach of [4]–[7] and the geometric approach of [8] and [9], our approach is algebraic. An earlier version of this note appears as [16].

Manuscript received August 17, 2006; revised May 29, 2007. Recommended by Associate Editor A. Astolfi.

J. B. Hoagg was with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140 USA (e-mail: jhoagg@umich.edu).

M. A. Santillo and D. S. Bernstein are with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI 48109-2140 USA (e-mail: santillo@umich.edu; dsbaero@umich.edu).

Digital Object Identifier 10.1109/TAC.2008.921526

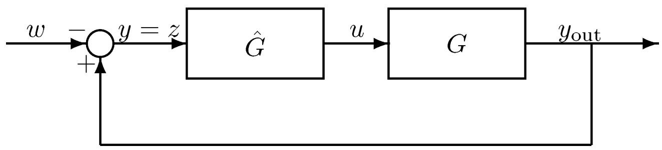

Fig. 1. SISO command following problem.

II. Problem Formulation

We consider the linear system

$$\mathcal{D}x(t) = Ax(t) + Bu(t) + D_1 w(t) \tag{1}$$
$$z(t) = E_1 x(t) + E_2 u(t) + E_0 w(t) \tag{2}$$
$$y(t) = Cx(t) + Du(t) + D_2 w(t) \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^{l_y}$ is the measured output available to the controller, $z \in \mathbb{R}^{l_z}$ is the performance, $u \in \mathbb{R}^{l_u}$ is the control, and $w \in \mathbb{R}^{l_w}$ is the exogenous input. To simultaneously consider continuous-time and discrete-time systems, let $\mathcal{D}$ denote the differential operator d/dt, the forward shift operator $\mathbf{q}$, or the delta operator $\delta = (\mathbf{q}-1)/h$, that is, $\mathcal{D}x(t) = dx(t)/dt$ for continuous time, $\mathcal{D}x(t) = x(t+1)$ for discrete time with the shift operator, and $\mathcal{D}x(t) = (1/h)[x(t+1) - x(t)]$ for discrete time with the delta operator, where $h > 0$ is the sampling period. The set τ of time arguments t depends on the operator $\mathcal{D}$, specifically,

$$\tau \stackrel{\triangle}{=} \begin{cases} [0,\infty), & \mathcal{D} = \frac{d}{dt} \\ \mathbb{Z}^+, & \mathcal{D} = \mathbf{q} \\ \{t : \frac{t}{h} \in \mathbb{Z}^+\}, & \mathcal{D} = \delta\,. \end{cases} \tag{4}$$

The problem objective is to construct a feedback controller that stabilizes the open-loop system (1)–(3) and regulates the performance z to 0 when the exogenous input w is generated by an unforced linear system. This control problem includes both disturbance rejection and command following objectives, where the exogenous signal w may contain components to be rejected and components to be followed. The problem can be restricted to command following by letting $D_1 = 0$, or can be restricted to disturbance rejection by letting $D_2 = 0$ and $E_0 = 0$. Mixed problems can be considered as well. As in the general problem formulation (1)–(3), all signals in these specialized problems may be multivariable.

As a special case, the classical SISO command following problem in Fig. 1 can be written in the form (1)–(3), where the plant G has the realization

$$G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

$D_1 = 0$, $E_1 = C$, $E_2 = D$, $E_0 = D_2 = -1$, and $\hat{G}$ is the feedback controller. Then, $z = y = Cx + Du + D_2 w = y_{\rm out} - w$ is the tracking error, where $y_{\rm out} = Cx + Du$. Similarly, the classical SISO disturbance rejection problem in Fig. 2 can be written in the form (1)–(3), where $D_1 = B$, $E_1 = C$, $E_2 = D$, $E_0 = D_2 = 0$, and

$$G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

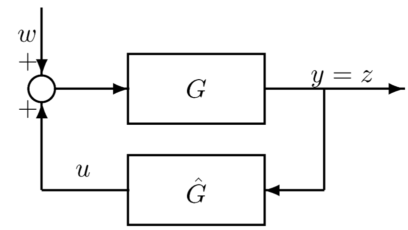

Fig. 2. SISO disturbance rejection problem.

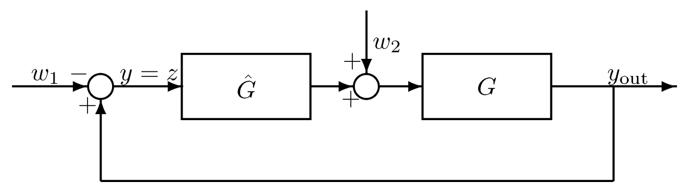

Fig. 3. Combined SISO command following and disturbance rejection problem.

Lastly, the combined SISO command following and disturbance rejection problem in Fig. 3 can be written in the form (1)–(3), where $D_1 = B$, $E_1 = C$, $E_2 = D$, $E_0 = D_2 = [1 \quad 0]$, and

$$G \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then, $z = y = Cx + Du + D_2 w = y_{\rm out} - w_1$ is the command following error.

III. Exosystem and Controller Construction

For each domain, we define the stable region

$$\mathcal{S} \stackrel{\triangle}{=} \begin{cases} \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda < 0\}, & \mathcal{D} = \frac{d}{dt} \\ \{\lambda \in \mathbb{C} : |\lambda| < 1\}, & \mathcal{D} = \mathbf{q} \\ \{\lambda \in \mathbb{C} : \frac{h}{2}\,|\lambda|^2 + \mathrm{Re}\,\lambda < 0\}, & \mathcal{D} = \delta \end{cases} \tag{5}$$

and the unstable region $\mathcal{U} \stackrel{\triangle}{=} \mathbb{C}\backslash\mathcal{S}$.

