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Average-preserving symmetries and energy equipartition in linear Hamiltonian systems

Sanjay P. Bhat · Dennis S. Bernstein

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Abstract This paper analyzes energy equipartition in linear Hamiltonian systems in a deterministic setting. We consider the group of phase space symmetries of a stable linear Hamiltonian system, and characterize the subgroup of symmetries whose elements preserve the time averages of quadratic functions along the trajectories of the system. As a corollary, we show that if the system has simple eigenvalues, then every symmetry preserves averages of quadratic functions. As an application of our results to linear undamped lumped-parameter systems, we provide a novel proof of the virial theorem, which states that the total energy is equipartitioned on the average between the kinetic energy and the potential energy. We also show that under the assumption of distinct natural frequencies, the time-averaged energies of two identical substructures of a linear undamped structure are equal. Examples are provided to illustrate the results.

Keywords Equipartition · Hamiltonian systems · Symmetry · Average-preserving symmetry · Virial theorem · Center subgroups

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S. P. Bhat (✉)
Department of Aerospace Engineering, Indian Institute of Technology,
Powai, Mumbai 400076, India
e-mail: bhat@aero.iitb.ac.in

D. S. Bernstein
Department of Aerospace Engineering, The University of Michigan,
Ann Arbor, MI 48109-2140, USA
e-mail: dsbaero@umich.edu

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1 Introduction

Undamped and thus conservative mechanical systems are Lyapunov stable, and thus have no steady-state energy distribution. The simplest example of an undamped single-degree-of-freedom oscillator shows that the system's energy is periodically converted from purely kinetic to purely potential and then back to purely kinetic. A two-degree-of-freedom system consisting of an interconnected pair of undamped oscillators exhibits similar behavior, with the energy alternately possessed by one oscillator and then the other. Yet, in classical statistical thermodynamics, a crystalline solid is modeled as a lattice of identical molecules undergoing undamped vibrations, whose degrees of freedom are assumed to satisfy the principle of equipartition of energy. Thus, despite the lack of a steady-state energy distribution in undamped systems, lattice models with a large number of oscillators provide the conceptual foundation for macroscopic energy transfer.

A model of the macroscopic dynamics is provided by the laws of thermodynamics, which govern the dynamics of heat. The laws of thermodynamics are inherently empirical, and the development of these laws and associated concepts has had a long and tortuous history, see [9,20]. From a systems perspective, thermodynamics is a theory of large scale systems, whose properties, in modern terminology, are a manifestation of emergent behavior.

Statistical mechanics has been very successful at predicting the macroscopic properties of a large-scale system from those of its microscopic constituents. However, statistical mechanics possesses two features that systems theorists and dynamicists have repeatedly attempted to understand rigorously [2,4,8,10,17,18,21,22]. The first is the use of stochastics to deduce the macroscopic properties of a system from the deterministic dynamics of its microscopic constituents. The second is the assumption of a large number of microscopic constituents. While the assumption of a large number of constituents is related to the use of asymptotic approximations like the Stirling approximation [19] or the central limit theorem [13], the use of stochastics arises from assuming the initial states to be random variables distributed according to canonical ensembles or distributions that remain invariant under the collective, conservative dynamics of the microscopic constituents.

The principle of equipartition of energy is a well-known result of classical statistical mechanics. It states that the average energies in any two degrees of freedom of a large-scale system in thermal equilibrium are equal, where the averages are ensemble averages taken with respect to one of the canonical ensembles on the state space. Under the assumption of ergodicity, the principle of equipartition of energy also implies that the time averages of the energies in any two degrees of freedom are equal. Physically, the principle of equipartition of energy implies that the temperature of each subsystem converges to the same value, thus underpinning the zeroth law of thermodynamics, i.e., that heat flows from hot to cold. The objective of this paper is to investigate equipartition in a deterministic setting for linear undamped systems whose number of degrees of freedom is not necessarily large.

