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EXPLICIT CONSTRUCTION OF QUADRATIC LYAPUNOV FUNCTIONS FOR THE SMALL GAIN, POSITIVITY, CIRCLE, AND POPOV THEOREMS AND THEIR APPLICATION TO ROBUST STABILITY. PART I: CONTINUOUS-TIME THEORY

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SUMMARY

The purpose of this paper is to construct Lyapunov functions to prove the key fundamental results of linear system theory, namely, the small gain (bounded real), positivity (positive real), circle, and Popov theorems. For each result a suitable Riccati-like matrix equation is used to explicitly construct a Lyapunov function that guarantees asymptotic stability of the feedback interconnection of a linear time-invariant system and a memoryless nonlinearity. Lyapunov functions for the small gain and positivity results are also constructed for the interconnection of two transfer functions. A multivariable version of the circle criterion, which yields the bounded real and positive real results as limiting cases, is also derived. For a multivariable extension of the Popov criterion, a Lure–Postnikov Lyapunov function involving both a quadratic term and an integral of the nonlinearity, is constructed. Each result is specialized to the case of linear uncertainty for the problem of robust stability. In the case of the Popov criterion, the Lyapunov function is a parameter-dependent quadratic Lyapunov function.

KEY WORDS Parameter-dependent Lyapunov functions Small gain Circle theorem Popov criterion

1. INTRODUCTION

One of the most basic issues in system theory is stability of feedback interconnections. Two of the most fundamental results concerning stability of feedback systems are the small-gain theorem and the positivity theorem.[1–14] Here we focus (in Sections 3 and 4) on the sufficiency aspect of these results. The small gain theorem implies that if G and G_c are asymptotically stable bounded-gain transfer functions such that $\| G \|_\infty \| G_c \|_\infty < 1$, then the feedback interconnection of G and G_c is asymptotically stable. Furthermore, the positivity theorem states that if G and G_c are (square) positive real transfer functions, one of which is strictly positive real, then the negative feedback interconnection of G and G_c is asymptotically stable.

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For robust stability, if G_c represents an uncertain perturbation, then it follows from the small gain theorem that an H_∞-norm bound on G implies robust stability in the presence of an H_∞-norm bound on G_c. Similarly, if the system uncertainty G_c can be cast as a positive real transfer function and G is strictly positive real, then the positivity theorem implies robust stability. Although the small gain theorem and positivity theorem are equivalent via the bilinear transformation,[5] positive real modelling of system uncertainty can be significantly less conservative than small gain modelling of system uncertainty.[15] This improvement is due to the fact that the small gain theorem is a norm-based result which captures gain uncertainty but ignores phase information. Since positive real transfer functions are phase bounded, the positivity theorem can exploit phase characteristics within a feedback interconnection.

Although the predominant approach to stability theory is Lyapunov's method,[4,11,16–25] most of the available proofs of the small gain and positivity theorems are based upon input–output properties and function-analytic methods.[1–3,5,7,12,26,27] In this paper we explicitly construct quadratic Lyapunov functions to prove sufficiency in special cases of these results. Specifically, sufficient conditions for asymptotic stability are obtained for a proper, but not necessarily strictly proper, bounded real (respectively, strongly positive real) transfer function in a positive feedback (respectively, negative feedback) configuration with a bounded real (respectively, positive real) time-varying memoryless nonlinearity. Specialization of these results to robust stability with *linear* time-varying bounded real and positive real (but otherwise unknown) plant uncertainty is also discussed.

Having addressed the small gain and positivity theorems, we then turn our attention (in Section 5) to the well-known circle criterion or circle theorem.[8,9,11,23,28–32] In a multivariable setting this result applies to sector-bounded nonlinearities and thus, upon appropriate specialization, generalizes (and includes as limiting cases) both the small gain and positivity results. Thus, for practical purposes, the circle theorem provides the means for incorporating both gain and phase aspects. The proof of the circle theorem given here is completely consistent with the proofs of the small gain and positivity results, thus providing a unified treatment of all these classical results.

Next we focus (in Section 6) on the Popov stability criterion.[33–42] Although often discussed in juxtaposition with the circle criterion, the Popov criterion is fundamentally distinct from the circle criterion with regard to its Lyapunov function foundation. Whereas the small gain, positivity, and circle results are based upon fixed quadratic Lyapunov functions, the Popov result is based upon a quadratic Lyapunov function that is a *function* of the sector-bounded nonlinearity. Thus, in effect, the Popov result guarantees stability by means of a *family* of Lyapunov functions. For robust stability, this situation corresponds to the construction of a parameter-dependent quadratic Lyapunov function as proposed in References 43 and 44. A key aspect of the Popov result is the fact that it does *not* apply to arbitrary time-varying uncertainties, which renders it less conservative than fixed quadratic Lyapunov function results (such as the small gain, positivity, and circle results) in the presence of real, constant parameter uncertainty.

Our proof of the Popov criterion is given in a form that is similar to the proofs of the small gain, positivity, and circle theorems. This unified presentation is intended to clarify relationships among these results. In Section 7 we return to the small gain and positivity theorems and consider the interconnection of two strictly proper dynamic systems. In each case a Lyapunov function is constructed to guarantee stability of the closed-loop system.

There are two main reasons for seeking Lyapunov-function proofs of the small gain, positivity, circle, and Popov theorems. First, these proofs help to build stronger ties between state-space and frequency-domain approaches to feedback system theory. And, second, these quadratic Lyapunov functions provide an algebraic basis in terms of matrix Riccati equations for the synthesis of robust feedback controllers.[45-54]

Although a general and unifying stability theory of nonlinear feedback interconnections using the concepts of passivity, dissipativeness, and non-expansivity is available (see References 55–62 for an excellent exposition of this subject), our aim is to construct explicit Lyapunov functions in terms of *single* algebraic Riccati equations that can be used for the synthesis of robust controllers. A reinterpretation of our results in terms of quadratic Lyapunov bounds consistent with the framework provided in Reference 50 is given in Section 8. This allows us to make explicit connections of these classical results with robust stability and H_2 performance analysis for state-space systems via quadratic fixed and parameter-dependent Lyapunov bounds in the spirit of Reference 50. It should be noted that the *stability* results presented in the present paper could be derived an special cases of energy storage functions of dissipative dynamical systems.[55-62] In fact Hill and Moylan[59] give a characterization of passivity for a broad class of nonlinear systems of the form

$$\dot{x} = F(x) + G(x)u,$$
$$y = H(x) + J(x)u.$$

However, no connections between the dissipative dynamical systems input–output approach and robust stability and H_2 performance for state-space systems are made in the above references. In this paper we provide such connections, give explicit uncertainty structure characterizations for the state-space models, and provide explicit uncertainty bounds in terms of single Riccati equations that can effectively be used for robust controller synthesis.

Although the results of this paper are confined to continuous-time systems, analogous results for discrete-time systems are given in Reference 54.

2. PRELIMINARIES

In this section we establish definitions and notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, let $()^{\mathrm{T}}$ denote transpose, and let I_n or I denote the $n \times n$ identity matrix. Furthermore, we write $\|\cdot\|_2$ for Euclidean norm, $\sigma_{\max}(\cdot)$ for the maximum singular value, and $M \geqslant 0$ ($M > 0$) to denote the fact that the Hermitian matrix M is nonnegative (positive) definite. In this paper a *real-rational matrix function* is a matrix whose elements are rational functions with real coefficients. Furthermore, a *transfer function* is a real-rational matrix function each of whose elements is *proper*, i.e., finite at $s = \infty$. A *strictly proper transfer function* is a transfer function that is zero at infinity. Finally, an *asymptotically stable transfer function* is a transfer function each of whose poles is in the open left half-plane. The space of asymptotically stable transfer functions is denoted by RH_∞, i.e., the real-rational subset of H_∞.[14] Let

$$G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

denote a state-space realization of a transfer function $G(s)$, that is, $G(s) = C(sI - A)^{-1}B + D$. The notation '$\overset{\min}{\sim}$' is used to denote a minimal realization. In addition, the parahermitian conjugate $G^{\sim}(s)$ of $G(s)$ has the realization

$$G^{\sim}(s) \sim \left[\begin{array}{c|c} -A^{\mathrm{T}} & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}} & D^{\mathrm{T}} \end{array}\right].$$

A transfer function $G(s)$ is *bounded real*[63,64] if (1) $G(s)$ is asymptotically stable and (2) $I-G^*(j\omega)G(j\omega)$ is nonnegative definite for all real ω. Equivalently, (2) can be replaced by (Reference 64, p. 307)(2′) $I-G^*(s)G(s)$ is nonnegative definite for $\mathrm{Re}[s]\geqslant 0$. Alternatively, a transfer function $G(s)$ is bounded real if and only if $G(s)$ is asymptotically stable and $\|G(s)\|_\infty \leqslant 1$. Furthermore, $G(s)$ is called *strictly bonded real* if (1) $G(s)$ is asymptotically stable and (2) $I-G^*(j\omega)G(j\omega)$ is positive definite for all real ω. Finally, note that if $G(s)$ is strictly bounded real (i.e., $\|G(s)\|_\infty < 1$) then $I-D^{\mathrm{T}}D>0$, where $D\triangleq G(\infty)$.

A square transfer function $G(s)$ is called *positive real* (Reference 64, p. 216) if(1) all poles of $G(s)$ are in the closed left half-plane and (2) $G(s)+G^*(s)$ is nonnegative definite for $\mathrm{Re}[s]>0$. A square transfer function $G(s)$ is called *strictly positive real*[65,66] if (1) $G(s)$ is asymptotically stable and (2) $G(j\omega)+G^*(j\omega)$ is positive definite for all real ω. Finally, a square transfer function $G(s)$ is *strongly positive real* if it is strictly positive real and $D+D^{\mathrm{T}}>0$, where $D\triangleq G(\infty)$. Recall that the minimal realization of a positive real transfer function is stable in the sense of Lyapunov.[67] Furthermore, strongly positive real implies strictly positive real, which further implies positive real.

For notational convenience in the paper, G will denote an $l\times m$ transfer function with input $u\in\mathbb{R}^m$, output $y\in\mathbb{R}^l$, and internal state $x\in\mathbb{R}^n$. Next we give two key lemmas concerning bounded real and positive real matrices.