Definition 3.1 The spectrum of A is $\mathrm{spec}(A) \stackrel{\triangle}{=} \{\lambda : \det(\lambda I - A) = 0\}$. A is asymptotically stable if $\mathrm{spec}(A) \subset \mathcal{S}$. A is antistable if $\mathrm{spec}(A) \subset \mathcal{U}$.

Let the exogenous signal w be the output of the linear system

$$\mathcal{D}x_w = A_w x_w \tag{6}$$
$$w = C_w x_w \tag{7}$$

where $x_w \in \mathbb{R}^{n_w}$, A_w is antistable, and (A_w, C_w) is observable. In the case $l_w = 1$, it follows that A_w is cyclic. If $l_w > 1$, we can assume, without the loss of generality, that A_w is cyclic. To show this, suppose that A_w is not cyclic, that is, derogatory. To demonstrate that there exists a cyclic system whose output is identical to the output of (6) and (7), consider the discrete-time derogatory system

$$q(k+1) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} q(k), \qquad w(k) = q(k) \tag{8}$$

where $q(k) \stackrel{\triangle}{=} [q_1(k) \quad q_2(k)]^T$. We consider two cases. First, suppose that $q_1(0) \neq 0$, and construct the system

$$q_r(k+1) = \lambda q_r(k), \qquad w_r(k) = \begin{bmatrix} 1 \\ \frac{q_2(0)}{q_1(0)} \end{bmatrix} q_r(k). \tag{9}$$

Then, with $q_r(0) = q_1(0)$, it follows that

$$w_r(k) = \begin{bmatrix} \lambda^k q_r(0) \\ \lambda^k q_2(0) \end{bmatrix} = \begin{bmatrix} \lambda^k q_1(0) \\ \lambda^k q_2(0) \end{bmatrix} = w(k). \tag{10}$$

Next, suppose that $q_2(0) \neq 0$ and construct the system

$$q_r(k+1) = \lambda q_r(k), \qquad w_r(k) = \begin{bmatrix} \frac{q_1(0)}{q_2(0)} \\ 1 \end{bmatrix} q_r(k). \tag{11}$$

Then, with $q_r(0) = q_2(0)$, it follows that

$$w_r(k) = \begin{bmatrix} \lambda^k q_1(0) \\ \lambda^k q_r(0) \end{bmatrix} = \begin{bmatrix} \lambda^k q_1(0) \\ \lambda^k q_2(0) \end{bmatrix} = w(k). \tag{12}$$

Next, consider the derogatory system

$$q(k+1) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} q(k), \qquad w(k) = q(k) \tag{13}$$

where $q(k) \triangleq [q_1(k) \quad q_2(k) \quad q_3(k)]^T$. Hence

$$q(k) = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^k \end{bmatrix} q(0), \qquad w(k) = q(k). \tag{14}$$

If $q_2(0) = 0$, then $q_2(k) \equiv 0$, and an equivalent system can be constructed as in (8). Hence, assume $q_2(0) \neq 0$ and consider the cyclic system

$$q_r(k+1) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} q_r(k), \qquad w_r(k) = C_r q_r(k) \tag{15}$$

where $q_r(k) \triangleq [q_{r1}(k) \quad q_{r2}(k)]^T$ and

$$C_r \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{q_3(0)}{q_2(0)} \end{bmatrix}. \tag{16}$$

Then, with $q_{r1}(0) = q_1(0)$ and $q_{r2}(0) = q_2(0)$, it follows that

$$\begin{aligned} w_r(k) &= \begin{bmatrix} \lambda^k q_{r1}(0) + k\lambda^{k-1} q_{r2}(0) \\ \lambda^k q_{r2}(0) \\ \lambda^k q_3(0) \end{bmatrix} \\ &= \begin{bmatrix} \lambda^k q_1(0) + k\lambda^{k-1} q_2(0) \\ \lambda^k q_2(0) \\ \lambda^k q_3(0) \end{bmatrix} = w(k). \end{aligned} \tag{17}$$

Note that, in all of these cases, the cyclic system is observable. If, however, the cyclic system is not observable, then the unobservable dynamics can be truncated. This construction extends to derogatory systems of arbitrary order.

Furthermore, analogous results hold in continuous time as well as in discrete time with the delta operator. To see this, note that, in continuous time, the state-transition matrix has the same form as in discrete time with the shift operator, where λ^k is replaced by $e^{\lambda t}$. For discrete time with the delta operator, note that, if A_s represents cyclic dynamics in discrete time with the shift operator, then $A_\delta \triangleq (1/h)A_s - I$ represents cyclic dynamics in discrete time with the delta operator. Therefore, without the loss of generality, A_w can be assumed to be cyclic. This assumption is invoked in the proof of Theorem 4.1.

We now consider the feedback controller

$$\mathcal{D}\hat{x} = \hat{A}\hat{x} + \hat{B}y \tag{18}$$

$$u = \hat{C}\hat{x} \tag{19}$$

where $\hat{x} \in \mathbb{R}^{\hat{n}}$. The closed-loop system (1)–(3) with the feedback controller (18), (19) is given by

$$\mathcal{D}\tilde{x} = \tilde{A}\tilde{x} + \tilde{D}w \tag{20}$$

$$z = \tilde{E}\tilde{x} + E_0 w \tag{21}$$

where

$$\tilde{A} \triangleq \begin{bmatrix} A & B\hat{C} \\ \hat{B}C & \hat{A} + \hat{B}D\hat{C} \end{bmatrix}, \qquad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ \hat{B}D_2 \end{bmatrix},$$
$$\tilde{E} \triangleq [\, E_1 \quad E_2\hat{C} \,], \qquad \tilde{x} \triangleq \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \tag{22}$$

The following result provides necessary and sufficient conditions for z to converge to 0 for arbitrary initial conditions. This result is presented in [9] for continuous-time systems, where a geometric proof is given. We provide an alternative proof that extends the result to discrete-time systems in the shift and delta domains.

