International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

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Online Publication Date: 01 January 1992

To cite this Article: Haddad, Wassim M., Bernstein, Dennis S., Huang, Hsing-Hsin and Halevi, Yoram (1992) 'Fixed-order sampled-data estimation', International Journal of Control, 55:1, 129 — 139

To link to this article: DOI: 10.1080/00207179208934229
URL: http://dx.doi.org/10.1080/00207179208934229

INT. J. CONTROL, 1992, VOL. 55, NO. 1, 129–139

Fixed-order sampled-data estimation

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For the Kalman filter-type sampled-data estimation problem utilizing an averaging A/D device, the equivalent discrete-time problem is shown to be of increased order. The fixed-structure optimal projection approach for reduced-order, discrete-time estimation is applied to the equivalent discrete-time problem in order to characterize reduced-order estimators.

Nomenclature

$I_r, 0_{r\times s}, 0_r$	$r \times r$ identity matrix, $r \times s$ zero matrix, and $r \times r$ zero matrix
$(\ )^{\mathrm{T}}$, tr	transpose, trace
$\mathbb{E}, \mathbb{R}, \mathbb{R}^{r\times s}$	expected value, real numbers, $r \times s$ real matrices
n, l, n_e, q	positive integers, $1 \leqslant n_e \leqslant n + l$
x, y, x_e, y_e	n, l, n_e, q-dimensional vectors
A, C	$n \times n$, $l \times n$ matrices
A_e, B_e, C_e, D_e	$n_e \times n_e$, $n_e \times l$, $q \times n_e$, $q \times l$ matrices
w_1, w_2	n, l-dimensional, zero-mean, continuous-time white noise processes
V_1	$n \times n$ non-negative-definite intensity of w_1
V_2	$l \times l$ positive-definite intensity of w_2
V_{12}	$n \times l$ cross intensity of w_1, w_2
R	$q \times q$ positive-definite matrix
L	$q \times n$ matrix
t, k	$t \in [0, \infty)$, discrete-time index, 1, 2, 3, ...

1. Introduction

Owing to the advances in digital computers, discrete-time filtering and control of continuous-time systems have been developed and used in numerous applications. In the present paper we consider a Kalman filter-type, sampled-data, problem. It is well known that the optimal discrete-time estimates of the dynamic states of a continuous-time model are given by a discrete-time Kalman filter, which is based on an equivalent discrete-time model. Thus, we derive here an equivalent discrete-time problem and apply the fixed-structure approach developed by Bernstein *et al.* (1986 b), and Haddad (1987), to obtain reduced-order filters.

Received 31 October 1989. Revised 8 February 1991.

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Specifically, we consider the case in which the measurements of the continuous-time system are corrupted by white noise and we develop an equivalent discrete-time model which employs an averaging-type A/D device, as did Åström (1970), Shats and Shaked (1989), and Bernstein, *et al.* (1986 a).

The goal of the present paper is to present a novel design procedure which is applicable to the equivalent discrete-time problem. Since the discrete-time model is of augmented order $n + l$ (n = number of states, l = number of sensors), we seek dynamic filters of reduced order. To this end we apply the optimal projection, fixed-structure, approach for discrete-time estimation of the equivalent discrete-time problem, in order to characterize optimal filters of order $n_e < n + l$. These equations are discussed in Bernstein *et al.* (1986 b), and derived in Haddad (1987). Since the sample interval in real-time estimation implementation depends directly upon the estimator order n_e, a reduced-order estimator can effectively increase the sample rate. Thus, the engineering trade-offs of performance against estimator order and sample interval can be investigated using the approach of the present paper. Finally, using the identities of Van Loan (1978) we derive formulae for integrals of matrix exponentials arising in the sampled-data/discrete-time conversion.

2. Sampled-data estimation problem and equivalent discrete-time formulation

In this section we state the fixed-order, sampled-data, estimation problem. In the problem formulation the sample interval h and the estimator order n_e are fixed and the optimization is performed over the estimator parameters (A_e, B_e, C_e, D_e). For design trade-off studies h and n_e can be varied and the problem can be solved for each pair of values of interest. Finally, we assume that the plant dynamics, A, is asymptotically stable. The case in which A may contain unstable modes (i.e. rigid body dynamics) is significantly more involved and is deferred to a future paper. For details on the unstable estimation problem, see Bernstein and Haddad (1989), and Haddad and Bernstein (1990).