Lemma 2.1

Let $M\in\mathbb{C}^{l\times m}$ and $N\in\mathbb{C}^{m\times l}$ be such that $\sigma_{\max}(M)\leqslant 1$ and $\sigma_{\max}(N)<1$. Then $\det[I_l-MN]\neq 0$.

Proof. Since $\sigma_{\max}(M)\leqslant 1$ and $\sigma_{\max}(N)<1$ it follows that $\rho(MN)\leqslant\sigma_{\max}(MN)\leqslant\sigma_{\max}(M)\sigma_{\max}(N)<1$, where $\rho(\cdot)$ denotes spectral radius. Hence $\det[I_l-MN]\neq 0$. □

Lemma 2.2

Let $M,N\in\mathbb{C}^{m\times m}$ be such that $M+M^*\geqslant 0$ and $N+N^*>0$. Then $\det(I_m+MN)\neq 0$.

Proof. First we show that N is invertible. Let $x\in\mathbb{C}^m$, $x\neq 0$, and $\lambda\in\mathbb{C}$ be such that $Nx=\lambda x$ and hence $x^*N^*=\bar{\lambda}x^*$ Then $x^*(N+N^*)x>0$ implies that $\mathrm{Re}\,\lambda>0$. Hence $\det N\neq 0$. Now define $S\triangleq N^{-1}+M$. Since $N^{-1}+N^{-*}=N^{-1}(N+N^*)N^{-*}$ it follows that $S+S^*>0$. Thus $\det S\neq 0$. Consequently, $\det(I_m+MN)=\det NS=(\det N)(\det S)\neq 0$. □

3. THE SMALL GAIN THEOREM

In this section we construct quadratic Lyapunov functions to prove sufficiency in the small gain theorem for the interconnection of a dynamic system and a static feedback gain. First, recall the bounded real lemma.[64]

Lemma 3.1 (Bounded real lemma)

$$G(s)\overset{\min}{\sim}\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$$

is bounded real if and only if there exist real matrices P, L, and W with P positive definite such that

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L, \tag{1}$$

$$0 = B^{\mathrm{T}}P + D^{\mathrm{T}}C + W^{\mathrm{T}}L, \tag{2}$$

$$0 = I - D^{\mathrm{T}}D - W^{\mathrm{T}}W. \tag{3}$$

Proof. Sufficiency follows from algebraic manipulation of (1)–(3) while necessity follows from spectral factorization theory. For details see Reference 64. □

Remark 3.1

If (1) is replaced by

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L + R, \tag{1'}$$

where $R \geqslant 0$, then (1)′–(3) imply that $G(s)$ is bounded real.

Suppose in Lemma 3.1 that $\sigma_{\max}(D) < 1$. Then since $I - D^{\mathrm{T}}D > 0$ and

$$W^{\mathrm{T}}W = I - D^{\mathrm{T}}D, \tag{4}$$

it follows that $W^{\mathrm{T}}W$ is nonsingular. Furthermore, (2) is equivalent to

$$W^{\mathrm{T}}L = -(B^{\mathrm{T}}P + D^{\mathrm{T}}C). \tag{5}$$

Using (5) and noting that $W(W^{\mathrm{T}}W)^{-1}W^{\mathrm{T}}$ is an orthogonal projection so that $I \geqslant W(W^{\mathrm{T}}W)^{-1} W^{\mathrm{T}}$ and hence $L^{\mathrm{T}}L \geqslant L^{\mathrm{T}}W(W^{\mathrm{T}}W)^{-1}W^{\mathrm{T}}L$, if follows from (1) that

$$0 \geqslant A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(W^{\mathrm{T}}W)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C \tag{6}$$

or, since $(W^{\mathrm{T}}W)^{-1} = (I - D^{\mathrm{T}}D)^{-1}$,

$$0 \geqslant A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C. \tag{7}$$

Thus, in this case conditions (1)–(3) are equivalent to the single Riccati inequality (7). The following result characterizes the bounded real property in terms of a Riccati equation.

Lemma 3.2

Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following statements are equivalent:

(i) A is asymptotically stable and $G(s)$ is strictly bounded real;
(ii) $I - D^{\mathrm{T}}D > 0$ and there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C + R. \tag{8}$$

Proof. (i) $\Rightarrow$ (ii): Using Proposition 3.2 of Reference 52 (i) implies that $I - D^{\mathrm{T}}D > 0$ and

$$I - B'^{\mathrm{T}}(-\mathrm{j}\omega I - A')^{-\mathrm{T}}C'^{\mathrm{T}}C'(\mathrm{j}\omega I - A')^{-1}B' > 0, \qquad \omega \in \mathbb{R}, \tag{9}$$

where

$$A' \triangleq A + BD^{\mathrm{T}}M^{-1}C, \tag{10}$$

$$B' \triangleq BN^{-1/2}, \tag{11}$$

$$C' \triangleq M^{-1/2}C, \tag{12}$$

$$M \triangleq I - DD^{\mathrm{T}}, \tag{13}$$

$$N \triangleq I - D^{\mathrm{T}}D. \tag{14}$$

Next, let $\varepsilon > 0$ be such that

$$I - B'^{\mathrm{T}}(-\mathrm{j}\omega I - A')^{-\mathrm{T}}(C'^{\mathrm{T}}C' + \varepsilon I)(\mathrm{j}\omega I - A')^{-1}B' > 0, \quad \omega \in \mathbb{R}. \tag{15}$$

Since $A' - B'N^{-1/2}D^{\mathrm{T}}C = A$ (note $MD = DN$) and A is asymptotically stable, it follows that (A', B') is stabilizable. Since also $(A', C'^{\mathrm{T}}C' + \varepsilon I)$ is observable, it follows from Lemma 5 of Reference 68 that there exists a real symmetric matrix P such that

$$0 = A'^{\mathrm{T}}P + PA' + PB'B'^{\mathrm{T}}P + C'^{\mathrm{T}}C' + \varepsilon I, \tag{16}$$

or, equivalently, using (10)–(14)

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C + \varepsilon I. \tag{17}$$

Now, since A is assumed to be asymptotically stable and $(B^{\mathrm{T}}P + D^{\mathrm{T}}C)\ (I - D^{\mathrm{T}}D)^{-1}$ $(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C + \varepsilon I > 0$, it follows from Lyapunov theory that $P > 0$, which establishes the existence of a positive definite matrix P satisfying (8).

(ii) ⇒ (i): Suppose (ii) holds. Note that

$$I - G^{\sim}(s)G(s) \sim \left[\begin{array}{cc|c} -A^{\mathrm{T}} & C^{\mathrm{T}}C & C^{\mathrm{T}}D \\ 0 & A & B \\ \hline B^{\mathrm{T}} & -D^{\mathrm{T}}C & I - D^{\mathrm{T}}D \end{array}\right]. \tag{18}$$

Next, it follows from (8) that there exists a positive definite matrix R such that

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C + R. \tag{19}$$

Applying the state-space transformation

$$S = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} \tag{20}$$

to (18) yields

$$I - G^{\sim}(s)G(s) \sim \left[\begin{array}{cc|c} A^{\mathrm{T}} & A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C & (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}} \\ 0 & A & B \\ \hline B^{\mathrm{T}} & -(B^{\mathrm{T}}P + D^{\mathrm{T}}C) & I - D^{\mathrm{T}}D \end{array}\right]$$

or, equivalently, using (19),

$$I - G^{\sim}(s)G(s)$$

$$\sim \left[\begin{array}{cc|c} -A^{\mathrm{T}} & -(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) - R & (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}} \\ 0 & A & B \\ \hline B^{\mathrm{T}} & -(B^{\mathrm{T}}P + D^{\mathrm{T}}C) & I - D^{\mathrm{T}}D \end{array}\right]$$

$$= N^{\sim}(s)N(s) + (I - D^{\mathrm{T}}D) - (B^{\mathrm{T}}P + D^{\mathrm{T}}C)E^{-2}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}},$$

where

$$E \triangleq [(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + R]^{1/2} > 0$$

and

$$N(s) \sim \left[\begin{array}{c|c} A & B \\ \hline E & -E^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}} \end{array}\right].$$

Noting that $(I - D^{\mathrm{T}}D) - (B^{\mathrm{T}}P + D^{\mathrm{T}}C)E^{-2}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}} > 0$ it follows that $I - G^*(\mathrm{j}\omega)G(\mathrm{j}\omega) > 0$, $\omega \in \mathbb{R}$. Next, note that since $P > 0$ and $(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C + R > 0$ it follows from (19) that A is asymptotically stable. □

Now we prove sufficiency of the small gain theorem for the feedback interconnection of a strictly bounded real transfer function and a norm-bounded memoryless time-varying nonlinearity. For convenience define the set

$$\Phi_{\mathrm{br}} \triangleq \{\phi: \mathbb{R}^l \times \mathbb{R}^+ \to \mathbb{R}^m : \|\phi(y,t)\|_2 \leqslant \|y\|_2, \quad y \in \mathbb{R}^l, \\ \text{a.a. } t \geqslant 0, \text{ and } \phi(y,\cdot) \text{ is Lebesgue measurable for all } y \in \mathbb{R}^l\}.$$

Theorem 3.1

Suppose

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is strictly bounded real. Then there exist positive-definite matrices P and R satisfying

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + C^{\mathrm{T}}C + R. \tag{21}$$

Furthermore, for all $\phi \in \Phi_{\mathrm{br}}$, the function $V(x) = x^{\mathrm{T}}Px$ is a Lyapunov function for the feedback interconnection of $G(s)$ and ϕ. Consequently, the feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi \in \Phi_{\mathrm{br}}$.