Lemma 3.1: Consider the closed-loop system (20)–(22) with the exogenous input (6), (7), and assume that $\tilde{A}$ is asymptotically stable. Then, for all initial conditions $\tilde{x}(0)$ and $x_w(0)$, $\lim_{t\to\infty} z(t) = 0$ if and only if there exists $S \in \mathbb{R}^{(n+\hat{n})\times n_w}$ such that

$$\tilde{A}S - SA_w = \tilde{D}C_w \tag{23}$$

$$\tilde{E}S = E_0 C_w. \tag{24}$$

Proof: The closed-loop system (20)–(22) with the exogenous input (6), (7) can be written as

$$\mathcal{D}x_s = A_s x_s \tag{25}$$

$$z = E_s x_s \tag{26}$$

where

$$A_s \triangleq \begin{bmatrix} \tilde{A} & \tilde{D}C_w \\ 0 & A_w \end{bmatrix}, \quad E_s \triangleq [\, \tilde{E} \quad E_0 C_w \,], \quad x_s \triangleq \begin{bmatrix} \tilde{x} \\ x_w \end{bmatrix}. \tag{27}$$

Since $\tilde{A}$ is asymptotically stable and A_w is antistable, the Sylvester equation (23) has a unique solution $S \in \mathbb{R}^{(n+\hat{n})\times n_w}$. Now define

$$Q \triangleq \begin{bmatrix} I & -S \\ 0 & I \end{bmatrix} \tag{28}$$

and consider the change of basis

$$\bar{A}_s \triangleq Q^{-1} A_s Q = \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_w \end{bmatrix} \tag{29}$$

$$\bar{E}_s \triangleq E_s Q = [\, \tilde{E} \quad -\tilde{E}S + E_0 C_w \,]. \tag{30}$$

To prove necessity in the continuous-time case, suppose that $\lim_{t\to\infty} z(t) = 0$ for all initial conditions, so that $\lim_{t\to\infty}[\tilde{E}e^{\tilde{A}t} + (-\tilde{E}S + E_0 C_w)e^{A_w t}] = 0$. Since $\tilde{A}$ is asymptotically stable, it follows that $\lim_{t\to\infty} \tilde{E}e^{\tilde{A}t} = 0$, and thus, $\lim_{t\to\infty}(-\tilde{E}S + E_0 C_w)e^{A_w t} = 0$. Since A_w is antistable, every nonzero entry of $(-\tilde{E}S + E_0 C_w)e^{A_w t}$ is either a constant or involves exponentials of t, where each coefficient of t has nonnegative real part. Therefore, $\lim_{t\to\infty}(-\tilde{E}S + E_0 C_w)e^{A_w t} = 0$ implies that $-\tilde{E}S + E_0 C_w = 0$.

To prove necessity in the discrete-time case with the shift operator, suppose that $\lim_{t\to\infty} z(t) = 0$ for all initial conditions, so that $\lim_{t\to\infty}[\tilde{E}\tilde{A}^t + (-\tilde{E}S + E_0 C_w)A_w^t] = 0$. Since $\tilde{A}$ is asymptotically stable, it follows that $\lim_{t\to\infty} \tilde{E}\tilde{A}^t = 0$, and thus, $\lim_{t\to\infty}(-\tilde{E}S + E_0 C_w)A_w^t = 0$. Since A_w is antistable, $\lim_{t\to\infty} A_w^t$ does not exist, and, for all $t \ge 0$, A_w^t is nonsingular. Assume that $-\tilde{E}S + E_0 C_w \neq 0$, let $\sigma_{\min}(\cdot)$ denote the minimum singular value, and let $\|\cdot\|_{\rm F}$ denote the Frobenius norm. Then, it follows that $0 = \lim_{t\to\infty} \|(-\tilde{E}S + E_0 C_w)A_w^t\|_{\rm F} \ge \| -\tilde{E}S + E_0 C_w\|_{\rm F} \lim_{t\to\infty} \sigma_{\min}(A_w^t) = \infty$, which is a contradiction. Therefore, $-\tilde{E}S + E_0 C_w = 0$.

To prove necessity in the discrete-time case with the delta operator, suppose that $\lim_{t\to\infty} z(t) = 0$ for all initial conditions, so that $\lim_{t\to\infty} \bar{E}_s (I + h\bar{A}_s)^{t/h} = 0$. Thus, $\lim_{t\to\infty}[\tilde{E}(I + h\tilde{A})^{t/h} + (-\tilde{E}S + E_0 C_w)\ (I + hA_w)^{t/h}] = 0$. Since $\tilde{A}$ is asymptotically stable, it follows that $\lim_{t\to\infty} \tilde{E}(I + h\tilde{A})^{t/h} = 0$, and thus $\lim_{t\to\infty}(-\tilde{E}S + E_0 C_w)(I + hA_w)^{t/h} = 0$. Since A_w is antistable, $\lim_{t\to\infty}(I + h\tilde{A})^{t/h}$ does not exist, and, for all $t \ge 0$, $(I + hA_w)^{t/h}$ is nonsingular. Assuming that $-\tilde{E}S + E_0 C_w \neq 0$, it follows that $0 = \lim_{t\to\infty} \left\|(-\tilde{E}S + E_0 C_w)\ (I + hA_w)^{t/h}\right\|_{\rm F} \ge \| -\tilde{E}S + E_0 C_w\|_{\rm F} \lim_{t\to\infty} \sigma_{\min}\left((I + hA_w)^{t/h}\right) = \infty$, which is a contradiction. Therefore, $-\tilde{E}S + E_0 C_w = 0$.

Conversely, since $\tilde{E}S - E_0 C_w = 0$, we have $z(t) = \tilde{E}e^{\tilde{A}t}\tilde{x}(0)$ in continuous time, $z(t) = \tilde{E}\tilde{A}^t\tilde{x}(0)$ in discrete time with the shift operator, and $z(t) = \tilde{E}(I + h\tilde{A})^{t/h}\tilde{x}(0)$ in discrete time with the delta operator. Since $\tilde{A}$ is asymptotically stable, $\lim_{t\to\infty} z(t) = 0$.

