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Optimal reduced-order state estimation for unstable plants

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The problem of optimal reduced-order steady-state state estimation is considered for the case in which the plant has unstable poles. In contrast to the standard full-order estimation problem involving a single algebraic Riccati equation, the solution to the reduced-order problem involves one modified Riccati equation and one Lyapunov equation coupled by a projection matrix. This projection is completely distinct from the projection obtained by Bernstein and Hyland (1985) for stable plants.

Notation and definitions

Note: All matrices have real entries

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r\times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
$I_n, (\ )^{\mathrm{T}}, 0_{r\times s}, 0_r$	$n\times n$ identity matrix, transpose, $r\times s$ zero matrix, $0_{r\times r}$
n, l, n_e, n_u, n_s, q	positive integers
x, y, x_e, x_u, x_s, y_e	n, l, n_e, n_u, n_s, q-dimensional vectors
A, C	$n\times n$, $l\times n$ matrices
A_u, A_{us}, A_s	$n_u\times n_u$, $n_u\times n_s$, $n_s\times n_s$ matrices
C_u, C_s	$l\times n_u$, $l\times n_s$ matrices
L, L_u, L_s	$q\times n$, $q\times n_u$, $q\times n_s$ matrices
R	$q\times q$ positive-definite matrix
A_e, B_e, C_e, D_e	$n_e\times n_e$, $n_e\times l$, $q\times n_e$, $q\times l$ matrices
t, k	$t\in[0,\infty)$, discrete-time index 1, 2, 3, ...
$\tilde{A}$	$A-\begin{bmatrix} I_{n_u} \\ 0_{n_s\times n_u}\end{bmatrix} B_e C$
$w_1(\cdot), w_2(\cdot)$	n, l-dimensional continuous-time or discrete-time white noise processes
V_1	$n\times n$ non-negative-definite intensity or covariance of $w_1(\cdot)$
V_2	$l\times l$ positive-definite intensity or covariance of $w_2(\cdot)$
V_{12}	$n\times l$ cross-intensity or cross-covariance of $w_1(\cdot)$, $w_2(\cdot)$
$\tilde{w}(\cdot)$	$w_1(\cdot)-\begin{bmatrix} I_{n_u} \\ 0_{n_s\times n_u}\end{bmatrix} B_e w_2(\cdot)$
$\tilde{V}$	$V_1 - V_{12}B_e^{\mathrm{T}}[I_{n_u}\quad 0_{n_u\times n_s}] - \begin{bmatrix} I_{n_u} \\ 0_{n_s\times n_u}\end{bmatrix} B_e V_{12}^{\mathrm{T}} + \begin{bmatrix} I_{n_u} \\ 0_{n_s\times n_u}\end{bmatrix} B_e V_2 B_e^{\mathrm{T}}[I_{n_u}\quad 0_{n_u\times n_s}]$

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1. Introduction

It has recently been shown that optimal reduced-order, steady-state state estimators can be characterized by means of an algebraic system of equations consisting of one modified Riccati equation and two modified Lyapunov equations coupled by a projection matrix. The solution given by Bernstein and Hyland (1985), however, was confined to problems in which the plant is asymptotically stable, while in practice it is often necessary to obtain estimators for plants with unstable modes. The purpose of the present paper is to obtain results similar to those of Bernstein and Hyland (1985) for unstable plants.

Intuitively, it is clear that finite, steady-state state-estimation error for unstable plants is achievable only when the estimator retains, or duplicates in some sense, the unstable modes. Roughly speaking, the solution given by Bernstein and Hyland (1985), is inapplicable to the unstable problem for the simple reason that the range of the projection matrix may not fully encompass the unstable subspace. Hence, in the present paper we derive a new reduced-order solution which is constrained to estimate all of the unstable states. Specifically, for a plant with an unstable subspace of dimension n_u, we characterize the optimal estimator of order n_u which observes all of the unstable states.

As in Bernstein and Hyland (1985), the solution is given in terms of an oblique projection (denoted in the present paper by μ) which characterizes the optimal estimator gains. Again in contrast to the one observer Riccati equation of the standard full-order theory, the optimal reduced-order estimator gains for an unstable plant are given by an algebraic system which, in the present case, consists of one modified Riccati equation and one Lyapunov equation coupled by the projection matrix μ.