Proof. First note that the feedback interconnection of $G(s)$ and ϕ corresponds to the state-space representation

$$\dot{x}(t) = Ax(t) + B\phi(y,t), \tag{22}$$

$$y(t) = Cx(t) + D\phi(y,t). \tag{23}$$

Since $G(s)$ is strongly bounded real it follows from Lemma 3.2 that there exist positive-definite matrices P and R such that (21) is satisfied. Next, we use the Lyapunov candidate $V(x) = x^{\mathrm{T}}Px$ to show that the feedback interconnection (22), (23) is asymptotically stable. The corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + \phi^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}PB\phi \tag{24}$$

or, equivalently, using (21)

$$\begin{aligned} \dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)^{\mathrm{T}}(I - D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)x \\ &- x^{\mathrm{T}}C^{\mathrm{T}}Cx + \phi^{\mathrm{T}}B^{\mathrm{T}}Px + x^{\mathrm{T}}PB\phi. \end{aligned} \tag{25}$$

Next, add and subtract $\phi^T\phi$, $2x^TC^TD\phi$, and $\phi^TD^TD\phi$ to and from (25) so that

$$\begin{aligned}\dot{V}(x) = &-x^TRx - x^T(B^TP + D^TC)^T(I - D^TD)^{-1}(B^TP + D^TC)x - x^TC^TCx \\ &+ \phi^TB^TPx + x^TPB\phi + \phi^T\phi - \phi^T\phi + x^TC^TD\phi + \phi^TD^TCx \\ &- x^TC^TD\phi - \phi^TD^TCx + \phi^TD^TD\phi - \phi^TD^TD\phi\end{aligned} \tag{26}$$

or, equivalently,

$$\begin{aligned}\dot{V}(x) = &-x^TRx - x^T(B^TP + D^TC)^T(I - D^TD)^{-1}(B^TP + D^TC)x \\ &+ x^T(B^TP + D^TC)^T\phi + \phi^T(B^TP + D^TC)x - \phi^T(I - D^TD)\phi \\ &+ \phi^T\phi - x^TC^TCx - \phi^TD^TD\phi - x^TC^TD\phi - \phi^TD^TCx.\end{aligned} \tag{27}$$

Grouping the appropriate terms in (27) yields

$$\dot{V}(x) = -x^TRx - z^Tz + \phi^T\phi - y^Ty, \tag{28}$$

where

$$z \triangleq (I - D^TD)^{-1/2}(B^TP + D^TC)x - (I - D^TD)^{1/2}\phi.$$

Since R is positive definite and $\phi^T\phi - y^Ty \leqslant 0$ for all $\phi \in \Phi_{br}$ it follows that $\dot{V}(x)$ is negative definite. Hence $V(x)$ is a Lyapunov function for the feedback interconnection of $G(s)$ and ϕ. □

Next, we specialize Theorem 3.1 to the feedback interconnection of a strictly bounded real transfer function and a linear bounded real gain. Hence consider the set $\mathscr{F}_{br}$ defined by

$$\mathscr{F}_{br} \triangleq \{F:\mathbb{R}^+ \to \mathbb{R}^{m\times l}: F(\cdot) \text{ is Lebesgue measurable and } \sigma_{max}(F(t)) \leqslant 1, \text{ a.a. } t \geqslant 0\}.$$

That is, $\mathscr{F}_{br}$ includes those ϕ in Φ_{br} of the form $\phi(y, t) = F(t)y$. The following corollary of Theorem 3.1 is thus immediate.

Corollary 3.1

If

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$$

is strictly bounded real, then the feedback interconnection of $G(s)$ and $F(\cdot)$ is asymptotically stable for all $F(\cdot) \in \mathscr{F}_{br}$.

Corollary 3.1 implies that $A + BF(\cdot)(I - DF(\cdot))^{-1}C$ is asymptotically stable in the sense that the zero solution of the linear time-varying system

$$\dot{x}(t) = (A + BF(t)(I - DF(t))^{-1}C)x(t) \tag{29}$$

is asymptotically stable. Recall from Lemma 2.1 that $(I - DF(t))^{-1}$ exists a.a. $t \geqslant 0$ since $\sigma_{max}(D) < 1$ and $\sigma_{max}(F(t)) \leqslant 1$, a.a. $t \geqslant 0$. This result thus implies robust stability with time-varying bounded real (but otherwise unknown) uncertainty. To make connections with robust stability consider the system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \tag{30}$$

where $\Delta A(\cdot) \in \mathscr{U}_{br}$ and $\mathscr{U}_{br}$ is the uncertainty set

$$\mathscr{U}_{br} \triangleq \{\Delta A(\cdot): \Delta A(t) = BF(t)(I - DF(t))^{-1}C, F(\cdot) \in \mathscr{F}_{br}\}.$$

Then it follows from Corollary 3.1 and (29) that the zero solution to (30) is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}_{\mathrm{br}}$. The set $\mathcal{U}_{\mathrm{br}}$ is a generalization of the uncertainty sets appearing in References 45, 46, 49 and 50 for robust controller analysis and synthesis. These uncertainty structures can be recovered by setting $D=0$ in $\mathcal{U}_{\mathrm{br}}$. The case $D \neq 0$ has not been treated previously. Finally, if we restrict our attention to constant matrices, then Corollary 3.1 implies that if $G(s)$ is strictly bounded real, then $A+BF(I-DF)^{-1}C$ is asymptotically stable for all F satisfying $\sigma_{\max}(F) \leqslant 1$.

4. THE POSITIVITY THEOREM

In this section we construct quadratic Lyapunov functions to prove the positivity theorem for the system interconnection considered in Section 3.

Lemma 4.1 (Positive real lemma)[67]

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is positive real if and only if there exist matrices P, L, and W with P positive such that

$$0 = A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L, \tag{31}$$

$$0 = B^{\mathrm{T}}P - C + W^{\mathrm{T}}L, \tag{32}$$

$$0 = D + D^{\mathrm{T}} - W^{\mathrm{T}}W. \tag{33}$$

Proof. As in the bounded real lemma, sufficiency follows from algebraic manipulations of (31)–(33) while necessity is a direct consequence of spectral factorization theory. For details see References 64 and 67. □

Remark 4.1

If (31) is replaced by

$$0 = A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L + R, \tag{31$'$}$$

where $R \geqslant 0$, then (31)$'$–(33) imply that $G(s)$ is positive real.

Suppose that $D + D^{\mathrm{T}} > 0$. Then, since

$$W^{\mathrm{T}}W = D + D^{\mathrm{T}}, \tag{34}$$

$W^{\mathrm{T}}W$ is nonsingular, and (32) implies

$$W^{\mathrm{T}}L = -(B^{\mathrm{T}}P - C). \tag{35}$$

Using (35) and noting as in Section 3 that $L^{\mathrm{T}}L \geqslant L^{\mathrm{T}}W(W^{\mathrm{T}}W)^{-1}W^{\mathrm{T}}L$, it follows from (31) that

$$0 \geqslant A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P - C)^{\mathrm{T}}(W^{\mathrm{T}}W)^{-1}(B^{\mathrm{T}}P - C) \tag{36}$$

or, since, $(W^{\mathrm{T}}W)^{-1} = (D + D^{\mathrm{T}})^{-1}$,

$$0 \geqslant A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(B^{\mathrm{T}}P - C). \tag{37}$$

Using the Riccati equation version of (37) to characterize positive realness, we have the following result.

Lemma 4.2.

Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following statements are equivalent:

(i) A is asymptotically stable and $G(s)$ is strongly positive real;

(ii) $D + D^{\mathrm{T}} > 0$ and there exist positive-definite matrices P and R such that

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(B^{\mathrm{T}}P - C) + R. \tag{38}$$

Proof. (i) $\Rightarrow$ (ii): Using the dual forms of Theorems 3.1 and 3.2, and Propositions 3.2 and 3.3 of Reference 52 it follows that the strongly positive real condition in (i) can be written as a strictly bounded real condition for a modified plant as

$$I - \hat{B}^{\mathrm{T}}(-\mathrm{j}\omega I + \hat{A})^{-\mathrm{T}}\hat{C}^{\mathrm{T}}\hat{C}(\mathrm{j}\omega I - \hat{A})^{-1}\hat{B} > 0, \qquad \omega \in \mathbb{R}, \tag{39}$$

where

$$\hat{A} \triangleq A - B(I + D)^{-1}C + B(I + D)^{-1}\hat{D}(I - \hat{D}\hat{D}^{\mathrm{T}})^{-1}(I + D)^{-1}C, \tag{40}$$

$$\hat{B} \triangleq \sqrt{2}B(I + D)^{-1}(I - \hat{D}\hat{D}^{\mathrm{T}})^{-1/2}, \tag{41}$$

$$\hat{C} \triangleq \sqrt{2}(I - \hat{D}\hat{D}^{\mathrm{T}})^{-1/2}(I + D)^{-1}C, \tag{42}$$

$$\hat{D} \triangleq (D - I)(D + I)^{-1}. \tag{43}$$

Note that since $(\hat{A}, \hat{B}, \hat{C})$ is bounded real, it follows that $\hat{A}$ and $A - B(I + D)^{-1}C$ are asymptotically stable (see Theorem 3.2 and Remark 3.1 of Reference 52 for details). Next, let $\varepsilon > 0$ be such that

$$I - \hat{B}^{\mathrm{T}}(-j\omega I - \hat{A})^{-\mathrm{T}}(\hat{C}^{\mathrm{T}}\hat{C} + \varepsilon I)(\mathrm{j}\omega I - \hat{A})^{-1}\hat{B} > 0, \qquad \omega \in \mathbb{R}. \tag{44}$$

Since $\hat{A} - \frac{1}{2}\hat{B}(I - \hat{D}^{\mathrm{T}}\hat{D})^{-1/2}\hat{D}^{\mathrm{T}}\hat{C} = A - B(I + D)^{-1}C$, and $A - B(I + D)^{-1}C$ is asymptotically stable, it follows that $(\hat{A}, \hat{B})$ is stabilizable. Since also, $(\hat{A}, \hat{C}^{\mathrm{T}}\hat{C} + \varepsilon I)$ is observable it follows from Lemma 5 of Reference 68 that there exists a real symmetric matrix P such that

$$0 = \hat{A}^{\mathrm{T}}P + P\hat{A} + P\hat{B}\hat{B}^{\mathrm{T}}P + \hat{C}^{\mathrm{T}}\hat{C} + \varepsilon I, \tag{45}$$

or, equivalently, using the dual of Proposition 3.3 of Reference 52

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(B^{\mathrm{T}}P - C) + \varepsilon I. \tag{46}$$

Now, since A is assumed to be asymptotically stable and $(B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}\cdot(B^{\mathrm{T}}P - C) + \varepsilon I > 0$, it follows from Lyapunov theory that $P > 0$ so that existence of a positive-definite matrix P satisfying (38) is established.