IV. Internal Model Control

We now consider the MIMO command following and disturbance rejection problem for the linear system (1)–(3). We provide sufficient conditions for the existence of a feedback controller (18), (19) that stabilizes (20)–(22) and regulates the performance variable z to 0.

To describe the form of this controller, consider the open-loop system (1)–(3) and, as in [4]–[7], cascade its output with an internal model of the exogenous dynamics

$$\mathcal{D}\hat{x}_1 = A_W\hat{x}_1 + B_W y \tag{31}$$

where $A_W \stackrel{\triangle}{=} I_{l_y} \otimes A_w \in \mathbb{R}^{n_w l_y \times n_w l_y}$, $B_W \stackrel{\triangle}{=} I_{l_y} \otimes B_w$, and $B_w \in \mathbb{R}^{n_w}$ is chosen such that (A_w, B_w) is controllable. The symbol $\otimes$ represents the Kronecker product. There exists B_w such that (A_w, B_w) is controllable since A_w is cyclic [17, Fact 5.12.6]. Note that the dynamics of (31) contain l_y copies of the exogenous dynamics A_w. The cascade (1)–(3) and (31) is

$$\mathcal{D}\begin{bmatrix} x_1 \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \end{bmatrix} + \begin{bmatrix} B \\ B_W D \end{bmatrix} u + \begin{bmatrix} D_1 \\ B_W D_2 \end{bmatrix} w \tag{32}$$

$$\begin{bmatrix} y \\ \hat{x}_1 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \end{bmatrix} + \begin{bmatrix} D \\ 0 \end{bmatrix} u + \begin{bmatrix} D_2 \\ 0 \end{bmatrix} w. \tag{33}$$

Next, for the augmented system (32) and (33), we consider a feedback controller of the form

$$\mathcal{D}\hat{x}_2 = A_c\hat{x}_2 + [\,B_{c1} \quad B_{c2}\,]\begin{bmatrix} y \\ \hat{x}_1 \end{bmatrix} \tag{34}$$

$$u = C_c\hat{x}_2 \tag{35}$$

where $A_c \in \mathbb{R}^{(n+n_w l_y)\times(n+n_w l_y)}$, $B_{c1} \in \mathbb{R}^{(n+n_w l_y)\times l_y}$, $B_{c2} \in \mathbb{R}^{(n+n_w l_y)\times n_w l_y}$, and $C_c \in \mathbb{R}^{l_u \times (n+n_w l_y)}$. Then, the closed-loop system consisting of (32)–(35), which is shown in Fig. 4, is given by

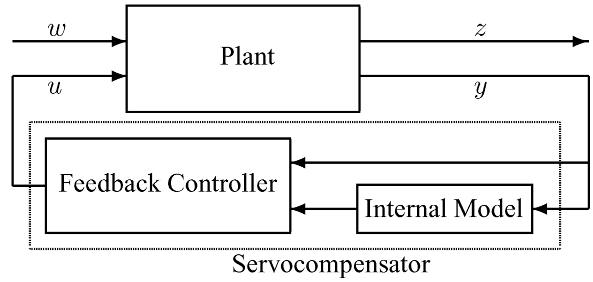

Fig. 4. Internal model control for multivariable command following and disturbance rejection.

$$\mathcal{D}\begin{bmatrix} x \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & BC_c \\ B_W C & A_W & B_W DC_c \\ B_{c1}C & B_{c2} & A_c + B_{c1}DC_c \end{bmatrix}\begin{bmatrix} x \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ B_W D_2 \\ B_{c1} D_2 \end{bmatrix} w \tag{36}$$

$$z = [\,E_1 \quad 0 \quad E_2 C_c\,]\begin{bmatrix} x \\ \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + E_0 w. \tag{37}$$

We now present the main result of this section. Note that stabilizability and detectability for delta-domain systems is defined in [15].

Theorem 4.1: Assume that the following conditions hold.

1) (A, B, C) is stabilizable and detectable.
2) $l_u \ge l_y$.
3) For all $\lambda \in \operatorname{spec}(A_w)$, rank $\left(\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}\right) = n + l_y$.
4) There exists $R \in \mathbb{R}^{l_z \times l_y}$ such that $z = Ry$.

Then, there exists a linear time-invariant controller of the form (18) and (19) such that the closed-loop system (20)–(22) is asymptotically stable, and, for all initial conditions $\tilde{x}(0)$ and $x_w(0)$, $\lim_{t\to\infty} z(t) = 0$. Furthermore, one such linear time-invariant controller is given by (18) and (19) with

$$\hat{A} \stackrel{\triangle}{=} \begin{bmatrix} A_W & 0 \\ B_{c2} & A_c \end{bmatrix}, \qquad \hat{B} \stackrel{\triangle}{=} \begin{bmatrix} B_W \\ B_{c1} \end{bmatrix}, \qquad \hat{C} \stackrel{\triangle}{=} [\,0 \quad C_c\,] \tag{38}$$

where A_c, B_{c1}, B_{c2}, and C_c are chosen such that (34) and (35) stabilize (32) and (33).