2.1. *Fixed-order, sampled-data estimation problem*

Given the nth order continuous-time system

$$\dot{x}(t) = Ax(t) + w_1(t), \quad t \in [0, \infty) \tag{2.1}$$

with continuous-time measurements

$$y(t) = Cx(t) + w_2(t) \tag{2.2}$$

design an n_eth-order discrete-time non-strictly proper estimator

$$x_e(k+1) = A_e x_e(k) + B_e y(k) \tag{2.3}$$

$$y_e(k) = C_e x_e(k) + D_e y(k) \tag{2.4}$$

which, with A/D averaged measurements

$$y(k) \triangleq \frac{1}{h} \int_{(k-1)h}^{kh} y(t)\, dt \tag{2.5}$$

and D/A zero-order-hold estimates

$$y_e(t) = y_e(k), \quad t \in [kh, (k+1)h] \tag{2.6}$$

minimizes the least squares estimation criterion

$$J(A_e, B_e, C_e, D_e) \triangleq \lim_{t \to \infty} \mathbb{E} \frac{1}{t} \int_0^t [Lx(s) - y_e(s)]^T R [Lx(s) - y_e(s)] \, ds \qquad (2.7)$$

In (2.7) the matrix L identifies the linear combinations Lx of states x whose estimates are desired. Furthermore, note that the feedthrough term D_e permits the utilization of a static least-squares estimator in conjunction with the dynamic estimator (A_e, B_e, C_e). The main result of this section concerns the propagation of the plant and the digitized measurements over one time step. For notational convenience in stating the main result define

$$H(s) \triangleq \int_0^s e^{Ar} \, dr \qquad (2.8)$$

Theorem 2.1

For the fixed order, sampled-data estimation problem, the plant dynamics (2.1), averaged measurements (2.5) and least-squares estimation criterion (2.7) have the equivalent discrete-time representations

$$x(k+1) = A'x'(k) + w'_1(k) \qquad (2.9)$$

$$y(k) = C'x(k-1) + w'_2(k-1) \qquad (2.10)$$

$$J(A_e, B_e, C_e, D_e) = \delta + \lim_{k \to \infty} \mathbb{E} \frac{1}{h} \int_0^t \{[Le^{As}x(k) - y_e(k)]^T R [Le^{As}x(k) - y_e(k)] - w_1'^T(s) L^T R y_e(k) - y_e(k) R L w_1'^T(s)\} \, ds \qquad (2.11)$$

where

$$A' \triangleq e^{Ah} \qquad (2.12)$$

$$C' \triangleq \frac{1}{h} CH(h) \qquad (2.13)$$

$$w'_1(k) \triangleq \int_0^h e^{A(h-s)} w_1(s + kh) \, ds \qquad (2.14)$$

$$w'_2(k-1) \triangleq \frac{1}{h} C \int_0^h \int_0^r e^{A(r-s)} w_1[s + (k-1)h] \, ds \, dr + \frac{1}{h} \int_0^h w_2[r + (k-1)h] \, dr \qquad (2.15)$$

$$\delta \triangleq \int_0^h \int_0^s e^{Ar} V_1 e^{A^T r} L^T R L \, dr \, ds \qquad (2.16)$$

$w'_1(k)$ and $w'_2(k)$ are zero-mean, white noise processes with

$$\mathbb{E} \begin{bmatrix} w'_1(k) \\ w'_2(k) \end{bmatrix} [w_1'^T(k) \quad w_2'^T(k)] = \begin{bmatrix} V'_1 & V'_{12} \\ V'^T_{12} & V'_2 \end{bmatrix} \qquad (2.17)$$

where

$$V_1' = \int_0^h e^{As} V_1 e^{A^{\mathrm{T}}s}\, ds$$

$$V_{12}' = \frac{1}{h}\int_0^h e^{As} V_1 e^{A^{\mathrm{T}}s}\, ds C^{\mathrm{T}} + \frac{1}{h} H(h) V_{12}$$

$$V_2' = \frac{1}{h} V_2 + \frac{1}{h^2} C \int_0^h H(s) V_1 H^{\mathrm{T}}(s)\, ds C^{\mathrm{T}}$$

$$+ \frac{1}{h^2} V_{12}^{\mathrm{T}} \int_0^h H^{\mathrm{T}}(s)\, ds C^{\mathrm{T}} + \frac{1}{h^2} C \int_0^h H(s)\, ds V_{12}.$$