(ii) $\Rightarrow$ (i): Suppose (ii) holds and note that

$$G(s) + G^{\sim}(s) \sim \left[\begin{array}{cc|c} -A^{\mathrm{T}} & 0 & -C^{\mathrm{T}} \\ 0 & A & B \\ \hline B^{\mathrm{T}} & C & D + D^{\mathrm{T}} \end{array}\right]. \tag{47}$$

From (38) it follows that there exists a positive-definite matrix R such that

$$0 = A^{\mathrm{T}}P + PA + (B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(B^{\mathrm{T}}P - C) + R. \tag{48}$$

Applying the state-space transformation

$$S=\begin{bmatrix} I & P \\ 0 & I \end{bmatrix}$$

to (47) we obtain

$$G(s)+G^{-}(s)\sim\left[\begin{array}{cc|c} -A^{T} & A^{T}P+PA & (B^{T}P-C)^{T} \\ 0 & A & B \\ \hline B^{T} & -(B^{T}P-C) & D+D^{T} \end{array}\right]$$

or, equivalently, using (48),

$$G(s)+G^{-}(s)\sim\left[\begin{array}{cc|c} -A^{T} & -(B^{T}P-C)^{T}(D+D^{T})^{-1}(B^{T}P-C)-R & (B^{T}P-C)^{T} \\ 0 & A & B \\ \hline B^{T} & -(B^{T}P-C) & D+D^{T} \end{array}\right] \quad (49)$$

$$=N^{-}(s)N(s)+D+D^{T}-(B^{T}P-C)E^{-2}(B^{T}P-C)^{T},$$

where

$$E\triangleq[(B^{T}P-C)^{T}(D+D^{T})^{-1}(B^{T}P-C)+R]^{1/2}>0$$

and

$$N(s)\sim\left[\begin{array}{c|c} A & B \\ \hline E & -E^{-1}(B^{T}P-C)^{T} \end{array}\right]. \quad (50)$$

Noting that $D+D^{T}-(B^{T}P-C)E^{-2}(B^{T}P-C)^{T}>0$ it follows from (49) that $G(j\omega)+G^{*}(j\omega)>0$, $\omega\in\mathbb{R}$. Next, note that since $P>0$ and $(B^{T}P-C)^{T}(D+D^{T})^{-1}(B^{T}P-C)+R>0$ it follows from (48) that A is asymptotically stable. □

We now prove the positivity theorem for the negative feedback interconnection of a strongly positive real transfer function and a memoryless time-varying nonlinearity. For the statement of the next result we define the set

$$\Phi_{\mathrm{pr}}\triangleq\{\phi:\mathbb{R}^{m}\times\mathbb{R}^{+}\to\mathbb{R}^{m}:\phi^{T}(y,t)y\geqslant 0,\quad y\in\mathbb{R}^{m},\ \text{a.a. } t\geqslant 0,$$
$$\text{and } \phi(y,\cdot) \text{ is Lebesgue measurable for all } y\in\mathbb{R}^{m}\}.$$

Theorem 4.1

Suppose

$$G(s)\overset{\min}{\sim}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is strongly positive real. Then there exist positive-definite matrices P and R satisfying

$$0=A^{T}P+PA+(B^{T}P-C)^{T}(D+D^{T})^{-1}(B^{T}P-C)+R. \quad (51)$$

Furthermore, for all $\phi\in\Phi_{\mathrm{pr}}$, the function $V(x)=x^{T}Px$ is a Lyapunov function for the negative feedback interconnection of $G(s)$ and ϕ. Consequently, the negative feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi\in\Phi_{\mathrm{pr}}$.

Proof. First note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ has the state-space representation

$$\dot{x}(t) = Ax(t) - B\phi(y,t), \tag{52}$$

$$y(t) = Cx(t) - D\phi(y,t). \tag{53}$$

Now it follows from Lemma 4.2 that if $G(s)$ is strongly positive real then there exist positive-definite matrices P and R such that (51) is satisfied. Next, we use the Lyapunov candidate $V(x) = x^{\mathrm{T}}Px$ to show that the feedback interconnection (52), (53) is asymptotically stable. The corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x - \phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi. \tag{54}$$

Now add and subtract $2\phi^{\mathrm{T}}Cx$ and $2\phi^{\mathrm{T}}D\phi$ to and from (54) so that

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}(B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(B^{\mathrm{T}}P - C)x\\ &-\phi^{\mathrm{T}}B^{\mathrm{T}}Px - x^{\mathrm{T}}PB\phi + 2\phi^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D\phi - \phi^{\mathrm{T}}D^{\mathrm{T}}\phi\\ &-2\phi^{\mathrm{T}}Cx + \phi^{\mathrm{T}}Cx + x^{\mathrm{T}}C^{\mathrm{T}}\phi\end{aligned} \tag{55}$$

or, equivalently,

$$\begin{aligned}\dot{V}(x) = &-x^{\mathrm{T}}Rx - x^{\mathrm{T}}(B^{\mathrm{T}}P - C)^{\mathrm{T}}(D + D^{\mathrm{T}})^{-1}(B^{\mathrm{T}}P - C)x\\ &-\phi^{\mathrm{T}}(B^{\mathrm{T}}P - C)^{\mathrm{T}}\phi - \phi^{\mathrm{T}}(B^{\mathrm{T}}P - C)x - \phi^{\mathrm{T}}(D + D^{\mathrm{T}})\phi\\ &-2\phi^{\mathrm{T}}(Cx - D\phi).\end{aligned} \tag{56}$$

Grouping the appropriate terms in (56) yields

$$\dot{V}(x) = -x^{\mathrm{T}}Rx - z^{\mathrm{T}}z - 2\phi^{\mathrm{T}}y, \tag{57}$$

where

$$z \triangleq -(D + D^{\mathrm{T}})^{-1/2}(B^{\mathrm{T}}P - C)x - (D + D^{\mathrm{T}})^{1/2}\phi.$$

Since R is positive definite and $\phi^{\mathrm{T}}(y,t)y \geq 0$ for all $\phi \in \Phi_{\mathrm{pr}}$ it follows that $\dot{V}(x)$ is negative definite. Hence $V(x)$ is a Lyapunov function for the feedback interconnection of $G(s)$ and ϕ. □

Next, we specialize Theorem 4.1 to the feedback interconnection of a strongly positive real transfer function and a linear gain $F(t)$ satisfying $F(t) + F^{\mathrm{T}}(t) \geq 0$. Hence define

$$\mathscr{F}_{\mathrm{pr}} \triangleq \{F: \mathbb{R}^+ \to \mathbb{R}^{m\times m}: F(\cdot) \text{ is Lebesgue measurable and } F(t) + F^{\mathrm{T}}(t) \geq 0, \quad \text{a.a. } t \geq 0\}.$$

Corollary 4.1

If

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

is strongly positive real, then the negative feedback interconnection of $G(s)$ and $F(\cdot)$ is asymptotically stable for all $F(\cdot) \in \mathscr{F}_{\mathrm{pr}}$.

As in the bounded real case, Corollary 4.1 guarantees robust stability for the system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \tag{58}$$

where $\Delta A(\cdot) \in \mathscr{U}_{\mathrm{pr}}$ and $\mathscr{U}_{\mathrm{pr}}$ is the uncertainty set

$$\mathscr{U}_{\mathrm{pr}} \triangleq \{\Delta A(\cdot): \Delta A(t) = -BF(t)(I + DF(t))^{-1}C, F(\cdot) \in \mathscr{F}_{\mathrm{pr}}\}.$$

Note that it follows from Lemma 2.2 that $(I+DF(\cdot))^{-1}$ exists, a.a. $t \geqslant 0$, since $D+D^{\mathrm{T}}>0$ and $F(t)+F^{\mathrm{T}}(t) \geqslant 0$, a.a. $t \geqslant 0$. The key feature of the uncertainty set $\mathscr{U}_{\mathrm{pr}}$ is the fact that $BF(t)(I+DF(t))^{-1}C$ also involves a positive real condition. To see this note that if $D+D^{\mathrm{T}}>0$ and $F(t)+F^{\mathrm{T}}(t) \geqslant 0$, then

$$F(t)(I+DF(t))^{-1}+[F(t)(I+DF(t))^{-1}]^{\mathrm{T}}$$
$$=(I+DF(t))^{-\mathrm{T}}[F(t)+F^{\mathrm{T}}(t)+F^{\mathrm{T}}(t)(D+D^{\mathrm{T}})F(t)]\,(I+DF(t))^{-1} \geqslant 0.$$

As shown in References 51 and 52, a natural characterization of uncertainty that can be captured by $\mathscr{U}_{\mathrm{pr}}$ arises in lightly damped structures with uncertain modal data.

Finally, if we restrict our attention to constant matrices F, then Corollary 4.1 implies that if $G(s)$ is strongly positive real, then $A-BF(I+DF)^{-1}C$ is asymptotically stable for all F satisfying $F+F^{\mathrm{T}} \geqslant 0$.