Proof: First, we show that the augmented system (32)–(33) with $w = 0$ is stabilizable and detectable. Let $s \in \mathcal{U}$ and $\lambda \in \operatorname{spec}(A_w) \subset \mathcal{U}$. Since (A, B) is stabilizable, it follows that

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} A - sI & B & 0 \\ B_W C & B_W D & A_W - sI \end{bmatrix} \\ &\quad\ge \operatorname{rank}\begin{bmatrix} A - \lambda I & B & 0 \\ B_W C & B_W D & A_W - \lambda I \end{bmatrix} \\ &\quad\ge \operatorname{rank}\left(\begin{bmatrix} I_n & 0 & 0 \\ 0 & B_W & A_W - \lambda I \end{bmatrix}\begin{bmatrix} A - \lambda I & B & 0 \\ C & D & 0 \\ 0 & 0 & I_{l_y n_w} \end{bmatrix}\right). \end{aligned} \tag{39}$$

Conditions (2) and (3) imply that

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B & 0 \\ C & D & 0 \\ 0 & 0 & I_{l_y n_w} \end{bmatrix} = n + l_y + l_y n_w$$

which is full row rank. Therefore,

$$n + l_y n_w \geq \operatorname{rank}\begin{bmatrix} A-sI & B & 0 \\ B_W C & B_W D & A_W - sI \end{bmatrix}$$

$$\geq \operatorname{rank}\begin{bmatrix} I_n & 0 & 0 \\ 0 & B_W & A_W - \lambda I \end{bmatrix}. \quad (40)$$

Since (A_W, B_W) is controllable, $\operatorname{rank}\begin{bmatrix} I_n & 0 & 0 \\ 0 & B_W & A_W - \lambda I \end{bmatrix} = n + l_y n_w$ and thus

$$\operatorname{rank}\begin{bmatrix} A-sI & B & 0 \\ B_W C & B_W D & A_W - sI \end{bmatrix} = n + l_y n_w. \quad (41)$$

Hence,

$$\left(\begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}, \begin{bmatrix} B \\ B_W D \end{bmatrix}\right)$$

is stabilizable. Furthermore, since (A, C) is detectable, it follows that

$$\left(\begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix}, \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}\right)$$

is detectable.

Since (32) and (33) are stabilizable and detectable, there exist observer-based controllers that stabilize the augmented systems (32) and (33). Hence, consider the controllers (34) and (35) with the parameters A_c, B_{c1}, B_{c2}, C_c chosen to stabilize (32) and (33). The closed-loop system consisting of (32) and (33) and the feedback controller (34) and (35) is thus asymptotically stable and is given by (36) and (37). Furthermore, the closed-loop system (36) and (37) is in the form of the closed-loop system (20)–(22) with the controller (18) and (19), where

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

and $(\hat{A}, \hat{B}, \hat{C})$ are given by (38). Therefore, the closed-loop system (20)–(22) with (38) is asymptotically stable.

Next, we show that $\lim_{t\to\infty} z(t) = 0$. Define

$$T \triangleq \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & I_{n+l_y n_w} \\ 0 & I_{l_y n_w} & 0 \end{bmatrix} \quad (42)$$

and consider the change of basis $\bar{x} = T\tilde{x}$. In the new basis, the closed-loop system (20)–(22) with (38) has the form

$$\mathcal{D}\bar{x} = \bar{A}\bar{x} + \bar{D}w \quad (43)$$

$$z = \bar{E}\bar{x} + E_0 w \quad (44)$$

where

$$\bar{A} \triangleq T\tilde{A}T^{-1} = \begin{bmatrix} A & BC_c & 0 \\ B_{c1}C & A_c + B_{c1}DC_c & B_{c2} \\ B_W C & B_W DC_c & A_W \end{bmatrix} \quad (45)$$

$$\bar{D} \triangleq T\tilde{D} = \begin{bmatrix} D_1 \\ B_{c1}D_2 \\ B_W D_2 \end{bmatrix} \quad (46)$$

$$\bar{E} \triangleq \tilde{E}T^{-1} = [\,E_1 \quad E_2 C_c \quad 0\,]. \quad (47)$$

Next, let

$$S \triangleq \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

be the unique solution to the Sylvester equation

$$\bar{A}S - SA_w = \bar{D}C_w. \quad (48)$$

Now, it follows from Lemma A1 with

$$F_1 = \begin{bmatrix} A & BC_c \\ B_{c1}C & A_c + B_{c1}DC_c \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ B_{c2} \end{bmatrix}, \quad F_3 = A_w,$$

$$G = B_w, \quad H = [\,C \quad DC_c\,], \quad J = \begin{bmatrix} D_1 C_w \\ B_{c1}D_2 C_w \end{bmatrix},$$

and $K = D_2 C_w$ that $[\,C \quad DC_c\,]S_1 = D_2 C_w$. Condition (4) implies that there exists $R \in \mathbb{R}^{l_z \times l_y}$ such that $E_1 = RC, E_2 = RD$, and $E_0 = RD_2$. Therefore, $[\,E_1 \quad E_2 C_c\,]S_1 = E_0 C_w$, or equivalently

$$\bar{E}S = E_0 C_w. \quad (49)$$

Thus, there exists S satisfying (48) and (49), and Lemma 3.1 implies that $\lim_{t\to\infty} z(t) = 0$.

V. Discussion of Necessity

Theorem 4.1 provides sufficient conditions for the existence of a linear time-invariant controller that stabilizes (1)–(3) in the differential, shift, and delta domains and regulates the performance z to 0. The case $z = y$ is considered in [5]–[7], where it is claimed that conditions (1)–(3) are necessary and sufficient for continuous-time systems. However, it is possible to construct examples for which (2) and (3) are not necessary. For example, consider the SISO disturbance rejection problem

$$y = G(s)(u + w), \qquad G(s) \triangleq \frac{s^2 + \alpha^2}{p(s)} \quad (50)$$

where $z = y$, $\alpha \in \mathbb{R}$, $\deg p(s) \geq 2$, and $p(s)$ does not have roots at $\pm \jmath\alpha$. Furthermore, assume that w is the output of the linear system (6) and (7), where A_w has the characteristic polynomial $p_w(s) = s^2 + \alpha^2$. Therefore, for every minimal realization of $G(s)$, condition (3) does not hold since $\pm\jmath\alpha$ are eigenvalues of A_w and zeros of $G(s)$. However, consider the feedback controller $u = -\hat{G}(s)y = -(\hat{q}(s)/\hat{p}(s))y$, where $\hat{q}(s)$ and $\hat{p}(s)$ are selected so that $\tilde{p}(s) \triangleq p_w(s)\hat{q}(s) + p(s)\hat{p}(s)$ is Hurwitz. Then, the final value theorem implies that