The proof of this theorem is a straightforward calculation involving integrals of white noise signals, and hence is omitted. See Bernstein *et al.* (1986 a) for further discussion.

Note that (2.10) shows that the averaged measurements depend upon delayed samples of the state. Thus, by augmenting the discretized state equation (2.9) to include these measurements, it is possible to state the original sampled-data problem as a discrete-time problem involving non-noisy measurements. □

Corollary 2.1

With the notation

$$\hat{x}(k) \triangleq \begin{bmatrix} x'(k) \\ y(k) \end{bmatrix}, \quad \hat{A} \triangleq \begin{bmatrix} A' & 0_{n\times l} \\ C' & 0_l \end{bmatrix}, \quad \hat{C} \triangleq [0_{l\times n} \quad I_l]$$

$$\hat{w}(k) \triangleq \begin{bmatrix} w_1'(k) \\ w_2'(k) \end{bmatrix}, \quad \hat{V} \triangleq \begin{bmatrix} V_1' & V_{12}' \\ V_{12}'^{\mathrm{T}} & V_2' \end{bmatrix}, \quad \hat{L} \triangleq [Le^{As} \quad 0_{q\times l}]$$

the fixed-order, sampled-data estimation problem is equivalent to the following discrete-time problem. Given the $(n+l)$th-order discrete-time system

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{w}(k) \tag{2.18}$$

with discrete-time measurements

$$\hat{y}(k) = \hat{C}\hat{x}(k) \tag{2.19}$$

design an n_eth-order discrete-time estimator of the form (2.3), (2.4) which minimizes

$$J(A_e, B_e, C_e, D_e) = \delta + \lim_{k\to\infty} \mathbb{E}\frac{1}{h}\int_0^h \{[\hat{L}\hat{x}(k) - y_e(k)]^{\mathrm{T}} R[\hat{L}\hat{x}(k) - y_e(k)]$$
$$- w_1'^{\mathrm{T}}(s) L^{\mathrm{T}} R y_e(k) - y_e^{\mathrm{T}}(k) R L w_1'(s)\}\, ds \tag{2.20}$$

□

Remark 2.1

Note that the equivalent discrete-time least squares estimation criterion involves a constant offset δ which serves as a lower bound on the sampled-data performance due to the discretization process. □

Remark 2.2

Note that the measurements $\hat{y}(k)$ are noise free, however, due to the discrete-time setting this singularity is not as serious as singular measurement noise in continuous-time settings where the Kalman filter gains are expressed in terms of the inverse of the measurement noise intensity. However, as in non-strictly proper continuous-time estimation with non-noisy measurements this formulation leads to a static projection matrix v defined below. See Haddad and Bernstein (1987) and Halevi (1989) for further details.

Remark 2.3

The increase in plant order from n to $n+l$ is due to the discretization process. Since discrete-time, steady-state, Kalman filter theory yields a possibly unwieldy $(n+l)$th-order filter, we seek reduced-order filters. Note that in this context an nth-order estimator can be regarded as being of reduced order.

Next, note that the augmented equivalent discrete-time system (2.18), (2.19), (2.3), (2.4) can be written as

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k) + \tilde{w}(k) \tag{2.21}$$

where

$$\tilde{x}(k) \triangleq \begin{bmatrix} \hat{x}(k) \\ x_e(k) \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} \hat{A} & 0 \\ B_e\hat{C} & A_e \end{bmatrix}, \quad \tilde{w}(k) \triangleq \begin{bmatrix} \hat{w}(k) \\ 0 \end{bmatrix}$$