5. THE CIRCLE CRITERION

In this section we construct quadratic Lyapunov functions to prove a multivariable generalization of the circle criterion. Application of this result to robust stability with respect to sector-bounded time-varying uncertainty is also discussed. Although proofs of the circle criterion based upon quadratic Lyapunov functions appear in the literature,[24,31] these proofs are confined to strictly proper systems with a single loop nonlinearity. Notable exceptions include References 23, 32 and 61 which provide multiloop extensions for strictly proper plants. However, the nonlinearities considered in these references are confined to scalar sector boundaries k_1, k_2. We remove these constraints and address the multivariable case for proper systems. To begin, we define the set Φ_{c} of sector-bounded time-varying memoryless nonlinearities. Let $K_1, K_2 \in \mathbb{R}^{m\times l}$ be given matrices and define

$$\Phi_{\mathrm{c}} \triangleq \{\phi:\mathbb{R}^{l}\times\mathbb{R}^{+}\rightarrow\mathbb{R}^{m}: [\phi(y,t)-K_1y]^{\mathrm{T}}[\phi(y,t)-K_2y] \leqslant 0,$$
$$y\in\mathbb{R}^{m},\ \text{a.a. } t\geqslant 0, \text{ and } \phi(y,\cdot) \text{ is Lebesgue measurable for all } y\in\mathbb{R}^{m}\}.$$

Note that for the scalar case $m=l=1$, the sector condition characterizing Φ_{c} is equivalent to the more familiar condition

$$K_1y^2 \leqslant \phi(y,t)y \leqslant K_2y^2, \qquad y\in\mathbb{R}, \qquad \text{a.a. } t\geqslant 0. \tag{59}$$

Theorem 5.1

Suppose $[I+K_2G(s)]\,[I+K_1G(s)]^{-1}$ is strongly positive real, where

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then there exist positive-definite matrices P and R satisfying

$$\begin{aligned}
0 = {} & [A-B(I+K_1D)^{-1}K_1C]^{\mathrm{T}}P+P[A-B(I+K_1D)^{-1}K_1C] \\
& +[(K_2-K_1)(I+DK_1)^{-1}C-(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]^{\mathrm{T}} \\
& \times[2I+(K_2-K_1)(I+DK_1)^{-1}D+D^{\mathrm{T}}(I+DK_1)^{-\mathrm{T}}(K_2-K_1)^{\mathrm{T}}]^{-1} \\
& \times[(K_2-K_1)(I+DK_1)^{-1}C-(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]+R.
\end{aligned} \tag{60}$$

Furthermore, for all $\phi\in\Phi_{\mathrm{c}}$, the function $V(x)=x^{\mathrm{T}}Px$ is a Lyapunov function for the negative feedback interconnection of $G(s)$ and ϕ. Consequently, the negative feedback interconnection of $G(s)$ and ϕ in asymptotically stable for all $\phi\in\Phi_{\mathrm{c}}$.

Proof. First note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot,\cdot)$ has the state-space representation (52) and (53). It thus follows that

$$[I+K_2G(s)]\,[I+K_1G(s)]^{-1}=I+(K_2-K_1)\,[I+G(s)K_1]^{-1}G(s). \tag{61}$$

Now, noting that $[I+G(s)K_1]^{-1}G(s)$ corresponds to a plant $G(s)$ with feedback gain K_1, it follows from feedback interconnection manipulations that a minimal realization for $I+(K_2-K_1)\,[I+G(s)K_1]^{-1}G(s)$ is given by

$$\left[\begin{array}{c|c} A-B(I+K_1D)^{-1}K_1C & B(I+K_1D)^{-1} \\ \hline (K_2-K_1)(I+DK_1)^{-1}C & I+(K_2-K_1)(I+DK_1)^{-1}D \end{array}\right].$$

Note that $(I+K_1D)^{-1}$ exists since by assumption $[I+K_2G(s)]\,[I+K_1G(s)]^{-1}$ is positive real and $D=G(\infty)$. Now it follows from Lemma 4.2 that since $[I+K_2G(s)]\,[I+K_1G(s)]^{-1}$ is strongly positive real there exist positive-definite matrices P and R such that (60) is satisfied. Next define the Lyapunov candidate $V(x)=x^{\mathrm{T}}Px$ and let $\phi\in\Phi_c$. Then we obtain

$$\dot V(x)=x^{\mathrm{T}}(A^{\mathrm{T}}P+PA)x-\phi^{\mathrm{T}}B^{\mathrm{T}}Px-x^{\mathrm{T}}PB\phi \tag{62}$$

or, equivalently, using (60),

$$\begin{aligned}\dot V(x)=&-x^{\mathrm{T}}Rx-x^{\mathrm{T}}Qx+x^{\mathrm{T}}C^{\mathrm{T}}K_1^{\mathrm{T}}(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}Px\\ &+x^{\mathrm{T}}PB(I+K_1D)^{-1}K_1Cx-\phi^{\mathrm{T}}B^{\mathrm{T}}Px-x^{\mathrm{T}}PB\phi,\end{aligned} \tag{63}$$

where

$$\begin{aligned}Q\triangleq{}&[(K_2-K_1)(I+DK_1)^{-1}C-(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]^{\mathrm{T}}\\ &\times[2I+(K_2-K_1)(I+DK_1)^{-1}D+D^{\mathrm{T}}(I+DK_1)^{-\mathrm{T}}(K_2-K_1)^{\mathrm{T}}]^{-1}\\ &\times[(K_2-K_1)(I+DK_1)^{-1}C-(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P].\end{aligned}$$

Next, add and subtract

$$\begin{gathered}2[(I+K_1D)\phi-K_1Cx]^{\mathrm{T}}[(I+K_1D)\phi-K_1Cx],\\ 2[(I+K_1D)\phi-K_1Cx]^{\mathrm{T}}(K_2-K_1)(I+DK_1)^{-1}Cx,\\ 2[(I+K_1D)\phi-K_1Cx]^{\mathrm{T}}(K_2-K_1)(I+DK_1)^{-1}D[(I+K_1D)\phi-K_1Cx]\end{gathered}$$

to and from (63) so that (after some algebraic manipulation)

$$\begin{aligned}\dot V(x)=&-x^{\mathrm{T}}Rx-x^{\mathrm{T}}Qx\\ &+[(K_2-K_1)(I+DK_1)^{-1}Cx-(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}Px]^{\mathrm{T}}[(I+K_1D)\phi-K_1Cx]\\ &+[(I+K_1D)\phi-K_1Cx]^{\mathrm{T}}[(K_2-K_1)(I+DK_1)^{-1}Cx-(I+K_1D)^{-\mathrm{T}}B^{\mathrm{T}}Px]\\ &-[(I+K_1D)\phi-K_1Cx]^{\mathrm{T}}\\ &\times[2I+(K_2-K_1)(I+DK_1)^{-1}D+D^{\mathrm{T}}(I+DK_1)^{-1}(K_2-K_1)^{\mathrm{T}}]\\ &\times[(I+K_1D)\phi-K_1Cx]\\ &+2[(I+K_1D)\phi-K_1Cx]^{\mathrm{T}}\{(I+K_1D)\phi-K_1Cx-(K_2-K_1)(I+DK_1)^{-1}Cx\\ &+(K_2-K_1)(I+DK_1)^{-1}D[(I+K_1D)\phi-K_1Cx]\}.\end{aligned} \tag{64}$$

Grouping the appropriate terms in (64) yields

$$\dot V(x)=-xRx-z^{\mathrm{T}}z+2(\phi-K_1y)^{\mathrm{T}}(\phi-K_2y), \tag{65}$$

where

$$\begin{aligned} z \triangleq\ & [2I + (K_2 - K_1)(I + DK_1)^{-1}D + D^{\mathrm{T}}(I + DK_1)^{-\mathrm{T}}(K_2 - K_1)^{\mathrm{T}}]^{-1/2} \\ & \times [(K_2 - K_1)(I + DK_1)^{-1}C - (I + K_1D)^{-\mathrm{T}}B^{\mathrm{T}}P]\,x \\ & - [2I + (K_2 - K_1)(I + DK_1)^{-1}D + D^{\mathrm{T}}(I + DK_1)^{-\mathrm{T}}(K_2 - K_1)]^{1/2} \\ & \times [(I + K_1D)\phi - K_1Cx]. \end{aligned}$$

Since R is positive definite and $(\phi - K_1y)^{\mathrm{T}}(\phi - K_2y) \leqslant 0, \phi \in \Phi_{\mathrm{c}}$, it follows that $\dot{V}(x)$ is negative definite. Hence $V(x)$ is a Lyapunov function for the negative feedback interconnection of $G(s)$ and ϕ. □

Remark 5.1

Note that the condition $[I + K_2G(s)]\,[I + K_1G(s)]^{-1}$ strongly positive real in the statement of Theorem 5.1 is equivalent to $\mathrm{Re}[I + K_2G(\mathrm{j}\omega)]\,[I + K_1G(\mathrm{j}\omega)]^{-1} > 0$ for $\omega \in \mathbb{R}$ which is the classical representation of the circle criterion.[11,13] Furthermore, if K_1 and K_2 are diagonal, then the conditions of Theorem 5.1 can be verified by using the multivariable Nyquist criterion. Specifically, by examining the number of counterclockwise encirclements of the zero point of the image of the clockwise Nyquist contour under the mapping $\det[I + K_1G(s)]$, the stability of the closed-loop system can be related to the number of unstable poles of $G(s)$. For further details (in the SISO case) see Reference 13.

Remark 5.2

Considerable simplification can be achieved in (60) by setting $D = 0$ which corresponds to a strictly proper $G(s)$. In this case, (60) becomes

$$0 = (A - BK_1C)^{\mathrm{T}}P + P(A - BK_1C) + \tfrac{1}{2}[(K_2 - K_1)C - B^{\mathrm{T}}P]^{\mathrm{T}}[(K_2 - K_1)C - B^{\mathrm{T}}P] + R, \quad (66)$$

or, equivalently,

$$0 = A^{\mathrm{T}}P + PA + \tfrac{1}{2}[(K_1 + K_2)C - B^{\mathrm{T}}P]^{\mathrm{T}}[(K_1 + K_2)C - B^{\mathrm{T}}P] - C^{\mathrm{T}}(K_2^{\mathrm{T}}K_1 + K_1^{\mathrm{T}}K_2)C + R. \quad (67)$$

Note that if $K_2^{\mathrm{T}}K_1 + K_1^{\mathrm{T}}K_2 \leqslant 0$, then it follows from (67) that a necessary condition for absolute stability of the negative feedback interconnection of $G(s)$ and ϕ is that A be Hurwitz. In the scalar case this simply corresponds to $K_1 < 0 < K_2$ in which case $\phi \equiv 0$ is an admissible nonlinearity.