In [7] a deterministic averaging approach was used to analyze equipartition in collections of identical, undamped coupled oscillators. It was shown that equipartition of energy holds for a pair of identical, coupled oscillators with *distinct* coupled frequencies. This result shows that, with time averaging, energy flows from the initially higher energy oscillator to the initially lower energy oscillator, thereby verifying the zeroth law of thermodynamics for a pair of coupled oscillators. In addition, numerical evidence was presented to suggest that an analogous result holds for a collection of coupled oscillators.

Reference [16] adopted a deterministic averaging approach in a behavioral framework to analyze equipartition in an oscillatory system comprised of two identical subsystems coupled together in a symmetric manner. It was shown that if the coupling renders the motion of each of the two subsystems observable through the variables of the other, then, along every motion of the coupled system, the time average of any given quadratic functional of the variables of one subsystem equals the time average of the same functional of the variables of the other subsystem. In particular, it follows that the energies of the two subsystems are equal on the average. An equipartition result in a time-averaged sense is also given in [1, Sec. 3.7] under the assumption of ergodicity. Reference [10] contains results which show that compartmental systems modeled in terms of energy exchange satisfying natural axioms exhibit energy equipartition in the steady state (but not necessarily in a time-averaged sense).

Intuitively, one expects a system composed of coupled subsystems to exhibit equipartition of energy only if the subsystems are symmetrically related, i.e., the state variables of one subsystem transform to those of the other subsystem under a state-space symmetry S that leaves the dynamics of the overall coupled system unchanged. Moreover, the energies of such symmetrically related subsystems, which are quadratic functions on the full state space, transform into one another under the symmetry transformation S. Hence, the average energies of the two identical subsystems are equal if the symmetry S that relates the two subsystems also preserves averages of all quadratic functions.

To formalize the above ideas we consider the Lie group $\mathbf{G}_A$ of phase space symmetries of a linear Hamiltonian system $\dot{y} = Ay$. Thus $\mathbf{G}_A$ is the set of all symplectic transformations that leave the dynamics of the system invariant. As expected from Noether's theorem [3, Appendix 5], the Lie algebra of $\mathbf{G}_A$ is the set of all linear Hamiltonian systems whose Hamiltonian functions are the quadratic integrals of motion of the original system $\dot{y} = Ay$.

In Sect. 4, we consider averages of quadratic functions along the solutions of a stable Hamiltonian system. In Sect. 5, we identify symmetries of the Hamiltonian system that preserve averages of quadratic functions. Our main result says that the symmetries in $\mathbf{G}_A$ that preserve the average of every quadratic function form the subgroup $\mathfrak{G}_A$ of $\mathbf{G}_A$ whose elements are also symmetries of every linear Hamiltonian system whose Hamiltonian function is an integral of motion of the original system.

In Sect. 6, we characterize the groups $\mathbf{G}_A$ and $\mathfrak{G}_A$ along with their Lie algebras and describe their structure in terms of the eigenstructure of A. In particular, we show that the subgroup $\mathfrak{G}_A$ of average-preserving symmetries is the center subgroup of $\mathbf{G}_A$ consisting of those symmetries of the system that commute with every other symmetry of the system. As a corollary of our characterization, we show that if the system has distinct eigenvalues, then $\mathfrak{G}_A = \mathbf{G}_A$, i.e., every symmetry preserves averages. This corollary partly justifies the assumption of distinct coupled natural frequencies used in [7].

In Sect. 7, we apply our main result to undamped linear lumped-parameter systems. We show that the kinetic and potential energies of such a system, considered as quadratic functions on the phase space, transform into one another under a phase space symmetry of the system that also commutes with every other phase space symmetry. An application of our main result thus yields a novel proof of the virial theorem for linear systems, which states that the average kinetic and potential energies of an undamped linear mechanical system are equal. We also specialize our main result to configuration space symmetries, i.e., orthogonal transformations on the configuration space that leave the mass and stiffness matrices invariant. We show that under the assumption of distinct natural frequencies, every configuration-level symmetry preserves the average of a quadratic function of positions or velocities. We provide an example to illustrate how this corollary can be used to deduce equipartition among identical subsystems of an undamped system. We also provide an example to show that equipartition does not necessarily hold when the assumption of distinct natural frequencies fails.