The cost can be expressed in terms of the augmented second-moment matrix. □

Proposition 2.1

$\tilde{A}$ is asymptotically stable if and only if A_e is asymptotically stable. In this case, the state estimation error criterion (2.20) is given by

$$J(A_e, B_e, C_e, D_e) = \delta + \operatorname{tr}[\tilde{Q}\tilde{R} - 2[D_e^{\mathrm{T}}R(\hat{V}_1 + \hat{V}_{12})]] \tag{2.22}$$

where the steady-state covariance

$$\tilde{Q}(k) \triangleq \lim_{k\to\infty} \mathbb{E}[\tilde{x}(k)\tilde{x}^{\mathrm{T}}(k)] \tag{2.23}$$

exists and satisfies the algebraic Lyapunov equation

$$\tilde{Q} = \tilde{A}\tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V} \tag{2.24}$$

and

$$\tilde{R} \triangleq \begin{bmatrix} R_1 & R_{12} \\ R_{12}^{\mathrm{T}} & R_2 \end{bmatrix}, \quad \tilde{V} \triangleq \begin{bmatrix} \hat{V} & 0 \\ 0 & 0 \end{bmatrix}$$

$$R_1 \triangleq \frac{1}{h}\int_0^h [\hat{L} - D_e\hat{C}]^{\mathrm{T}}R[\hat{L} - D_e\hat{C}]\,ds$$

$$R_{12} \triangleq \hat{C}^{\mathrm{T}}D_e^{\mathrm{T}}RC_e - \hat{L}^{\mathrm{T}}RC_e, \quad R_2 \triangleq C_e^{\mathrm{T}}RC_e$$

$$\hat{L} \triangleq \left[\frac{1}{h}LH(h) \;\; 0_{q\times l}\right] = [L' \;\; 0_{q\times l}]$$

$$\hat{V}_1 \triangleq \frac{1}{h^2}L\int_0^h H(s)V_1H^{\mathrm{T}}(s)\,ds\,C^{\mathrm{T}}, \quad \hat{V}_{12} \triangleq \frac{1}{h^2}L\int_0^h H(s)\,ds\,V_{12}$$

□

3. Necessary conditions for the equivalent discrete-time problem

We now apply the optimal projection equations for discrete-time estimation to the equivalent discrete-time problem in order to obtian necessary conditions for optimality. The following lemma is needed before stating the main results.

Lemma 3.1

Let $\hat{Q}$, $\hat{P}$ be $(n+l)\times(n+l)$ non-negative-definitive matrices and assume rank $(\hat{Q}\hat{P}) = n + l$. Then, there exist $n_e \times (n \times l)$ matrices G, Γ and $n_e \times n_e$ invertible matrix M, which are unique except for a change of basis in $\mathbb{R}^{n_e}$, such that the product $\hat{Q}\hat{P}$ can be factored as

$$\hat{Q}\hat{P} = G^{\mathrm{T}} M \Gamma \tag{3.1}$$

$$\Gamma G^{\mathrm{T}} = I_{n_e} \tag{3.2}$$

Furthermore, the $(n+l)\times(n+l)$ matrices

$$\tau \triangleq G^{\mathrm{T}}\Gamma \tag{3.3}$$

$$\tau_\perp \triangleq I_{(n+l)} - \tau \tag{3.4}$$

are idempotent and have rank n_e and $(n+l) - n_e$, respectively. □

Proof

For the proof, see Hyland and Bernstein (1985). □

To guarantee that J is finite and independent of initial conditions, we restrict our consideration to the open set

$$\mathcal{S} \triangleq \{(A_e, B_e, C_e, D_e)\colon A_e \text{ is stable and } (A_e, B_e, C_e) \text{ is minimal}\}$$

The following main result gives the necessary conditions that characterize solutions to the fixed-order, sampled-data, estimation problem. For convenience in stating this result define

$$\nu \triangleq Q\hat{C}^{\mathrm{T}}(\hat{C}Q\hat{C}^{\mathrm{T}})^{-1}\hat{C}, \quad \nu_\perp \triangleq I_{(n+l)} - \nu$$

for arbitrary $Q \in \mathbb{R}^{(n+l)\times(n+l)}$.