Next, as in Sections 3 and 4, we specialize the results of Theorem 5.1 to robust stability of a linear time-invariant plant with a linear time-varying uncertainty. To this end we have the following immediate result. Define

$$\mathscr{F}_{\mathrm{c}} \triangleq \{F: \mathbb{R}^+ \to \mathbb{R}^{m \times l}: F(\cdot) \text{ is Lebesgue measurable and } [F(t) - K_1]^{\mathrm{T}}[F(t) - K_2] \leqslant 0, \quad \text{a.a. } t \geqslant 0\}$$

and consider the system

$$\dot{x}(t) = (A + \Delta A(t))x(t), \quad (68)$$

where $\Delta A(\cdot) \in \mathcal{U}_c$ and the uncertainty set $\mathcal{U}_c$ is defined by

$$\mathcal{U}_c \triangleq \{\Delta A(\cdot)\colon \Delta A(t) = -BF(t)(I + DF(t))^{-1}C,\ F(\cdot) \in \mathcal{F}_c\}.$$

Then it follows from Theorem 5.1, with $\phi(y,t) = F(t)y$, that the zero solution to (68) is asymptotically stable for all $\Delta A(\cdot) \in \mathcal{U}_c$. Note that a simpler uncertainty structure is obtained by setting $D = 0$ in $\mathcal{U}_c$. As shown in Remark 5.2 this results in considerable simplification of (60). Finally, it is useful to note that if $K_1 = -K$ and $K_2 = I$, then $\mathcal{U}_c = \mathcal{U}_{br}$, while if $K_1 = 0$ and $K_2 = \infty$, then $\mathcal{U}_c = \mathcal{U}_{pr}$.

6. THE POPOV CRITERION

In this section we construct Lyapunov functions to prove the Popov criterion for a multivariable plant containing an arbitrary number of memoryless time-invariant nonlinearities. Specialization of this result to robust stability with respect to time-invariant linear plant uncertainty is also considered. To begin we define the set Φ_P characterizing a class of sector-bounded time-invariant memoryless nonlinearities. Let $K \in \mathbb{R}^{m\times m}$ be a given positive-definite matrix and define

$$\Phi_P \triangleq \{\phi\colon \mathbb{R}^m \to \mathbb{R}^m\colon \phi^{\mathrm{T}}(y)[K^{-1}\phi(y) - y] \leq 0, \quad y \in \mathbb{R}^m \text{ and } \phi(y) = [\phi_1(y_1), \phi_2(y_2), \dots, \phi_m(y_m)]^{\mathrm{T}}\}.$$

In the special case that $K = \operatorname{diag}[k_1, k_2, \dots, k_m]$, $k_i > 0$, $i = 1, \dots, m$, it follows that each component $\phi_i(y_i)$ of ϕ satisfies

$$0 \leq \phi_i(y_i)y_i \leq k_i y_i^2, \qquad y_i \in \mathbb{R}, \qquad i = 1, 2, \dots, m. \tag{69}$$

Note that the components of ϕ are assumed to be decoupled.

Theorem 6.1

Suppose there exists a nonnegative-definite diagonal matrix N such that $K^{-1} + (I + Ns)G(s)$ is strongly positive real, where

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

Then there exist positive-definite matrices P and R satisfying

$$0 = A^{\mathrm{T}}P + PA + (C + NCA - B^{\mathrm{T}}P)^{\mathrm{T}} \times [(K^{-1} + NCB) + (K^{-1} + NCB)^{\mathrm{T}}]^{-1}(C + NCA - B^{\mathrm{T}}P) + R. \tag{70}$$

Furthermore, for all $\phi \in \Phi_P$, the function

$$V(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m}\int_0^{y_i} \phi_i(\sigma)N_i\, \mathrm{d}\sigma \tag{71}$$

is a Lyapunov function for the negative feedback interconnection of $G(s)$ and ϕ. Consequently, the negative feedback interconnection of $G(s)$ and ϕ is asymptotically stable for all $\phi \in \Phi_P$.

Proof. First note that the negative feedback interconnection of $G(s)$ and $\phi(\cdot)$ has the state-space representation

$$\dot{x}(t) = Ax(t) - B\phi(y), \tag{72}$$

$$y = Cx(t). \tag{73}$$

Next, since $sG(s)$ has a realization

$$sG(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right],$$

it follows that $K^{-1} + (I + Ns)G(s)$ has minimal realization (using cascade state-space manipulations)

$$K^{-1} + (I + Ns)G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C + NCA & NCB + K^{-1} \end{array}\right].$$

Now it follows from Lemma 4.2 that since $K^{-1} + (I + Ns)G(s)$ is strongly positive real there exist positive-definite matrices P and R such that (70) is satisfied. Next, for $\phi \in \Phi_P$ define the Lyapunov candidate

$$V(x) = x^T Px + 2 \sum_{i=1}^{m} \int_0^{y_i} \phi_i(\sigma) N_i \, d\sigma.$$

Note that since P is positive definite and $\phi \in \Phi_P$, $V(x)$ is positive definite for all nonzero x. Thus, the corresponding Lyapunov derivative is given by

$$\dot{V}(x) = x^T(A^T P + PA)x - \phi^T B^T Px - x^T PB\phi + 2 \sum_{i=1}^{m} \phi_i(y_i) N_i \dot{y}_i$$

or, equivalently, using (70)

$$\begin{aligned}\dot{V}(x) = &-x^T Rx - x^T(C + NCA - B^T P)^T[(K^{-1} + NCB) + (K^{-1} + NCB)^T]^{-1} \\ &\times (C + NCA - B^T P)x - x^T PB\phi - \phi^T B^T Px + 2\phi^T(y)N\dot{y}.\end{aligned} \tag{74}$$

Next, since $\dot{y} = C\dot{x} = CAx - CB\phi$, (74) becomes

$$\begin{aligned}\dot{V}(x) = &-x^T Rx - x^T(C + NCA - B^T P)^T[(K^{-1} + NCB) + (K^{-1} + NCB)^T]^{-1} \\ &\times (C + NCA - B^T P)x - x^T(B^T P - NCA)^T\phi - \phi^T \\ &\times (B^T P - NCA)x - \phi^T(NCB + B^T C^T N)\phi.\end{aligned} \tag{75}$$

Adding and subtracting $2\phi^T Cx$ and $2\phi^T K^{-1}\phi$ to and from (75) yields

$$\begin{aligned}\dot{V}(x) = &-x^T Rx - x^T(C + NCA - B^T P)^T[(K^{-1} + NCB) + (K^{-1} + NCB)^{-1} \\ &\times (C + NCA - B^T P)x - \phi^T[B^T P - (C + NCA)]x - x^T[B^T P - (C + NCA)]^T\phi \\ &-\phi^T[(K^{-1} + NCB) + (K^{-1} + NCB)^T]\phi + 2\phi^T(K^{-1}\phi - Cx)\end{aligned} \tag{76}$$

or, equivalently,

$$\dot{V}(x) = -x^T Rx - z^T z + 2\phi^T[K^{-1}\phi - y], \tag{77}$$

where

$$\begin{aligned}z \triangleq\ &[(K^{-1} + NCB) + (K^{-1} + NCB)^T]^{-1/2}(C + NCA - B^T P)x \\ &- [(K^{-1} + NCB) + (K^{-1} + NCB)^T]^{1/2}\phi.\end{aligned}$$

Since R is positive definite and $\phi^T[K^{-1}\phi - y] \leqslant 0$ it follows that $\dot{V}(x)$ is negative definite.

Hence $V(x)$ is a Lyapunov function for the negative feedback interconnection of $G(s)$ and ϕ. □

Remark 6.1

A similar proof of the generalized Popov criterion is given in Reference 36 using the three equation form of the positive real lemma.

Note that because of the integral term in (71), the Lyapunov function $V(x)$ is not generally quadratic. However, we now specialize to linear parameter uncertainty in which case $V(x)$ is quadratic. To see this, define the set

$$\mathscr{F}_{\mathrm{P}} \triangleq \{F \in \mathbb{R}^{m \times m} : F = \mathrm{diag}[F_1, \ldots, F_m], 0 \leqslant F_i \leqslant k_i, i = 1, \ldots, m\}$$

of constant diagonal matrices F where $k_1, \ldots, k_m$ are positive constants. Next consider the system

$$\dot{x}(t) = (A + \Delta A)x(t), \tag{78}$$

where the constant matrix ΔA satisfies $\Delta A \in \mathscr{U}_{\mathrm{P}}$ and where $\mathscr{U}_{\mathrm{P}}$ is defined by

$$\mathscr{U}_{\mathrm{P}} \triangleq \{\Delta A : \Delta A = -BFC, F \in \mathscr{F}_{\mathrm{P}}\}$$

with B and C given matrices denoting the structure of the uncertainty. It now follows from Theorem 6.1 by setting $\phi(y) = Fy = FCx$ that $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}_{\mathrm{P}}$. Note that if $k_i \to \infty$, $i = 1, \ldots, m$, then $\mathscr{U}_{\mathrm{P}}$ becomes

$$\mathscr{U}_{\mathrm{P}} = \{\Delta A : \Delta A = -BFC, F = \mathrm{diag}[F_1, \ldots, F_m], F_i \geqslant 0, i = 1, \ldots, m\}.$$

The main difference between the results of this section and the previous sections is that the elements of the set $\mathscr{U}_{\mathrm{p}}$ are constant rather than time varying. This is due to the Lyapunov function that establishes robust stability, i.e.,

$$V(x) = x^{\mathrm{T}}Px + 2\sum_{i=1}^{m} \int_0^{y_i} F_i \sigma N_i \, \mathrm{d}\sigma, \qquad y_i = C_i x,$$

or, equivalently,

$$V(x) = x^{\mathrm{T}}Px + x^{\mathrm{T}}C^{\mathrm{T}}FNCx = x^{\mathrm{T}}Px + \sum_{i=1}^{m} F_i N_i x^{\mathrm{T}} C_i^{\mathrm{T}} C_i x,$$

where C_i denotes the ith row of C.

Note that this quadratic Lyapunov function is parameter-dependent, that is, it is a function of the uncertain parameters. Consequently the uncertain parameters are not allowed to be arbitrarily time-varying. Such Lyapunov functions are generally less conservative than constant Lyapunov functions[43,44] when the uncertain parameters are known to be constant. In contrast, the results of the previous sections are established by parameter-independent quadratic Lyapunov functions that guarantee robust stability with respect to *time-varying* parameter variations.