2 Preliminaries

We begin by reviewing some terminologies and notions from symplectic geometry. The reader may refer to [3, Sec. 41] and [14, Ch. II] for further details.

For each positive integer n, we let I_n denote the identity matrix of size $n \times n$, and let

$$J_{2n} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Given n, the *canonical symplectic form* on $\mathbb{R}^{2n}$ is the nondegenerate, skew-symmetric, bilinear form given by $(x, y) \mapsto x^{\mathrm{T}} J_{2n} y$. For every n, $\mathbb{R}^{2n}$ together with the canonical symplectic form is a *symplectic space* [14, Ch. II].

Two subspaces $\mathbb{V}_1$ and $\mathbb{V}_2$ of $\mathbb{R}^{2n}$ are *skew-orthogonal* if $x_1^{\mathrm{T}} J_{2n} x_2 = 0$ for all $x_1 \in \mathbb{V}_1$ and $x_2 \in \mathbb{V}_2$.

A *symplectic subspace* of $\mathbb{R}^{2n}$ is a linear subspace $\mathbb{W}$ of $\mathbb{R}^{2n}$ such that, for every nonzero $x \in \mathbb{W}$, there exists $y \in \mathbb{W}$ such that $x^{\mathrm{T}} J_{2n} y \neq 0$. Every symplectic subspace of $\mathbb{R}^{2n}$ has even dimension [14, p. 43].

Suppose $\mathbb{V}$ is a $2r$-dimensional symplectic subspace of $\mathbb{R}^{2n}$. A basis $\{x_1, \ldots, x_{2r}\}$ for the subspace $\mathbb{V}$ is a *symplectic basis* if $x_i^{\mathrm{T}} J_{2n} x_j = (J_{2n})_{ij}$ for all $i, j \in \{1, \ldots, 2r\}$. The standard basis is a symplectic basis for $\mathbb{R}^{2n}$. Every symplectic subspace of $\mathbb{R}^{2n}$ has a symplectic basis [14, Cor. II.B.2].

A matrix $S \in \mathbb{R}^{2n \times 2n}$ is *symplectic* if $S^{\mathrm{T}} J_{2n} S = J_{2n}$, and *Hamiltonian* if $J_{2n} S$ is symmetric. We denote by $\mathrm{Sp}(n)$ the set of all $2n \times 2n$ symplectic matrices. $\mathrm{Sp}(n)$ is a $n(2n+1)$-dimensional Lie group [15, p. 8]. The Lie algebra of $\mathrm{Sp}(n)$ is the set $\mathrm{sp}(n)$ of $2n \times 2n$ Hamiltonian matrices [5, Prop. 11.5.5]. A matrix $B \in \mathrm{sp}(n)$ is *stable* if every eigenvalue of B is semisimple and has zero real part.

For every n, we denote the $\frac{1}{2}n(n+1)$-dimensional real vector space of all $n \times n$ real symmetric matrices by $\mathrm{Sym}(n)$, the set of all $n \times n$ real special orthogonal matrices by $\mathrm{SO}(n)$, and the $\frac{1}{2}n(n-1)$-dimensional real vector space of all $n \times n$ real skew-symmetric matrices by $\mathrm{so}(n)$. $\mathrm{SO}(n)$ is a $\frac{1}{2}n(n-1)$-dimensional Lie group, and has the Lie algebra $\mathrm{so}(n)$ [5, Prop. 11.5.5].

Two groups G_1 and G_2 are *isomorphic* if there exists a bijection $\varphi : G_1 \to G_2$ such that $\varphi(S_1S_2) = \varphi(S_1)\varphi(S_2)$ for all $S_1, S_2 \in G_1$.

The *center subgroup* of a group G is the set of elements in G that commute with all other elements of G under the group operation. The *center subalgebra* of a Lie algebra g is the set of elements in g that commute with all other elements of g [15, p. 40] under the Lie bracket.