Theorem 3.1

Suppose A' is stable and $(A_e, B_e, C_e, D_e) \in \mathcal{S}$ solves the fixed-order, sampled-data estimation problem. Then there exist $(n+l)\times(n+l)$ non-negative definite Q, $\hat{Q}$ and $\hat{P}$ such that A_e, B_e, C_e, D_e are given by

$$A_e = \Gamma\hat{A}\nu_\perp G^{\mathrm{T}} \tag{3.5}$$

$$B_e = \Gamma\hat{A}Q\hat{C}^{\mathrm{T}}(\hat{C}Q\hat{C}^{\mathrm{T}})^{-1} \tag{3.6}$$

$$C_e = (\tilde{L} - D_e\hat{C})G^{\mathrm{T}} \tag{3.7}$$

$$D_e = (\tilde{L}Q\hat{C}^{\mathrm{T}} + \hat{V}_1 + \hat{V}_{12})(\hat{C}Q\hat{C}^{\mathrm{T}})^{-1} \tag{3.8}$$

and such that Q, $\hat{Q}$ and $\hat{P}$ satisfy

$$Q = \hat{A}Q\hat{A}^{\mathrm{T}} + \hat{V} + \tau_{\perp}(\hat{A}\hat{Q}\hat{A}^{\mathrm{T}} + \hat{A}vQv^{\mathrm{T}}\hat{A}^{\mathrm{T}})\tau_{\perp}^{\mathrm{T}} - \hat{A}vQv^{\mathrm{T}}\hat{A}^{\mathrm{T}} \tag{3.9}$$

$$\hat{Q} = \tau_{\perp}(\hat{A}\hat{Q}\hat{A}^{\mathrm{T}} + \hat{A}vQv^{\mathrm{T}}\hat{A}^{\mathrm{T}})\tau_{\perp}^{\mathrm{T}} \tag{3.10}$$

$$\hat{P} = \tau^{\mathrm{T}}v_{\perp}^{\mathrm{T}}\hat{A}^{\mathrm{T}}\hat{P}\hat{A}v_{\perp}\tau + \tau^{\mathrm{T}}(\tilde{L} - D_{\mathrm{e}}\hat{C})^{\mathrm{T}}R(\tilde{L} - D_{\mathrm{e}}\hat{C})\tau \tag{3.11}$$

Furthermore, the minimal cost is given by

$$J(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}}) = \delta + \operatorname{tr}[QR_1 + \hat{Q}[R_1 - (\tilde{L} - D_{\mathrm{e}}\hat{C})^{\mathrm{T}}R(\tilde{L} - D_{\mathrm{e}}\hat{C})] - 2[D_{\mathrm{e}}^{\mathrm{T}}R(\hat{V}_1 + \hat{V}_{12})]] \tag{3.12}$$

□

Next, we present a partial converse of the necessary conditions. First, we need the following definition.

Definition 3.1

An estimator $(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}})$ is an *extremal* of the optimal fixed-order sampled-data estimation problem if it satisfies the stationary conditions (3.9)–(3.11). □

Theorem 3.2

Suppose there exist non-negative definite matrices Q, $\hat{Q}$, $\hat{P}$ satisfying (3.9)–(3.11) and $(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}})$ satisfying (3.5)–(3.8). Then the estimator $(A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}, D_{\mathrm{e}})$ is an extremal of the optimal fixed-order, sampled-data estimation problem. Furthermore, the following are equivalent

(i) A_{e} is stable

(ii) $\left(\begin{bmatrix} \hat{A} & 0 \\ \hat{B}_{\mathrm{e}}C & A_{\mathrm{e}} \end{bmatrix}, \begin{bmatrix} \hat{V} & 0 \\ 0 & 0 \end{bmatrix}^{1/2}\right)$ is stabilizable.

In this case $(A_{\mathrm{e}}, B_{\mathrm{e}})$ is controllable and $(A_{\mathrm{e}}, C_{\mathrm{e}})$ is observable. □

The proofs of Theorems 3.1 and 3.2 follow as a special case from the reduced-order, discrete-time, estimation proofs given by Haddad (1987).