It is important to stress that the solution derived in the present paper is fundamentally different from the solution obtained by Bernstein and Hyland (1985), for two reasons. First, the estimator obtained by Bernstein and Hyland (1985) was characterized by three matrix equations (in variables Q, $\hat{Q}$ and $\hat{P}$) while the solution obtained herein involves two matrix equations (in variables Q and P). And, second, since the projection μ arising in the present paper depends upon P, it is completely distinct from the projection τ given by Bernstein and Hyland (1985), which depends upon $\hat{Q}$ and $\hat{P}$. Hence the results of the present paper neither generalize, nor are a special case of, the results of Bernstein and Hyland (1985).

In applying the results of the present paper we note that the solution is applicable to problems in which the unstable subspace also includes additional stable modes. Indeed, the only constraint in applying the theory is that the observed subspace includes all the unstable poles. To clarify this point (see § 2 and § 3 for notation), we note that all unstable poles of A must be contained in A_u, but A_u may also contain an arbitrary number of selected stable poles. Thus, the estimator derived in the present paper can be viewed as a subspace-constrained observer-estimator.

Finally, the result given herein is only a partial solution to the reduced-order estimation problem. Specifically, a reduced-order estimator which includes all of the unstable modes and optimal combinations of a fixed number of stable modes should involve both projections τ and μ and four matrix equations in variables Q, P, $\hat{Q}$ and $\hat{P}$. This problem is addressed in Haddad and Bernstein (1989). When this result is specialized to the full-order case the two projections merge to form the identity and the four matrix equations collapse to the single observer Riccati equation. A third projection ν due to singular measurement noise and static estimation can also be incorporated (Haddad and Bernstein 1987, Halevi 1989). This general solution remains the subject of current research.

We consider the reduced-order estimation problem for continuous-time plants in § 2. In § 3 the corresponding discrete-time problem is considered. For stable plants the reduced-order discrete-time solution was given by Bernstein *et al.* (1986).

2. Problem statement and main theorem

Reduced-order state-estimation problem

Given the nth-order observed system

$$\dot{x}(t) = Ax(t) + w_1(t) \tag{2.1}$$

$$y(t) = Cx(t) + w_2(t) \tag{2.2}$$

design an n_eth-order state estimator

$$\dot{x}_e(t) = A_e x_e(t) + B_e y(t) \tag{2.3}$$

$$y_e(t) = C_e x_e(t) \tag{2.4}$$

which minimizes the state-estimation error criterion

$$J(A_e, B_e, C_e) \triangleq \lim_{t\to\infty} \mathbb{E}[Lx(t) - y_e(t)]^{\mathrm{T}} R[Lx(t) - y_e(t)] \tag{2.5}$$

In this formulation the plant is partitioned into possibly unstable and stable subsystems. Thus, letting $x(t) = [x_u^{\mathrm{T}}(t) \quad x_s^{\mathrm{T}}(t)]^{\mathrm{T}}$ and $w_1(t) = [w_{1u}^{\mathrm{T}}(t) \quad w_{1s}^{\mathrm{T}}(t)]^{\mathrm{T}}$, (2.1) can be written as

$$\begin{bmatrix} \dot{x}_u(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} A_u & A_{us} \\ 0_{n_s \times n_u} & A_s \end{bmatrix} \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} w_{1u}(t) \\ w_{1s}(t) \end{bmatrix} \tag{2.6}$$

where $A_u \in \mathbb{R}^{n_u \times n_u}$ is possibly unstable, $A_s \in \mathbb{R}^{n_s \times n_s}$ is asymptotically stable, and the measurement equation (2.2) becomes

$$y(t) = [C_u \quad C_s] \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} + w_2(t) \tag{2.7}$$

Furthermore, the matrix L, which is partitioned as

$$L = [L_u \quad L_s] \tag{2.8}$$

identifies the states or linear combinations of states whose estimates are desired. The dimension n_e of the estimator state x_e is fixed to be equal to the order of the unstable part of the system, i.e. $n_e = n_u$. Thus, the goal of the Reduced-Order State-Estimation Problem is to design an estimator of order n_u which yields quadratically optimal estimates of specified linear combinations of states of the system. As mentioned in § 1, A_u includes all unstable modes of A as well as an arbitrary number of selected stable modes of A.