Later in this paper, we will have to consider the group $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ of all symplectic, special orthogonal matrices. $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ is a closed subgroup of the Lie group $\mathrm{Sp}(n)$, and hence a Lie group [15, Thm. 2.3.6]. It is easy to show that the Lie algebra of $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ is the set $\mathrm{sp}(n) \cap \mathrm{so}(2n)$ of Hamiltonian, skew-symmetric matrices. It is also easy to show that $S \in \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ if and only if $S \in \mathrm{Sp}(n)$ and $SJ_{2n} = J_{2n}S$. Similarly, $B \in \mathrm{sp}(n) \cap \mathrm{so}(2n)$ if and only if $B \in \mathrm{sp}(n)$ and $BJ_{2n} = J_{2n}B$. It follows that, if $S, B \in \mathbb{R}^{2n\times 2n}$ are partitioned as

$$S = \begin{bmatrix} R_1 & R_2 \\ -R_2 & R_1 \end{bmatrix}, \quad B = \begin{bmatrix} G & P \\ -P & G \end{bmatrix}, \tag{1}$$

where $R_1, R_2, G, P \in \mathbb{R}^{n\times n}$, then $S \in \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ if and only if $R_1^{\mathrm{T}}R_1 + R_2^{\mathrm{T}}R_2 = I_n$ and $R_2^{\mathrm{T}}R_1 - R_1^{\mathrm{T}}R_2 = 0$, while $B \in \mathrm{sp}(n) \cap \mathrm{so}(2n)$ if and only if $G \in \mathrm{so}(n)$ and $P \in \mathrm{Sym}(n)$. This last fact allows us to conclude that $\mathrm{sp}(n) \cap \mathrm{so}(2n)$ is a n^2-dimensional Lie algebra. Consequently, $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ is a n^2-dimensional Lie group.

Our first result characterizes the center subgroup of $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ for later use. The proof is given in Appendix.

Proposition 2.1 *The following statements are equivalent.*

(i) *S belongs to the center subgroup of* $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$.
(ii) *$S \in \mathrm{Sp}(n)$ and $SB = BS$ for all $B \in \mathrm{sp}(n) \cap \mathrm{so}(2n)$.*
(iii) *There exists $t \in \mathbb{R}$ such that $S = e^{J_{2n}t}$.*

Proposition 2.1 implies that the center subgroup of $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ is the one-parameter subgroup generated by J_{2n}. Our next result shows that the center subalgebra of $\mathrm{sp}(n) \cap \mathrm{so}(2n)$ is $\mathrm{span}\{J_{2n}\}$. The proof is given in Appendix.

Proposition 2.2 *The center subalgebra of* $\mathrm{sp}(n) \cap \mathrm{so}(2n)$ *is* $\mathrm{span}\{J_{2n}\}$.

Our last two results of this section characterize the center subgroup and subalgebra of $\mathrm{Sp}(n)$ and $\mathrm{sp}(n)$, respectively. The proofs of both results are given in Appendix.

Proposition 2.3 *The following statements are equivalent.*

(i) *S belongs to the center subgroup of* $\mathrm{Sp}(n)$.
(ii) *$S \in \mathrm{Sp}(n)$ and $SB = BS$ for all $B \in \mathrm{sp}(n)$.*
(iii) $S \in \{I_{2n}, -I_{2n}\}$.

Proposition 2.4 *The center subalgebra of* $\mathrm{sp}(n)$ *is* $\{0\}$.

Given subsets $G_1, \ldots, G_k$ of $\mathbb{R}^{n\times n}$, we use $\bigotimes_{i=1}^{k} G_i$ to denote the set $\{S_1 \cdots S_k : S_i \in G_i, i = 1, \ldots, k\}$ of all possible ordered products formed by taking exactly one element from each G_i. If $\mathbb{V}_1, \ldots, \mathbb{V}_k$ are linear subspaces of $\mathbb{R}^{2n}$ such that $\mathbb{V}_i \cap \mathbb{V}_j = \{0\}$ for all distinct $i, j \in \{1, \ldots, k\}$, then we denote the direct sum of $\mathbb{V}_1, \ldots, \mathbb{V}_k$ by $\bigoplus_{i=1}^{k} \mathbb{V}_i$.