Remark 3.1

Theorem 3.1 can immediately be specialized to the more restrictive problem in which the estimator is strictly proper. This can be done by ignoring (3.8) and replacing the last term in (3.11) by $\tilde{L}^{\mathrm{T}}R\tilde{L}$. □

Remark 3.2

In the full-order case $n_{\mathrm{e}} = n + l$, the projection τ becomes the identity and (3.10) and (3.11) play no role. In this case $G^{\mathrm{T}}\Gamma = \Gamma G^{\mathrm{T}} = I_{(n+l)}$ and thus G and Γ can be chosen to be the identity. In this case the estimator is characterized by

$$A_{\mathrm{e}} = \hat{A}v_{\perp} \tag{3.13}$$

$$B_{\mathrm{e}} = \hat{A}Q\hat{C}^{\mathrm{T}}(\hat{C}Q\hat{C}^{\mathrm{T}})^{-1} \tag{3.14}$$

$$C_c = \tilde{L} - D_c\hat{C} \tag{3.15}$$

$$D_c = (\tilde{L}Q\hat{C}^{\mathrm{T}} + \hat{V}_1 + \hat{V}_{12})(\hat{C}Q\hat{C}^{\mathrm{T}})^{-1} \tag{3.16}$$

where Q satisfies

$$Q = \hat{A}Q\hat{A}^{\mathrm{T}} - \hat{A}\nu Q\nu^{\mathrm{T}}\hat{A}^{\mathrm{T}} + \hat{V} \tag{3.17}$$

□

4. Numerical evaluation of integrals involving matrix exponentials

To evaluate the exponential/integral expressions appearing in Theorem 2.1, we utilize the approach given by Van Loan (1978). This approach eliminates the need for integration by computing the matrix exponential of appropriate block matrices. For details on numerical matrix exponentiation see Moler and Van Loan (1978).

Proposition 4.1

Consider the following partitioned matrix exponentials of order $(3n+l)\times(3n+l)$ and $(3n)\times(3n)$, respectively.

$$\begin{bmatrix} F_1 & F_2 & F_3 & F_4 \\ 0_n & F_5 & F_6 & F_7 \\ 0_n & 0_n & F_8 & F_9 \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & I_l \end{bmatrix} \triangleq \exp \begin{bmatrix} -A & I_n & 0_n & 0_{n\times l} \\ 0_n & -A & V_1 & V_{12} \\ 0_n & 0_n & A^{\mathrm{T}} & C^{\mathrm{T}} \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & 0_l \end{bmatrix} h$$

$$\begin{bmatrix} F_{25} & F_{26} & F_{27} & F_{28} \\ 0_n & F_{29} & F_{30} & F_{31} \\ 0_n & 0_n & F_{32} & F_{33} \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & I_l \end{bmatrix} \triangleq \exp \begin{bmatrix} -A^{\mathrm{T}} & L^{\mathrm{T}}RL & 0_n & 0_{n\times l} \\ 0_n & I_n & 0_n & 0_{n\times l} \\ 0_n & 0_n & A^{\mathrm{T}} & L^{\mathrm{T}} \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & 0_l \end{bmatrix} h$$

$$\begin{bmatrix} F_{10} & F_{11} & F_{12} \\ 0_n & F_{13} & F_{14} \\ 0_n & 0_n & F_{15} \end{bmatrix} \triangleq \exp \begin{bmatrix} -A & I_n & 0_n \\ 0_n & -A & V_1 \\ 0_n & 0_n & A \end{bmatrix} h$$

$$\begin{bmatrix} F_{16} & F_{17} & F_{18} & F_{19} \\ 0_n & F_{20} & F_{21} & F_{22} \\ 0_n & 0_n & F_{23} & F_{24} \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & I_l \end{bmatrix} \triangleq \exp \begin{bmatrix} -A & I_n & 0_n & 0_{n\times l} \\ 0_n & -A & V_1 & 0_{n\times l} \\ 0_n & 0_n & A^{\mathrm{T}} & C^{\mathrm{T}} \\ 0_{l\times n} & 0_{l\times n} & 0_{l\times n} & 0_l \end{bmatrix} h$$