Since A_u may contain unstable modes, define the error state $z(t) \triangleq x_u(t) - x_e(t)$ satisfying

$$\dot{z}(t) = (A_u - B_e C_u)x_u(t) - A_e x_e(t) + (A_{us} - B_e C_s)x_s(t) + w_{1u}(t) - B_e w_2(t) \tag{2.9}$$

Note that the explicit dependence of the error states $z(t)$ on the unstable states $x_u(t)$ can be eliminated by constraining

$$A_e = A_u - B_e C_u \tag{2.10}$$

so that (2.9) becomes

$$\dot{z}(t) = (A_u - B_e C_u) z(t) + (A_{us} - B_e C_s) x_s(t) + w_{1u}(t) - B_e w_2(t) \tag{2.11}$$

Similarly, the explicit dependence of the estimation error (2.5) on the unstable states $x_u(t)$ can be eliminated by setting

$$C_e = L_u \tag{2.12}$$

Now (2.9)–(2.11) yield

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{w}(t) \tag{2.13}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} z(t) \\ x_s(t) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A_u - B_e C_u & A_{us} - B_e C_s \\ 0_{n_s \times n_u} & A_s \end{bmatrix}$$

and $\tilde{w}(t)$ and its intensity $\tilde{V}$ are given in § 1.

To guarantee that J is finite, consider the set of asymptotically stable reduced-order estimators

$$S \triangleq \{(A_e, B_e, C_e) \colon A_e = A_u - B_e C_u \text{ is asymptotically stable}\}$$

so that $\tilde{A}$ is asymptotically stable. Of course, S is non-empty if (A_u, C_u) is detectable. Furthermore, for non-degeneracy we restrict our attention to the set of admissible estimators

$$S^+ \triangleq \{(A_e, B_e, C_e) \in S \colon (A_e, C_e) \text{ is observable}\}$$

where A_e and C_e are given by (2.10) and (2.12). Also, for arbitrary $Q \in \mathbb{R}^{n \times n}$ define the notation

$$Q_a \triangleq QC^{\mathrm{T}} + V_{12}.$$

Theorem 2.1

Suppose $(A_e, B_e, C_e) \in S^+$ solves the Reduced-Order State-Estimation Problem with constraints (2.10) and (2.12). Then there exist $n \times n$ non-negative-definite matrices Q, P such that A_e, B_e, C_e are given by

$$A_e = \Phi(A - Q_a V_2^{-1} C) F^{\mathrm{T}} \tag{2.14}$$

$$B_e = \Phi Q_a V_2^{-1} \tag{2.15}$$

$$C_e = LF^{\mathrm{T}} \tag{2.16}$$

and such that Q, P satisfy

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q_a V_2^{-1} Q_a^{\mathrm{T}} + \mu_\perp Q_a V_2^{-1} Q_a^{\mathrm{T}} \mu_\perp^{\mathrm{T}} \tag{2.17}$$

$$0 = (A - \mu Q_a V_2^{-1} C)^{\mathrm{T}} P + P(A - \mu Q_a V_2^{-1} C) + L^{\mathrm{T}} R L \tag{2.18}$$

where

$$P = \begin{bmatrix} P_u & P_{us} \\ P_{us}^{\mathrm{T}} & P_s \end{bmatrix} \in \mathbb{R}^{(n_u + n_s) \times (n_u + n_s)} \tag{2.19}$$

$$F \triangleq [I_{n_u} \quad 0_{n_u \times n_s}], \quad \Phi \triangleq [I_{n_u} \quad P_u^{-1} P_{us}] \tag{2.20}$$

$$\mu \triangleq F^{\mathrm{T}}\Phi = \begin{bmatrix} I_{n_u} & P_u^{-1}P_{us} \\ 0_{n_s \times n_u} & 0_{n_s} \end{bmatrix}, \quad \mu_\perp \triangleq I_n - \mu \tag{2.21}$$

Furthermore, the minimal cost is given by

$$J(A_e, B_e, C_e) = \operatorname{tr} QL^{\mathrm{T}}RL \tag{2.22}$$

Proof

See Appendix A.

Remark 2.1

Note that since $\Phi F^{\mathrm{T}} = I_{n_u}$ the $n \times n$ matrix μ which couples the modified Riccati equation (2.17) and the Lyapunov equation (2.18) is idempotent, i.e. $\mu^2 = \mu$. Note also that rank $\mu = n_u$. This projection is completely distinct from the projection τ given by Bernstein and Hyland (1985).

Remark 2.2

In the full-order case $n_u = n$, Theorem 2.1 corresponds to the standard steady-state Kalman filter result. To see this, formally set $\Phi = F = \mu = I_n$ and $\mu_\perp = 0_n$ so that (2.18) is superfluous and (2.17) specializes to the standard observer Riccati equation.

Remark 2.3

Note that (2.14) and (2.16) are merely restatements of (2.10) and (2.12). Furthermore, (2.15) implies that $\tilde{A} = A - \mu Q_a V_2^{-1} C$ so that the coefficient of P in (2.18) is asymptotically stable.