Finally, given square matrices $A_1, \ldots, A_k$, we denote by $\mathrm{diag}(A_1, \ldots, A_k)$ the block diagonal matrix

$$\begin{bmatrix} A_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & A_k \end{bmatrix}.$$

3 Integrals of motion and symmetries of linear Hamiltonian systems

In this paper, we consider a linear Hamiltonian system, i.e., a system having the state-space description

$$\dot{y}(t) = Ay(t), \tag{2}$$

where $y(t) \in \mathbb{R}^{2n}$ and $A \in \mathrm{sp}(n)$. The *Hamiltonian function* of the system (2) is the quadratic function $x \mapsto x^{\mathrm{T}} Hx$, where the matrix $H \overset{\mathrm{def}}{=} -J_{2n}A = J_{2n}^{-1}A = A^{\mathrm{T}}J_{2n}$ is symmetric.

Next, define $\mathcal{L}_A : \mathrm{Sym}(2n) \to \mathrm{Sym}(2n)$ by

$$\mathcal{L}_A(P) \overset{\mathrm{def}}{=} A^{\mathrm{T}}P + PA.$$

For each $P \in \mathrm{Sym}(2n)$, the quadratic function $x \mapsto x^{\mathrm{T}}\mathcal{L}_A(P)x$ is the Lie derivative of the quadratic function $x \mapsto x^{\mathrm{T}}Px$ along trajectories of (2), i.e., for every solution $y(\cdot)$ of (2) and for every $P \in \mathrm{Sym}(2n)$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t} y^{\mathrm{T}}(t)Py(t) = y^{\mathrm{T}}(t)\mathcal{L}_A(P)y(t)$ for all $t \in \mathbb{R}$. If $P \in \mathrm{kernel}\, \mathcal{L}_A$, then $A^{\mathrm{T}}P + PA = 0$, and thus the quadratic function $x \mapsto x^{\mathrm{T}}Px$ is an integral of motion for the system (2).

Next, we introduce the set $\mathbf{G}_A \overset{\mathrm{def}}{=} \{S \in \mathrm{Sp}(n) : S^{-1}AS = A\}$ of linear symplectic transformations with respect to which the dynamics (2) are invariant in the sense that $z(\cdot) = S^{-1}y(\cdot)$ satisfies (2) for every $S \in \mathbf{G}_A$ and every $y(\cdot)$ satisfying (2). It will be convenient to refer to each element of $\mathbf{G}_A$ as a *symmetry* of (2). It is easy to show that $\mathbf{G}_A = \{S \in \mathrm{Sp}(n) : S^{\mathrm{T}}HS = H\}$ so that $\mathbf{G}_A$ is also the set of symplectic transformations that leave the quadratic Hamiltonian function $x \mapsto x^{\mathrm{T}}Hx$ invariant. $\mathbf{G}_A$ is clearly a group. Our next result asserts that $\mathbf{G}_A$ is a Lie group. Furthermore, the Lie algebra of $\mathbf{G}_A$ is the Lie algebra of linear Hamiltonian systems $\dot{x} = Bx$ on $\mathbb{R}^{2n}$, each of whose quadratic Hamiltonian function $x^{\mathrm{T}}J_{2n}^{-1}Bx$ is a quadratic integral of motion of (2).