Then

$$A' = F_8^{\mathrm{T}}, \quad C' = \frac{1}{h}F_9^{\mathrm{T}}, \quad \delta = \frac{1}{h}\operatorname{tr}(L^{\mathrm{T}}RLF_{15}F_{12}^{\mathrm{T}})$$

$$V_1' = F_8^{\mathrm{T}}F_6, \quad V_{12}' = \frac{1}{h}F_8^{\mathrm{T}}F_7$$

$$V_2' = \frac{1}{h}\left(V_2 + \frac{1}{h}CF_8^{\mathrm{T}}F_4 + \frac{1}{h}F_4^{\mathrm{T}}F_8C^{\mathrm{T}} - \frac{1}{h}CF_8^{\mathrm{T}}F_{19}\right)$$

$$R_1 = \frac{1}{h}F_{29}^{\mathrm{T}}F_{26}, \quad L' = \frac{1}{h}F_{33}^{\mathrm{T}}$$

□

The proof of the above proposition involves straightforward manipulations of matrix exponentials and hence is omitted.

5. Illustrative numerical example

In this section we present a numerical example involving a simply supported Euler–Bernoulli beam. The partial differential equation for the transverse deflection $w(x, t)$ is given by

$$m(x)w_{tt}(x, t) = -[EI(x)w_{xx}(x, t)]_{xx} + f(x, t)$$

$$w(x, t)\,|_{x=0,L} = 0, \quad EI(x)w_{xx}(x, t)\,|_{x=0,L} = 0$$

where $m(x)$ is the mass per unit length, $EI(x)$ is the flexural rigidity with E denoting Young's modulus of elasticity and $I(x)$ denoting the cross-sectional area moment of inertia about an axis of the plane of vibration and passing through the centre of the cross-sectional area. Finally, $f(x, t)$ is a disturbance acting on the beam. Assuming uniform beam properties, the modal decomposition of this system has the form

$$w(x, t) = \sum_{r=1}^{\infty} W_r(x)q_r(t)$$

$$\int_0^L mW_r^2(x)\, dx = 1$$

$$W_r(x) = (2/mL)^{1/2} \sin \frac{r\pi x}{L}$$

where, assuming uniform proportional damping, the modal coordinates q_r satisfy

$$\ddot{q}_r(t) + 2\zeta\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = \int_0^L f(x, t)W_r(x), \quad r = 1, 2, \dots$$

For simplicity assume $L = \pi$ and $m = EI = 2/\pi$ so that $(2/mL)^{1/2} = 1$. Furthermore, we assume a sensor located at $x = 0{\cdot}65\pi$, and a point force disturbance located at $x = 0{\cdot}40\pi$. As inputs to the estimator design we choose to weight the performance of the beam displacement at $x = 0{\cdot}65\pi$. Finally, modelling the first five modes and defining the plant state as $x = [q_1, \dot{q}_1, \dots, q_5, \dot{q}_5]^{\mathrm{T}}$, the resulting state space model and problem data are

$$A = \underset{i=1,\dots,5}{\text{block-diag}} \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \quad \omega_i = i^2, \quad i = 1, \dots, 5, \quad \zeta = 0{\cdot}05$$

$$C = [0{\cdot}9877 \;\; 0 \;\; -0{\cdot}3090 \;\; 0 \;\; -0{\cdot}8910 \;\; 0 \;\; 0{\cdot}5878 \;\; 0 \;\; 0{\cdot}7071 \;\; 0]$$

$$L = [0{\cdot}8910 \;\; 0 \;\; -0{\cdot}8090 \;\; 0 \;\; -0{\cdot}1564 \;\; 0 \;\; 0{\cdot}9511 \;\; 0 \;\; -0{\cdot}7071 \;\; 0]$$

$$D = [0 \;\; 0{\cdot}9511 \;\; 0 \;\; 0{\cdot}5878 \;\; 0 \;\; -0{\cdot}5878 \;\; 0 \;\; -0{\cdot}9511 \;\; 0 \;\; 0]^{\mathrm{T}}$$

$$V_1 = DD^{\mathrm{T}}, \quad V_2 = 0{\cdot}001, \quad R = 1$$

For the full-order case, $n_e = 11$, discrete-time, sampled-data, estimators were obtained using (3.17) and Proposition 4.1 for continuous-time to discrete-time conversions. A straightforward iterative scheme was employed for solving (3.17). The results are summarized as follows. The Table shows respective estimation error costs for sampling intervals of 10 Hz, 30 Hz, and 60 Hz sensors. Corresponding simulation plots for error states 3 and 4 are shown in Figs 1 and 2.