$$\lim_{t\to\infty} z(t) = \lim_{s\to 0} s\frac{G(s)}{1 + G(s)\hat{G}(s)}\mathcal{L}(w(t))$$

$$= \lim_{s\to 0} s\frac{p_w(s)\hat{p}(s)}{\tilde{p}(s)}\frac{q_w(s)}{p_w(s)} = 0 \quad (51)$$

where $\mathcal{L}(\cdot)$ is the Laplace transform and $\mathcal{L}(w(t)) = q_w(s)/p_w(s)$. In this case, every stabilizing controller drives the performance to zero because the disturbance frequency corresponds to the zeros of the open-loop system.

Appendix A

Lemma 5.1: Let $F_1 \in \mathbb{R}^{q\times q}$, $F_2 \in \mathbb{R}^{q\times mp}$, $F_3 \in \mathbb{R}^{m\times m}$, $G \in \mathbb{R}^{m\times 1}$, $H \in \mathbb{R}^{p\times q}$, $J \in \mathbb{R}^{q\times m}$, and $K \in \mathbb{R}^{p\times m}$. Assume that

$$\operatorname{spec}\left(\begin{bmatrix} F_1 & F_2 \\ (I_p \otimes G)H & I_p \otimes F_3 \end{bmatrix}\right) \cap \operatorname{spec}(F_3) = \emptyset$$

and the pair (F_3, G) is controllable. Let

$$S \triangleq \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$$

be the unique solution to the Sylvester equation

$$\begin{bmatrix} F_1 & F_2 \\ (I_p \otimes G)H & I_p \otimes F_3 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} - \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} F_3 = \begin{bmatrix} J \\ (I_p \otimes G)K \end{bmatrix}. \tag{A1}$$

Then,

$$HS_1 = K. \tag{A2}$$

Proof: The Sylvester equation (A1) is equivalent to

$$F_1S_1 + F_2S_2 - S_1F_3 = J \tag{A3}$$

$$(I_p \otimes G)HS_1 + (I_p \otimes F_3)S_2 - S_2F_3 = (I_p \otimes G)K. \tag{A4}$$

Next let $S_2 = \begin{bmatrix} S_{2,1} \\ \vdots \\ S_{2,p} \end{bmatrix}$, where, for all $i = 1,\ldots,p$, $S_{2,i} \in \mathbb{R}^{m\times m}$. It follows from (A4) that, for all $i = 1,\ldots,p$, $F_3S_{2,i} - S_{2,i}F_3 = G\Lambda_i$, where $\Lambda_i \triangleq e_i(K - HS_1)$ and $e_i \triangleq [\, 0_{1\times(i-1)} \quad 1 \quad 0_{1\times(p-i)} \,]$.

Let $M \in \mathbb{R}^{m\times m}$ be such that $\bar{F} \triangleq M^{-1}F_3M$ is in Jordan canonical form, that is, for some $\mu \le m$, $\bar{F} = \operatorname{diag}(\bar{F}_1,\ldots,\bar{F}_\mu)$, where, for $j = 1,\ldots,\mu$,

$$\bar{F}_j \triangleq \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_j \end{bmatrix} \in \mathbb{R}^{f_j\times f_j}$$

and $\lambda_j \in \operatorname{spec}(F_3)$. Furthermore, define

$$\bar{G} \triangleq M^{-1}G = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_\mu \end{bmatrix} \tag{A5}$$

and, for $i = 1,\ldots,p$, define

$$\bar{S}_i \triangleq M^{-1}S_{2,i}M = \begin{bmatrix} \bar{S}_{i,11} & \cdots & \bar{S}_{i,1\mu} \\ \vdots & \ddots & \vdots \\ \bar{S}_{i,\mu 1} & \cdots & \bar{S}_{i,\mu\mu} \end{bmatrix} \tag{A6}$$

$$\bar{\Lambda}_i \triangleq \Lambda_i M = [\, \phi_{i,1} \quad \cdots \quad \phi_{i,m} \,] \tag{A7}$$

where, for $j = 1,\ldots,\mu$, $\bar{G}_j \in \mathbb{R}^{f_j\times 1}$, and, for $i = 1,\ldots,p$ and for $j,k = 1,\ldots,\mu$, $\bar{S}_{i,jk} \in \mathbb{R}^{f_j\times f_k}$. Therefore, for $i = 1,\ldots,p$, premultiplying $F_3S_{2,i} - S_{2,i}F_3 = G\Lambda_i$ by M^{-1} and postmultiplying by M yields

$$\bar{F}\bar{S}_i - \bar{S}_i\bar{F} = \bar{G}\bar{\Lambda}_i. \tag{A8}$$

Substituting (A5)–(A6) into (A8) and considering only the block-diagonal terms imply that, for all $i = 1,\ldots,p$ and for all $j = 1,\ldots,\mu$,

$$\bar{F}_j\bar{S}_{i,jj} - \bar{S}_{i,jj}\bar{F}_j = \bar{G}_j\bar{\Lambda}_iE_j \tag{A9}$$

where $E_j \triangleq \begin{bmatrix} 0_{(f_1+\cdots+f_{j-1})\times f_j} \\ I_{f_j} \\ 0_{(f_{j+1}+\cdots+f_\mu)\times f_j} \end{bmatrix}$ and $f_0 = 0$.

Next, for all $i = 1,\ldots,p$, and for all $j = 1,\ldots,\mu$, let $\bar{S}_{i,jj} = \begin{bmatrix} s_{ij,1,1} & \cdots & s_{ij,1,f_j} \\ \vdots & \ddots & \vdots \\ s_{ij,f_j,1} & \cdots & s_{ij,f_j,f_j} \end{bmatrix}$, so that (A10) holds, as shown at the bottom of the page.