7. DYNAMIC FEEDBACK INTERCONNECTIONS

In this section we consider the feedback interconnection of dynamic systems. Specifically, we give explicit constructions of quadratic Lyapunov functions for the small gain and positivity theorems. For simplicity of exposition we shall only consider strictly proper transfer functions.

The interpretation of the following results could correspond to a dynamic plant under feedback with a dynamic compensator. First we consider the feedback interconnection of two dynamic bounded real transfer functions.

Theorem 7.1

Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad \text{and} \quad G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

be asymptotically stable transfer functions. If $G(s)$ and $G_c(s)$ are bounded real, then the feedback interconnection of $G(s)$ and $G_c(s)$ is stable in the sense of Lyapunov, that is, the linear system with dynamics matrix

$$\tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}$$

is stable in the sense of Lyapunov. If, in addition, $G_c(s)$ is strictly bounded real, then $\tilde{A}$ is asymptotically stable.

Proof. It follows from the bounded real lemma that there exist positive-definite matrices P, P_c and matrices L, W, L_c, and W_c such that

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + L^{\mathrm{T}}L, \tag{79}$$

$$0 = B^{\mathrm{T}}P + W^{\mathrm{T}}L, \tag{80}$$

$$0 = I - W^{\mathrm{T}}W, \tag{81}$$

$$0 = A_c^{\mathrm{T}}P_c + P_cA_c + C_c^{\mathrm{T}}C_c + L_c^{\mathrm{T}}L_c, \tag{82}$$

$$0 = B_c^{\mathrm{T}}P_c + W_c^{\mathrm{T}}L_c, \tag{83}$$

$$0 = I - W_c^{\mathrm{T}}W_c. \tag{84}$$

Next, we prove the stability of

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t)$$

by constructing a Lyapunov equation of the form

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}, \tag{85}$$

where, conformally with $\tilde{A}$, the matrices $\tilde{P}$ and $\tilde{R}$ are partitioned as

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P^{\mathrm{T}}_{12} & P_2 \end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^{\mathrm{T}} & R_2 \end{bmatrix}. \tag{86}$$

Expanding (85) with $\tilde{P}$ and $\tilde{R}$ given by (86) yields

$$0 = A^{\mathrm{T}}P_1 + C^{\mathrm{T}}B_c^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_1A + P_{12}B_cC + R_1, \tag{87}$$

$$0 = A^{\mathrm{T}}P_{12} + C^{\mathrm{T}}B_c^{\mathrm{T}}P_2^{\mathrm{T}} + P_{12}A_c + P_1BC_c + R_{12}, \tag{88}$$

$$0 = A_c^{\mathrm{T}}P_2 + C_c^{\mathrm{T}}B^{\mathrm{T}}P_{12} + P_2A_c + P_{12}^{\mathrm{T}}BC_c + R_2. \tag{89}$$

Setting

$$R_1 = L^{\mathrm{T}}L + C^{\mathrm{T}}C, \qquad R_{12} = L^{\mathrm{T}}WC_c + C^{\mathrm{T}}W_c^{\mathrm{T}}L_c, \qquad R_2 = L_c^{\mathrm{T}}L_c + C_c^{\mathrm{T}}C_c,$$

(87)–(89) are satisfied by

$$P_1 = P, \qquad P_{12} = 0, \qquad P_2 = P_c. \tag{90}$$

To see that (88) is satisfied note that (80) and (82) imply that

$$\begin{aligned} C^{\mathrm{T}}B_c^{\mathrm{T}}P_2 + P_1BC_c + R_{12} &= C^{\mathrm{T}}B_c^{\mathrm{T}}P_c + PBC_c + L^{\mathrm{T}}WC_c + C^{\mathrm{T}}W_c^{\mathrm{T}}L_c \\ &= -C^{\mathrm{T}}W_c^{\mathrm{T}}L_c - L^{\mathrm{T}}WC_c + L^{\mathrm{T}}WC_c + C^{\mathrm{T}}W_c^{\mathrm{T}}L_c \\ &= 0. \end{aligned}$$

With (90) $\tilde{P}$ is given by

$$\tilde{P} = \begin{bmatrix} P & 0 \\ 0 & P_c \end{bmatrix} > 0. \tag{91}$$

It now follows that with $V(\tilde{x}) \triangleq \tilde{x}^{\mathrm{T}}\tilde{P}\tilde{x}$,

$$\dot{V}(\tilde{x}) = -\tilde{x}^{\mathrm{T}}\tilde{R}\tilde{x} \leqslant 0 \tag{92}$$

since

$$\tilde{R} = \begin{bmatrix} L^{\mathrm{T}}L + C^{\mathrm{T}}C & L^{\mathrm{T}}WC_c + C^{\mathrm{T}}W_c^{\mathrm{T}}L_c \\ C_c^{\mathrm{T}}W^{\mathrm{T}}L + L_c^{\mathrm{T}}W_cC & L_c^{\mathrm{T}}L_c + C_c^{\mathrm{T}}C_c \end{bmatrix} = \begin{bmatrix} L^{\mathrm{T}} \\ C_c^{\mathrm{T}}W \end{bmatrix}[L\ WC_c] + \begin{bmatrix} C^{\mathrm{T}}W_c^{\mathrm{T}} \\ L_c^{\mathrm{T}} \end{bmatrix}[W_cC\ L_c] \geqslant 0. \tag{93}$$

Hence $\tilde{A}$ is stable in the sense of Lyapunov. Since by assumption (C, A) and (C_c, A_c) are observable, it follows from the PBH test that $(\tilde{R}, \tilde{A})$ is observable. Now it follows from Lemma 12.2 of Reference 69 that $\tilde{A}$ is asymptotically stable. □

Remark 7.1

Note that the Lyapunov function guaranteeing stability of the feedback interconnection of $G(s)$ and $G_c(s)$ has a particular internal structure that is inherited from the Lyapunov functions for $G(s)$ and $G_c(s)$.

Finally, we consider the feedback interconnection of two dynamic positive real transfer functions. The Lyapunov function proof given below provides an alternative approach to hyperstability concepts which yields the same results.[10]

Theorem 7.2

Let

$$G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad \text{and} \quad G_c(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

be asymptotically stable transfer functions. If $G(s)$ and $G_c(s)$ are positive real, then the negative feedback interconnection of $G(s)$ and $G_c(s)$ is stable in the sense of Lyapunov, that is, the linear system with dynamics matrix

$$\tilde{A} \triangleq \begin{bmatrix} A & -BC_c \\ B_cC & A_c \end{bmatrix}$$

is stable in the sense of Lyapunov. If, in addition, $G_c(s)$ is strictly positive real, then $\tilde{A}$ is asymptotically stable.

Proof. It follows from Lemma 4.1 that there exist positive-definite matrices P, P_c and matrices L and L_c such that

$$0 = A^{\mathrm{T}}P + PA + L^{\mathrm{T}}L, \tag{94}$$

$$0 = B^{\mathrm{T}}P - C, \tag{95}$$

$$0 = A_c^{\mathrm{T}}P_c + P_cA_c + L_c^{\mathrm{T}}L_c, \tag{96}$$

$$0 = B_c^{\mathrm{T}}P_c - C_c. \tag{97}$$

Once again, we prove stability of

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) \tag{98}$$

by constructing a Lyapunov equation of the form

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}, \tag{99}$$

where

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^{\mathrm{T}} & R_2 \end{bmatrix}. \tag{100}$$

Expanding (99) yields

$$0 = A^{\mathrm{T}}P_1 + C^{\mathrm{T}}B_c^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_1A + P_{12}B_cC + R_1, \tag{101}$$

$$0 = A^{\mathrm{T}}P_{12} + C^{\mathrm{T}}B_c^{\mathrm{T}}P_2^{\mathrm{T}} + P_{12}A_c - P_1BC_c + R_{12}, \tag{102}$$

$$0 = A_c^{\mathrm{T}}P_2 - C_c^{\mathrm{T}}P_{12} + P_2A_c - P_{12}^{\mathrm{T}}BC_c + R_2. \tag{103}$$

Setting

$$R_1 = L^{\mathrm{T}}L, \qquad R_{12} = 0, \qquad R_2 = L_c^{\mathrm{T}}L_c, \tag{104}$$

(101)–(103) are satisfied by

$$P_1 = P, \qquad P_{12} = 0, \qquad P_2 = P_c. \tag{105}$$

To see that (102) is satisfied note that (95) and (97) imply

$$\begin{aligned} C^{\mathrm{T}}B_c^{\mathrm{T}}P_2 - P_1BC_c &= C^{\mathrm{T}}B_c^{\mathrm{T}}P_c - PBC_c \\ &= C^{\mathrm{T}}C_c - C^{\mathrm{T}}C_c \\ &= 0. \end{aligned}$$

Hence, with (105), $\tilde{P}$ is given by

$$\tilde{P} = \begin{bmatrix} P & 0 \\ 0 & P_c \end{bmatrix} > 0. \tag{106}$$

It now follows that with $V(\tilde{x}) = \tilde{x}^{\mathrm{T}}\tilde{P}\tilde{x}$,

$$\dot{V}(\tilde{x}) = -\tilde{x}^{\mathrm{T}}\tilde{R}\tilde{x} \leqslant 0 \tag{107}$$

since

$$\tilde{R} = \begin{bmatrix} L^{\mathrm{T}}L & 0 \\ 0 & L_c^{\mathrm{T}}L_c \end{bmatrix} \geqslant 0 \tag{108}$$

and thus $\tilde{A}$ is stable in the sense of Lyapunov. If $G_c(s)$ is strictly positive real, then the invariant set theorem[70] can be used to prove asymptotic stability. □

8. CONNECTIONS BETWEEN THE CLASSICAL SYSTEM-THEORETIC CRITERIA AND ROBUST STABILITY AND PERFORMANCE FOR STATE-SPACE SYSTEMS

In this paper we have constructed Lyapunov functions for the small gain (bounded real), positivity (positive real), circle, and Popov theorems. Each result was applied to the interconnection of a linear time-invariant transfer function and a memoryless nonlinearity. Lyapunov functions for the small gain and positivity results were also constructed for the interconnection of two transfer functions. Each result was then specialized to the problem of robust stability involving linear uncertainty resulting in robustness tests in terms of single Riccati equations that bound the respective uncertainty structures considered. Even though not explicitly discussed in the previous sections, the results of this paper also apply to the problem of robust H_2 performance over the class of plant variations. To see this we consider a simple reinterpretation of the results of this paper using the framework of Reference 50.