Proposition 3.1 $\mathbf{G}_A$ *is a closed Lie subgroup of* $\mathrm{Sp}(n)$*, and its Lie algebra is* $\mathbf{g}_A \stackrel{\text{def}}{=} \{B \in \mathrm{sp}(n) : AB = BA\}$. *Furthermore,*

$$\mathbf{g}_A = \{J_{2n}P : P \in \text{kernel } \mathcal{L}_A\}.$$

Proof The first statement follows from (1) of Theorem I.1.2.1 of [15] by noting that $\mathbf{G}_A$ is the stabilizer of A under the action of the group $\mathrm{Sp}(n)$ on $\mathbb{R}^{2n\times 2n}$ given by $(S, B) \mapsto S^{-1}BS$. The second statement follows from the fact that $B \in \mathrm{sp}(n)$ satisfies $AB = BA$ if and only if $P \stackrel{\text{def}}{=} J_{2n}^{-1}B \in \text{kernel } \mathcal{L}_A$. □

Remark 3.1 The relationship between the symmetries and the integrals of motion of (2) pointed out in Proposition 3.1 is a restatement of the Hamiltonian version of Noether's theorem (see, for instance, [3, Appendix 5], [14, Thm. V.D.1]).

4 Time-averages of quadratic functions

In this section, we investigate time-averages of quadratic functions along the solutions of (2).

Henceforth, we assume that the Hamiltonian matrix A is stable. Thus all eigenvalues of A are semisimple and imaginary, and every solution of (2) is bounded. In particular, the matrix exponential $t \mapsto e^{At}$ is a bounded function.

Next, we define $\mathcal{A}_A : \mathrm{Sym}(2n) \to \mathrm{Sym}(2n)$ by

$$\mathcal{A}_A(Q) \stackrel{\text{def}}{=} \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{A^{\mathrm{T}}\tau} Q e^{A\tau}\, \mathrm{d}\tau. \tag{3}$$

For $x \in \mathbb{R}^{2n}$ and $Q \in \mathrm{Sym}(2n)$, $x^{\mathrm{T}}\mathcal{A}_A(Q)x$ is the average over $[0, \infty)$ of the quadratic function $z \mapsto z^{\mathrm{T}}Qz$ evaluated along the solution $y(\cdot)$ of (2) satisfying $y(0) = x$. Thus the elements of kernel $\mathcal{A}_A$ represent quadratic functions that have zero average along trajectories of (2).

The next result relates the property of having zero average along trajectories of (2) to the property of being an integral of motion of (2). More specifically, equation (4) below asserts that a quadratic function is an integral of motion of (2) if and only if it is the average of some quadratic function along the solutions of (2), while (5) asserts that a quadratic function has zero average along trajectories of (2) if and only if it is the Lie derivative of some quadratic function along the trajectories of (2). Equation (4) is also given as Proposition 15 in [16].

Proposition 4.1

$$\text{kernel } \mathcal{L}_A = \text{range } \mathcal{A}_A, \tag{4}$$
$$\text{kernel } \mathcal{A}_A = \text{range } \mathcal{L}_A. \tag{5}$$

Proof Let $Q \in \mathrm{Sym}(2n)$ and consider $P = \mathcal{A}_A(Q)$. We have

$$\mathcal{L}_A(P) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \left(A^{\mathrm{T}} e^{A^{\mathrm{T}}\tau} Q e^{A\tau} + e^{A^{\mathrm{T}}\tau} Q e^{A\tau} A\right) \mathrm{d}\tau$$

$$= \lim_{t\to\infty} \frac{1}{t} \int_0^t \tfrac{\mathrm{d}}{\mathrm{d}\tau} e^{A^{\mathrm{T}}\tau} Q e^{A\tau} \mathrm{d}\tau = \lim_{t\to\infty} \frac{1}{t} \left[e^{A^{\mathrm{T}}t} Q e^{At} - Q\right] = 0.$$

Thus range $\mathcal{A}_A \subseteq$ kernel $\mathcal{L}_A$. To show the reverse inclusion, let $Q \in$ kernel $\mathcal{L}_A$. Then $e^{A^{\mathrm{T}}\tau} Q e^{A\tau} = Q$ for $\tau = 0$ and $\frac{\mathrm{d}}{\mathrm{d}\tau} e^{A^{\mathrm{T}}\tau} Q e^{A\tau} = e^{A^{\mathrm{T}}\tau} \mathcal{L}_A(Q) e^{A\tau} = 0$ for all $\tau \geq 0$, so that $e^{A^{\mathrm{T}}\tau} Q e^{A\tau} = Q$ for all $\tau > 0$. Hence $\mathcal{A}_A(Q) = Q$ so that $Q \in$ range $\mathcal{A}_A$. Equation (4) now follows.