Zeroth estimator ($B_e = 0$)	18·6714
10 Hz estimator	1·6732
30 Hz estimator	0·2895
60 Hz estimator	0·1573

Estimation error costs.

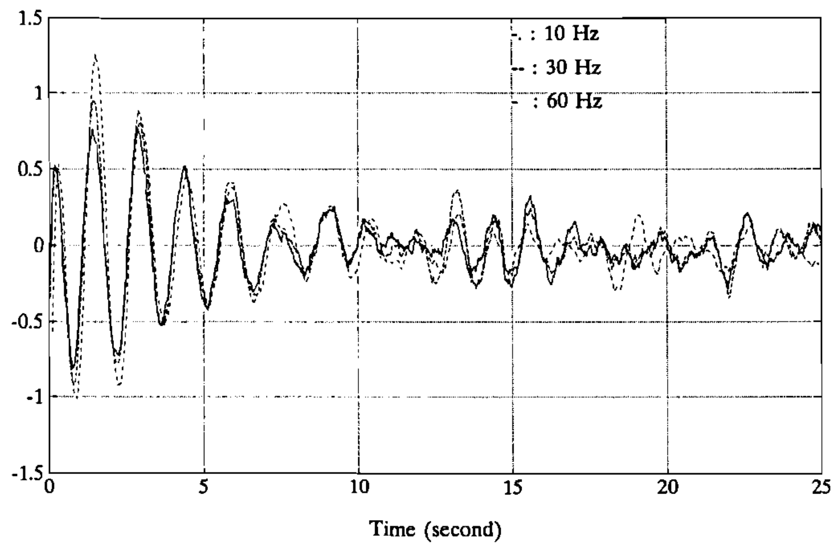

Figure 1. Error state 3 simulation.

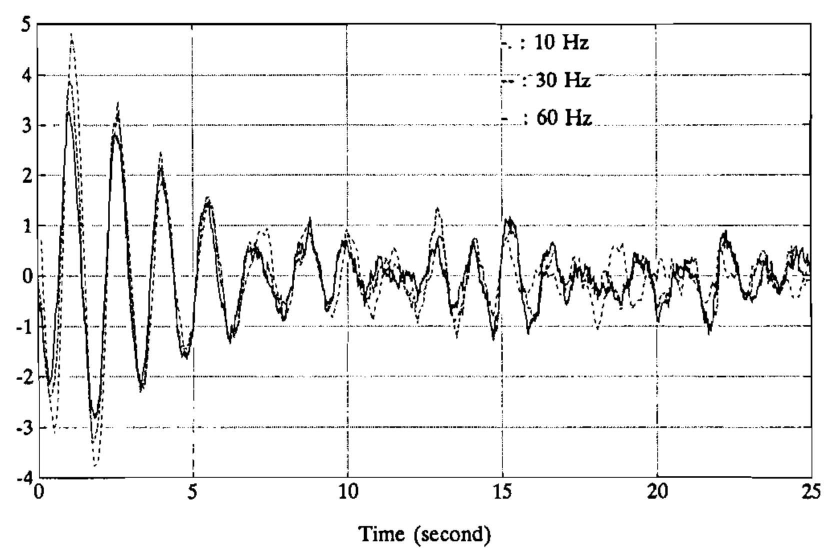

Figure 2. Error state 4 simulation.

6. Directions for further research

The following extensions and related developments immediately suggest themselves: reduced-order, discrete-time modelling of continuous-time systems (see Hyland and Bernstein 1985): robust sampled-data estimation of uncertain systems (see Bernstein and Hollot 1989): multi-rate sampling (see Glasson 1982, Andrisani and Fu-Gau 1987, Araki and Yamamoto 1986, and Haddad *et al.* 1990): alternative A/D and D/A devices and asynchronous sampling.

Acknowledgments

The work was supported in part by the Florida Institute of Technology Space Research Institute under a grant from TRDA and the Air Force Office of Scientific Research under Contract F49620-89-C-0011.

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