Consider the asymptotically stable linear system

$$\dot{x}(t) = Ax(t) \tag{109}$$

with quadratic Lyapunov function

$$V(x) = x^{\mathrm{T}}Px, \tag{110}$$

where the positive-definite matrix P is given by the Lyapunov equation

$$0 = A^{\mathrm{T}}P + PA + R, \tag{111}$$

where R is positive definite. In order to address additive disturbances for a system of the form

$$\dot{x}(t) = Ax(t) + w(t), \tag{112}$$

we shall utilize the dual equation

$$0 = AQ + QA^{\mathrm{T}} + V \tag{113}$$

in which A is replaced by A^{T} and where V is interpreted as the intensity of the disturbance w. In (113), the matrix Q can be viewed as a controllability Gramian or covariance matrix with associated quadratic (H_2) performance measure

$$J = \operatorname{tr} QR = \operatorname{tr} PV. \tag{114}$$

Now suppose A is uncertain so that (109) is replaced by

$$\dot{x}(t) = (A + \Delta A)x(t), \tag{115}$$

$\Delta A \in \mathcal{U}$, a set of perturbations. To determine whether $A + \Delta A$ remains stable, one may replace (111) by

$$0 = A^{\mathrm{T}}P + PA + \Omega(P) + R, \tag{116}$$

where $\Omega(\cdot)$ satisfies

$$\Delta A^{\mathrm{T}}P + P\,\Delta A \leqslant \Omega(P), \text{ for all } \Delta A \in \mathcal{U}, \tag{117}$$

and for all positive-definite P. It then follows by rewriting (116) as

$$0 = (A + \Delta A)^{\mathrm{T}}P + P(A + \Delta A) + \Omega(P) - (\Delta A^{\mathrm{T}}P + P\,\Delta A) + R \tag{118}$$

that $A + \Delta A$ is stable and that

$$P_{\Delta A} \leqslant P, \qquad \Delta A \in \mathcal{U}, \tag{119}$$

where $P_{\Delta A}$ satisfies

$$0 = (A + \Delta A)^{\mathrm{T}} P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R. \tag{120}$$

Thus tr PV provides a worst case bound for the actual H_2 performance tr $P_{\Delta A}V$.

Since the ordering induced by the cone of nonnegative-definite matrices is only a partial ordering, there does not exist an operator $\Omega(\cdot)$ satisfying (117) that is a least upper bound. Indeed, there are many alternative definitions for the bound $\Omega(\cdot)$.[50] In fact by reinterpreting the results of this paper one can deduce three such Ω-bounds. To illustrate this, assume for convenience that ΔA is of the form

$$\Delta A = BFC, \qquad \sigma_{\max}(F) \leqslant 1, \tag{121}$$

where F is an uncertain real matrix and B, C are known matrices denoting the structure of the uncertainty. Using the results of Corollary 3.1 it follows that

$$\Omega(P) = C^{\mathrm{T}}C + PBB^{\mathrm{T}}P \tag{122}$$

satisfies (117) with $\mathscr{U} = \mathscr{U}_{\mathrm{br}}$ (for the case $D = 0$). Thus, we can see that well-known system-theoretic criteria such as the bounded real, positivity, and circle theorems apply to the problem of robust stability and performance for the special case of linear uncertainty. In fact, using (116) with ΔA and $\Omega(P)$ given by (121) and (122) respectively forms the basis of robust H_∞ analysis for state-space systems.[49]

Although not immediately evident, a serious defect of this Ω-bound approach is the fact that stability is guaranteed even if ΔA is a function of t, as was seen throughout the paper. This observation follows from the fact that the Lyapunov derivative $\dot{V}(x) = V_x((t))(A + \Delta A(t))x(t)$ need only be negative pointwise. This of course leads to conservatism when ΔA is actually constant.[15,71] This is simply because time-varying parameter variations can destabilize a system even when the parameter variations are confined to a region in which constant variations are nondestabilizing. Hence, a robustness theory that accounts for time-varying uncertainty will be conservative when the uncertain parameters are actually constant.

A closer comparison between the bounded real uncertainty structure $\mathscr{U}_{\mathrm{br}}$ and the positive real uncertainty structure $\mathscr{U}_{\mathrm{pr}}$ shows that $\mathscr{U}_{\mathrm{br}}$ simply involves a magnitude constraint on the uncertainty while $\mathscr{U}_{\mathrm{br}}$ involves a magnitude and phase constraint on the uncertainty.[52] Specifically, even though the Ω-bound corresponding to positive-real theory guarantees robust stability with respect to time-varying uncertainty F the definiteness property of F places a bound on the phase variation of the uncertainty of $\pm 90°$. Hence, the Ω-bound for positivity theory can be viewed or a refined Ω-bound since it takes phase information into account and thereby will most likely be less conservative than the Ω-bound for bounded real theory (H_∞) for constant real parameter uncertainty.

Since, as discussed above, it is crucial to severely restrict the allowable time-variations of the uncertainty to address the constant plant uncertainty problem, an alternative approach to the phase information/real parameter uncertainty problem is to construct refined Lyapunov functions that are functions of the uncertain parameters. This brings us to the Popov criterion which was shown only applies to time-invariant nonlinearities. Recall that in the linear uncertainty case stability was established via a parameter-dependent Lyapunov function.

To demonstrate parameter-dependent Lyapunov functions in a manner consistent with the above discussion of Ω-bounds, we consider the Lyapunov function

$$V(x, \Delta A) = x^{\mathrm{T}}(P + P_0(\Delta A))x, \tag{123}$$

where the (parameter-independent) matrix P satisfies

$$0 = A^{\mathrm{T}}P + PA + \Omega_0(P) + R \tag{124}$$

and $P_0(\Delta A)$ is a specified function such that $P + P_0(\Delta A)$ is positive definite for all $\Delta A \in \mathscr{U}$. In contrast with the Ω-bound in (116), however, $\Omega_0(P)$ is *not* assumed to satisfy (117), but rather the more involved condition

$$\begin{aligned}\Delta A^{\mathrm{T}}P + P\,\Delta A &\leqslant \Omega_0(P) \\ &\quad - [A^{\mathrm{T}}P_0(\Delta A) + P_0(\Delta A)A + \Delta A^{\mathrm{T}}P_0(\Delta A) + P_0(\Delta A)\,\Delta A], \text{ for all } \Delta A \in \mathscr{U},\end{aligned} \tag{125}$$

and all nonnegative-definite P. Note that if $P_0(\Delta A)$ is identically zero, then (125) specializes to (117). The idea behind (125) is that although $\Omega_0(P)$ alone is insufficient to bound $\Delta A^{\mathrm{T}}P + P\,\Delta A$, the additional terms 'assist' in forming a bound. To see this, let $\Omega(P, \Delta A)$ denote the right-hand side of (125), which can be viewed as a *parameter-dependent* Ω*-bound*. Then (124) can be written as

$$0 = (A + \Delta A)^{\mathrm{T}}(P + P_0(\Delta A)) + (P + P_0(\Delta A))(A + \Delta A) + \Omega(P, \Delta A) - (\Delta A^{\mathrm{T}}P + P\,\Delta A) + R, \tag{126}$$

which implies that $A + \Delta A$ is stable for all $\Delta A \in \mathscr{U}$. Furthermore, subtracting (120) from (126) shows that $P_{\Delta A} \leqslant P + P_0(\Delta A)$ for all $\Delta A \in \mathscr{U}$ and thus $\operatorname{tr}[(P + \bar{P}_0)V]$ provides a worst case bound for the actual H_2 performance $\operatorname{tr} P_{\Delta A}V$ where $\bar{P}_0 \geqslant P_0(\Delta A)$ for $\Delta A \in \mathscr{U}$.

For practical purposes the form of the parameter-dependent Lyapunov function $V(x, \Delta A)$ is critical since the presence of a ΔA severely restricts the allowable time-varying uncertain parameters. That is, if $\Delta A(t)$ were permitted, then terms involving $\Delta\dot{A}(t)$ might subvert the negative definiteness of $\dot{V}(x, \Delta A)$. Hence, it can be seen that the generalized form of the classical Popov criterion can be reinterpreted to the problem of robust stability and performance by simply using a more refined Ω-bound framework.

Using the unified Ω-bound framework which provides an algebraic formulation in terms of matrix Riccati equations one can synthesize robust feedback controllers using the fixed-structure parameter optimization approach.[72] Control design applications for the bounded real structure are given in References 45, 46 and 49 while the positivity, Popov, and circle structure are considered in References 52–54 respectively.

9. FUTURE EXTENSIONS

We conclude by mentioning several extensions and open problems for future research.

1. It may be possible to weaken the strictly bounded real and strongly positive real assumptions of Theorems 3.1 and 4.1 by replacing Φ_{br} and Φ_{pr} with smaller sets. It appears possible to do this by adopting the uncertainty characterization used in References 52 and 73.
2. It would be of interest to generalize the Popov criterion to *coupled* multivariable nonlinearities.[55,59] Furthermore, although we assumed that $\phi \in \Phi_{\mathrm{P}}$ is time-invariant it remains to characterize the allowable time variation of the nonlinearity ϕ.
3. It appears possible to derive a single result that generalizes both the circle and Popov criteria.[74,75]
4. It is possible to restrict the allowable class of sector bounded nonlinearities to monotonic and odd monotonic classes resulting in less conservative robust stability tests when specialized to the linear constant uncertainty case.[76–79]

5. It is reasonable to conjecture that a Lyapunov function can be constructed to prove Kharitonov's theorem.[80] The required Lyapunov function would most likely be parameter-dependent and thus extend Kharitonov's theorem to the case of sector-bounded time-invariant nonlinearities.

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