3. Discrete-time formulation

Discrete-time reduced-order state-estimation problem

Given the nth-order observed system

$$x(k+1) = Ax(k) + w_1(k) \tag{3.1}$$

$$y(k) = Cx(k) + w_2(k) \tag{3.2}$$

design an n_eth-order state estimator

$$x_e(k+1) = A_e x_e(k) + B_e y(k) \tag{3.3}$$

$$y_e(k) = C_e x_e(k) + D_e y(k) \tag{3.4}$$

which minimizes the discrete-time state-estimation error criterion

$$\tilde{J}(A_e, B_e, C_e, D_e) \triangleq \lim_{k \to \infty} \mathbb{E}[Lx(k) - y_e(k)]^{\mathrm{T}} R[Lx(k) - y_e(k)] \tag{3.5}$$

Because of the discrete-time setting it is now possible as in Bernstein *et al.* (1986), to permit a static feedthrough term D_e in the estimator design. The gain D_e represents a static least squares estimator in conjunction with the dynamic estimator (A_e, B_e, C_e).

As in the continuous-time case, the plant is partitioned into stable and possibly unstable subsystems according to (2.6). Furthermore, an error state $z(k) \triangleq x_u(k) - x_e(k)$

is defined, A_e is constrained as in (2.10), and C_e is constrained to be $L_u - D_e C_u$. Thus, the augmented system consisting of the error states $z(k)$ and the stable states $x_s(k)$ becomes

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{w}(k) \tag{3.6}$$

where $\tilde{x}(k) \triangleq [z^{\mathrm{T}}(k) \quad x_s^{\mathrm{T}}(k)]^{\mathrm{T}}$.

To guarantee that J is finite and to obtain closed-form expressions for the estimator gains we restrict our attention to the sets

$$\hat{S} \triangleq \{(A_e, B_e, C_e, D_e): A_e = A_u - B_e C_u \text{ is asymptotically stable}\}$$

$$\hat{S}^+ \triangleq \{(A_e, B_e, C_e, D_e) \in \hat{S}: (A_e, C_e) \text{ is observable}\}$$

Also, for arbitrary $Q \in \mathbb{R}^{n \times n}$ define the notation

$$\hat{Q}_a \triangleq AQC^{\mathrm{T}} + V_{12}, \quad \hat{V}_2 \triangleq V_2 + CQC^{\mathrm{T}}$$

Theorem 3.1

Suppose $(A_e, B_e, C_e, D_e) \in \hat{S}^+$ solves the Discrete-Time Reduced-Order State-Estimation Problem. Then there exist $n \times n$ non-negative-definite Q, P such that A_e, B_e, C_e, D_e are given by

$$A_e = \Phi(A - \hat{Q}_a \hat{V}_2^{-1} C)F^{\mathrm{T}} \tag{3.7}$$

$$B_e = \Phi \hat{Q}_a \hat{V}_2^{-1} \tag{3.8}$$

$$C_e = (L - D_e C)F^{\mathrm{T}} \tag{3.9}$$

$$D_e = LQC^{\mathrm{T}} \hat{V}_2^{-1} \tag{3.10}$$

and such that Q, P satisfy

$$Q = AQA^{\mathrm{T}} + V_1 - \hat{Q}_a \hat{V}_2^{-1} \hat{Q}_a^{\mathrm{T}} + \mu_\perp \hat{Q}_a \hat{V}_2^{-1} \hat{Q}_a^{\mathrm{T}} \mu_\perp^{\mathrm{T}} \tag{3.11}$$

$$P = (A - \mu \hat{Q}_a \hat{V}_2^{-1} C)^{\mathrm{T}} P (A - \mu \hat{Q}_a \hat{V}_2^{-1} C) + (L - D_e C)^{\mathrm{T}} R (L - D_e C) \tag{3.12}$$

where F, Φ, μ and $\mu_\perp$ are defined by (2.19)–(2.21). Furthermore, the minimal cost is given by

$$\hat{J}(A_e, B_e, C_e, D_e) = \operatorname{tr}[(LQL^{\mathrm{T}} - D_e V_2 D_e^{\mathrm{T}})R] \tag{3.13}$$

Proof

See Appendix A.

Remark 3.1

If a strictly proper estimator is desired, then delete D_e in (3.9), (3.12) and (3.13).