It is easy to verify by direct substitution that $\mathcal{L}_A \circ \mathcal{A}_A = \mathcal{A}_A \circ \mathcal{L}_A$, while (4) implies that $\mathcal{L}_A \circ \mathcal{A}_A = 0$. Hence, it follows that range $\mathcal{L}_A \subseteq$ kernel $\mathcal{A}_A$. To show that equality holds, we note that $\dim(\text{range } \mathcal{L}_A) = n(2n+1) - \dim(\text{kernel } \mathcal{L}_A) = n(2n+1) - \dim(\text{range } \mathcal{A}_A) = \dim(\text{kernel } \mathcal{A}_A)$, where we have used (4). Since range $\mathcal{L}_A$ is contained in kernel $\mathcal{A}_A$ and has the same dimension as kernel $\mathcal{A}_A$, (5) follows. □

5 Average-preserving symmetries

In this section, we identify symmetries of (2) that preserve time averages of quadratic functions along solutions of (2). Specifically, we show that the set of average-preserving symmetries of (2) is the set $\mathfrak{G}_A \stackrel{\text{def}}{=} \cap_{B\in\mathbf{g}_A} \mathbf{G}_B$. Note that every element of $\mathfrak{G}_A$ is a symmetry of (2) that is also a symmetry of every linear Hamiltonian system $\dot{x} = Bx$ whose quadratic Hamiltonian function $x \mapsto x^{\mathrm{T}} J_{2n}^{-1} Bx$ is an integral of motion of (2).

Recall that the center subalgebra of $\mathbf{g}_A$ is the set $\mathfrak{g}_A \stackrel{\text{def}}{=} \{C \in \mathbf{g}_A : CB = BC \text{ for all } B \in \mathbf{g}_A\}$ of elements in $\mathbf{g}_A$ that commute with every element of $\mathbf{g}_A$. The center subalgebra $\mathfrak{g}_A$ is a commutative Lie subalgebra of $\mathbf{g}_A$. It is easy to show that $\mathfrak{g}_A = \cap_{B\in\mathbf{g}_A} \mathbf{g}_B$. The next result states that $\mathfrak{G}_A$ is a Lie subgroup of $\mathrm{Sp}(n)$, and that the Lie algebra of $\mathfrak{G}_A$ is $\mathfrak{g}_A$.

Proposition 5.1 *$\mathfrak{G}_A$ is a Lie subgroup of* $\mathrm{Sp}(n)$*, and its Lie algebra is* $\mathfrak{g}_A$.

Proof The result follows from Proposition 3.1 and Theorem I.1.4.2 of [15]. □

The following theorem, which is our main result, asserts that two quadratic functions related by a symmetry of (2) have equal averages along every solution of (2) if and only if the symmetry relating the two quadratic functions is contained in $\mathfrak{G}_A$.

Theorem 5.1 *Suppose $S \in \mathbf{G}_A$. Then $\mathcal{A}_A(S^{\mathrm{T}}QS) = \mathcal{A}_A(Q)$ for every $Q \in \mathrm{Sym}(2n)$ if and only if $S \in \mathfrak{G}_A$.*

Proof We begin by noting that, since $Se^{At} = e^{At}S$ for every t, it follows that $\mathcal{A}_A(S^{\mathrm{T}}QS) = S^{\mathrm{T}}\mathcal{A}_A(Q)S$ for every $Q \in \mathrm{Sym}(2n)$.