For all $j = 1,\ldots,\mu$, let $g_j \in \mathbb{R}$ denote the last entry of $\bar{G}_j$. For all $i = 1,\ldots,p$, and all $j = 1,\ldots,\mu$, combining (A7), (A9), and (A10) yields (A11), as shown at the bottom of the page, where $\sharp$ denotes an inconsequential entry.

Now, since (F_3, G) is controllable, it follows that $(\bar{F}, \bar{G})$ is controllable, and thus, for all $j = 1,\ldots,\mu$, $(\bar{F}_j, \bar{G}_j)$ is controllable. Therefore, for all $j = 1,\ldots,\mu$, $g_j \neq 0$.

First, consider $j = 1$. Since $g_j \neq 0$, inspecting the $(f_j, 1)$ entry of (A11) yields that, for all $i = 1,\ldots,p$, $\phi_{i,1+f_1+\cdots+f_{j-1}} = 0$, and thus, for all $k = 2,\ldots,f_j$, $s_{ij,k,1} = 0$. Now, since $g_j \neq 0$ and $s_{ij,f_j,1} = 0$, inspecting the $(f_j, 2)$ entry of (A11) yields that, for all $i = 1,\ldots,p$, $\phi_{i,1+f_1+\cdots+f_{j-1}+1} = 0$, and thus, for all $k = 3,\ldots,f_j$, $s_{ij,k,2} = 0$. Now, since $g_j \neq 0$ and $s_{ij,f_j,2} = 0$, inspecting the $(f_j, 3)$ entry of (A11) yields that, for all $i = 1,\ldots,p$, $\phi_{i,1+f_1+\cdots+f_{j-1}+2} = 0$, and thus, for all $k = 4,\ldots,f_j$, $s_{ij,k,3} = 0$. Continuing in this manner yields, for all $i = 1,\ldots,p$, $[\, \phi_{i,1+f_1+\cdots+f_{j-1}} \quad \cdots \quad \phi_{i,f_1+\cdots+f_j} \,] = 0$. Repeating this for all $j = 2,\ldots,\mu$ yields, for all $i = 1,\ldots,p$, $\bar{\Lambda}_i = 0$, which implies that $HS_1 - K = 0$, thus proving (A2). ■

$$\bar{F}_j\bar{S}_{i,jj} - \bar{S}_{i,jj}\bar{F}_j = \begin{bmatrix} s_{ij,2,1} & s_{ij,2,2} - s_{ij,1,1} & s_{ij,2,3} - s_{ij,1,2} & \cdots & s_{ij,2,f_j} - s_{ij,1,f_j-1} \\ s_{ij,3,1} & s_{ij,3,2} - s_{ij,2,1} & s_{ij,3,3} - s_{ij,2,2} & \cdots & s_{ij,3,f_j} - s_{ij,2,f_j-1} \\ s_{ij,4,1} & s_{ij,4,2} - s_{ij,3,1} & s_{ij,4,3} - s_{ij,3,2} & \cdots & s_{ij,4,f_j} - s_{ij,3,f_j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{ij,f_j,1} & s_{ij,f_j,2} - s_{ij,f_j-1,1} & s_{ij,f_j,3} - s_{ij,f_j-1,2} & \cdots & s_{ij,f_j,f_j} - s_{ij,f_j-1,f_j-1} \\ 0 & -s_{ij,f_j,1} & -s_{ij,f_j,2} & \cdots & -s_{ij,f_j,f_j-1} \end{bmatrix} \tag{A10}$$

$$\begin{aligned} &\begin{bmatrix} s_{ij,2,1} & s_{ij,2,2} - s_{ij,1,1} & s_{ij,2,3} - s_{ij,1,2} & \cdots & s_{ij,2,f_j} - s_{ij,1,f_j-1} \\ s_{ij,3,1} & s_{ij,3,2} - s_{ij,2,1} & s_{ij,3,3} - s_{ij,2,2} & \cdots & s_{ij,3,f_j} - s_{ij,2,f_j-1} \\ s_{ij,4,1} & s_{ij,4,2} - s_{ij,3,1} & s_{ij,4,3} - s_{ij,3,2} & \cdots & s_{ij,4,f_j} - s_{ij,3,f_j-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{ij,f_j,1} & s_{ij,f_j,2} - s_{ij,f_j-1,1} & s_{ij,f_j,3} - s_{ij,f_j-1,2} & \cdots & s_{ij,f_j,f_j} - s_{ij,f_j-1,f_j-1} \\ 0 & -s_{ij,f_j,1} & -s_{ij,f_j,2} & \cdots & -s_{ij,f_j,f_j-1} \end{bmatrix} \\ &\quad = \begin{bmatrix} \sharp \\ \vdots \\ \sharp \\ g_j \end{bmatrix} [\, \phi_{i,1+f_1+\cdots+f_{j-1}} \quad \cdots \quad \phi_{i,f_1+\cdots+f_j} \,] \end{aligned} \tag{A11}$$

ACKNOWLEDGMENT

We would like to thank the Associate Editor and reviewers for helpful comments.