Appendix A

Proof of Theorems 2.1 *and* 3.1

To analyse (2.13) define the second-moment matrix

$$Q(t) \triangleq \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{A 1}$$

which satisfies

$$\dot{Q}(t) = \tilde{A}Q(t) + Q(t)\tilde{A}^{\mathrm{T}} + \tilde{V}, \quad t \geqslant 0 \tag{A 2}$$

Since $(A_e, B_e, C_e) \in S$, $\tilde{A}$ is asymptotically stable and

$$Q \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)]$$

exists and satisfies

$$0 = \tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + \tilde{V} \tag{A 3}$$

Next note that (2.5) can be written as

$$J(A_e, B_e, C_e) = \operatorname{tr} QL^{\mathrm{T}}RL \tag{A 4}$$

To minimize (A 4) over the open set S^+ subject to the constraint (A 3), form the lagrangian

$$\mathscr{L}(B_e, Q, P, \lambda) \triangleq \operatorname{tr}[\lambda QL^{\mathrm{T}}RL + (\tilde{A}Q + Q\tilde{A}^{\mathrm{T}} + \tilde{V})P] \tag{A 5}$$

where the Lagrange multipliers $\lambda \geqslant 0$ and $P \in \mathbb{R}^{n\times n}$ are not both zero. Setting $\partial\mathscr{L}/\partial Q = 0$, $\lambda = 0$ implies $P = 0$ since $\tilde{A}$ is asymptotically stable. Hence, without loss of generality set $\lambda = 1$.

Now partition $n \times n$ P into $n_u \times n_u$, $n_u \times n_s$ and $n_s \times n_s$ sub-blocks as

$$P = \begin{bmatrix} P_u & P_{us} \\ P_{us}^{\mathrm{T}} & P_s \end{bmatrix} \tag{A 6}$$

Thus the stationarity conditions are given by

$$\frac{\partial\mathscr{L}}{\partial Q} = \tilde{A}^{\mathrm{T}}P + P\tilde{A} + L^{\mathrm{T}}RL = 0 \tag{A 7}$$

$$\frac{\partial\mathscr{L}}{\partial B_e} = P_u B_e V_2 - [P_u \quad P_{us}](QC^{\mathrm{T}} + V_{12}) = 0 \tag{A 8}$$

Expanding the $n_u \times n_u$ sub-block of (A 7) yields

$$0 = (A_u - B_e C_u)^{\mathrm{T}} P_u + P_u(A_u - B_e C_u) + L_u^{\mathrm{T}} R L_u \tag{A 9}$$

which, using (2.10) and (2.12), is equivalent to

$$0 = A_e^{\mathrm{T}} P_u + P_u A_e + C_e^{\mathrm{T}} R C_e \tag{A 10}$$

Thus, since $(A_e, B_e, C_e) \in S^+$, (A_e, C_e) is observable and it follows from (A 10) that P_u is positive-definite. Since P_u is invertible, define the $n_u \times n$ matrices

$$F \triangleq [I_{n_u} \quad 0_{n_u\times n_s}], \quad \Phi \triangleq [I_{n_u} \quad P_u^{-1} P_{us}] \tag{A 11}$$

and the $n \times n$ matrix $\mu \triangleq F^{\mathrm{T}}\Phi$. Note that since $\Phi F^{\mathrm{T}} = I_{n_u}$, μ is idempotent, i.e. $\mu^2 = \mu$.

Next note that (A 8) and (A 11) imply (2.15). Similarly, (2.14) is equivalent to (2.10) with B_e given by (2.15). Finally, (2.16) is a restatement of (2.12). Now, using the expression for B_e, $\tilde{A}$ and $\tilde{V}$ become

$$\tilde{A} = A - \mu Q_a V_2^{-1} C \tag{A 12}$$

$$\tilde{V} = V_1 - V_{12} V_2^{-1} Q_a^{\mathrm{T}} \mu^{\mathrm{T}} - \mu Q_a V_2^{-1} V_{12}^{\mathrm{T}} + \mu Q_a V_2^{-1} Q_a^{\mathrm{T}} \mu^{\mathrm{T}} \tag{A 13}$$

Finally, (2.17) and (2.18) follow from (A 3) and (A 7) using (A 12) and (A 13).

For the discrete-time problem define the second-moment matrix

$$Q(k) \triangleq \mathbb{E}[\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)]$$

which satisfies

$$Q(k+1) = \tilde{A}Q(k)\tilde{A}^{\mathrm{T}} + \tilde{V} \tag{A 14}$$

Since $\tilde{A}$ is asymptotically stable

$$Q \triangleq \lim_{k \to \infty} \mathbb{E}[\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)]$$

exists and satisfies

$$Q = \tilde{A}Q\tilde{A}^{\mathrm{T}} + \tilde{V} \tag{A 15}$$

The remainder of the proof follows as above for the continuous-time case.

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