REFERENCES

[1] C. D. Johnson, "Accommodation of external disturbances in linear regulator and servomechanism problems," *IEEE Trans. Autom. Control*, vol. 16, no. 6, pp. 635–644, Dec. 1971.
[2] E. J. Davison and H. W. Smith, "Pole assignment in linear time-invariant multivariable systems with constant disturbances," *Automatica*, vol. 7, pp. 489–498, 1971.
[3] P. C. Young and J. C. Willems, "An approach to the multivariable servomechanism problem," *Int. J. Control*, vol. 15, pp. 961–979, 1972.
[4] E. J. Davison, "The output control of linear time-invariant multivariable systems with unmeasurable arbitrary disturbances," *IEEE Trans. Autom. Control*, vol. 17, no. 5, pp. 621–630, Oct. 1972.
[5] E. J. Davison, "A generalization of the output control of linear multivariable systems with unmeasurable arbitrary disturbances," *IEEE Trans. Autom. Control*, vol. 20, no. 6, pp. 788–792, Dec. 1975.
[6] E. J. Davison and A. Goldenberg, "Robust control of a general servomechanism problem: The servo compensator," *Automatica*, vol. 11, pp. 461–471, 1975.
[7] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Autom. Control*, vol. AC-21, no. 1, pp. 25–34, Feb. 1976.
[8] B. A. Francis, A. Sebakhy, and W. M. Wonham, "Synthesis of multivariable regulators: The internal model principle," *J. Appl. Math. Optim.*, vol. 1, pp. 64–86, 1974.
[9] B. A. Francis and W. M. Wonham, "The internal model principle for linear multivariable regulators," *J. Appl. Math. Optim.*, vol. 2, pp. 170–194, 1975.
[10] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*. New York: Academic, 1974.
[11] G. Bengtsson, "Output regulation and internal models—A frequency domain approach," *Automatica*, vol. 13, pp. 333–345, 1977.
[12] K. J. Åström and B. Wittenmark, *Adaptive Control*. 2nd ed. Reading, MA: Addison-Wesley, 1995.
[13] A. G. Sparks and D. S. Bernstein, "Optimal tradeoff between H_2 performance and tracking accuracy in servocompensator synthesis," *AIAA J. Guid. Control Dyn.*, vol. 18, pp. 1239–1243, 1995.
[14] A. G. Sparks and D. S. Bernstein, "Optimal rejection of stochastic and deterministic disturbances," *ASME J. Dyn. Sys. Meas. Control*, vol. 119, pp. 140–143, 1997.
[15] R. H. Middleton and G. C. Goodwin, *Digital Control and Estimation: A Unified Approach*. Englewood Cliffs, NJ: Prentice-Hall, 1990.
[16] J. B. Hoagg and D. S. Bernstein, "Deadbeat internal model control for command following and disturbance rejection for discrete-time systems," in *Proc. Amer. Control Conf.*, Minneapolis, MN, 2006, pp. 194–199.
[17] D. S. Bernstein, *Matrix Mathematics*. Princeton, NJ: Princeton Univ. Press, 2005.

Variance Analysis of a Cross-Covariance Matching Method for Continuous-Time ARX Parameter Estimation

Magnus Mossberg, *Senior Member, IEEE*

***Abstract*—A method for estimating the parameters of a continuous-time autoregressive exogenous process from discrete-time data is analyzed. The method consists of fitting an expression for the cross-covariance function, parameterized by the unknown parameters, to sample cross-covariances. The main contribution of the note is the derivation of an approximate expression for the covariance matrix of the estimated parameter vector.**

***Index Terms*—Continuous-time ARX, covariance matrix, cross-covariance function, estimation.**

I. INTRODUCTION

The autoregressive exogenous (ARX) model is often used in model-based control design of discrete-time stochastic systems. In the same way, the continuous-time ARX (CARX) model is a useful standard model for the continuous-time case. The CARX model is defined as

$$A(p)y(t) = B(p)u(t) + e(t) \tag{1}$$

where $A(p) = p^n + a_1 p^{n-1} + \cdots + a_n$, $B(p) = b_1 p^{n-1} + \cdots + b_n$, and $\mathrm{E}\{e(t)e(s)\} = \sigma_e^2 \delta(t-s)$. Here, p denotes the differentiation operator, $y(t)$ is the output signal, $u(t)$ is the input signal, $e(t)$ is a continuous-time white noise source, and $\delta(\cdot)$ is the Dirac delta function. The output signal $y(t)$ can be expressed as the sum of a deterministic term $y_d(t)$ and a stochastic term $y_s(t)$

$$y(t) = \frac{B(p)}{A(p)}u(t) + \frac{1}{A(p)}e(t) \stackrel{\Delta}{=} y_d(t) + y_s(t)$$

where $y_s(t)$ is well defined, $n-1$ times differentiable, and given by a stochastic differential equation. Also, note that the spectral density $\phi_{y_s}(\omega) = \sigma_e^2/|A(i\omega)|^2$ of $y_s(t)$ is modeled well using description (1). The problem studied here is to estimate the CARX parameters

$$\boldsymbol{\theta}_0 = [a_1 \quad \cdots \quad a_n \quad b_1 \quad \cdots \quad b_n]^T$$

from the discrete-time data $\mathcal{D} = \{u(kh), y(kh)\}_{k=1}^N$, where h denotes the sampling interval. The problem is treated in [1], where the derivatives are approximated by carefully chosen discrete-time differences. This gives a linear regression from which the parameter vector can be obtained by the least squares method. The bias given by the method is proportional to the sampling interval. The approach is extended to the case of irregularly sampled data in [2]. In [3], a method based on sample cross-covariances is presented. The main idea of the method is to fit an expression, parameterized by the unknown parameters, for the cross-covariance function between $y(t)$ and $u(t)$ to a cross-covariance function estimated from $\mathcal{D}$. This note derives an approximate covariance matrix of the estimated parameter vector given by the method presented in [3], and it is shown that the method is consistent. A possible advantage with the method is robustness to measurement noise, since a cross-covariance function can be estimated with high accuracy from large data sets even in the presence of measurement noise. More material on estimation of continuous-time stochastic system parameters from discrete-time data can be found in [4] and the references therein.

Manuscript received December 5, 2006; revised August 27, 2007. Recommended by Associate Editor J.-F. Zhang.
The author is with the Department of Electrical Engineering, Karlstad University, SE-651 88 Karlstad, Sweden (e-mail: magnus.mossberg@kau.se).
Digital Object Identifier 10.1109/TAC.2008.919554