To show sufficiency, suppose $S \in \mathfrak{G}_A$ and let $Q \in \mathrm{Sym}(2n)$. Proposition 4.1 implies that $\mathcal{A}_A(Q) \in \mathrm{kernel}\, \mathcal{L}_A$, so that $J_{2n}\mathcal{A}_A(Q) \in \mathbf{g}_A$ by Proposition 3.1. It now follows by the definition of $\mathfrak{G}_A$ that $S \in \mathbf{G}_{J_{2n}\mathcal{A}_A(Q)}$, i.e., $S^{\mathrm{T}}\mathcal{A}_A(Q)S = \mathcal{A}_A(Q)$. Hence, it follows that $\mathcal{A}_A(S^{\mathrm{T}}QS) = \mathcal{A}_A(Q)$.

Next, to prove necessity, suppose $\mathcal{A}_A(S^{\mathrm{T}}QS) = \mathcal{A}_A(Q)$ for every $Q \in \mathrm{Sym}(2n)$, and let $B \in \mathbf{g}_A$. Define $Q \overset{\text{def}}{=} J_{2n}^{-1}B \in \mathrm{Sym}(2n)$. Proposition 3.1 implies that $Q \in \mathrm{kernel}\, \mathcal{L}_A$, so that $e^{A^{\mathrm{T}}\tau}Qe^{A\tau} = Q$ for all $\tau \geq 0$, and hence $\mathcal{A}_A(Q) = Q$. We now have $S^{\mathrm{T}}QS = S^{\mathrm{T}}\mathcal{A}_A(Q)S = \mathcal{A}_A(S^{\mathrm{T}}QS) = \mathcal{A}_A(Q) = Q$, which implies that $SB = BS$, i.e., $S \in \mathbf{G}_B$. Since $B \in \mathbf{g}_A$ was chosen arbitrarily, it follows that $S \in \mathfrak{G}_A$. □

6 Structure of the Lie groups $\mathbf{G}_A$ and $\mathfrak{G}_A$

In this section, we characterize the Lie groups $\mathbf{G}_A$ and $\mathfrak{G}_A$ as well as their Lie algebras.

Let $\pm\jmath\beta_1, \ldots, \pm\jmath\beta_r$ be the distinct nonzero eigenvalues of the stable Hamiltonian matrix A, and let $m_1, \ldots, m_r$ be the corresponding algebraic multiplicities. Let $2m_0 \geq 0$ be the algebraic multiplicity of the zero eigenvalue $\beta_0 \overset{\text{def}}{=} 0$ of A, so that $2(m_0 + \cdots + m_r) = 2n$.

For every $i \in \{0, \ldots, r\}$, let $\mathbb{V}_i \overset{\text{def}}{=} \{x \in \mathbb{R}^{2n} : A^2x = -\beta_i^2 x\}$. For every $i \in \{1, \ldots, r\}$, the subspace $\mathbb{V}_i$ is the real eigenspace of A associated with the eigenvalue pair $\pm\jmath\beta_i$. Since A is stable, each of its eigenvalues is semisimple, and hence, for every $i \in \{1, \ldots, r\}$, the subspace $\mathbb{V}_i$ has dimension $2m_i$. The subspace $\mathbb{V}_0$, which is the eigenspace associated with the zero eigenvalue, has dimension $2m_0$. It is easy to show that each of the subspaces $\mathbb{V}_0, \ldots, \mathbb{V}_r$ is an invariant subspace of every element of $\mathbf{G}_A$ and $\mathbf{g}_A$. Given distinct $i, j \in \{0, \ldots, r\}$, the subspaces $\mathbb{V}_i$ and $\mathbb{V}_j$ are skew orthogonal, i.e., $x^{\mathrm{T}}J_{2n}y = 0$ for every $x \in \mathbb{V}_i$ and every $y \in \mathbb{V}_j$ [14, Lem. II.D.1]. Consequently, each of these subspaces is a symplectic subspace of $\mathbb{R}^{2n}$ [14, Prop. II.B.4]. Finally, $\mathbb{R}^{2n} = \bigoplus_{i=0}^{r} \mathbb{V}_i$.

Next, for each $i = 0, 1, \ldots, r$, we define $\mathbf{G}_i \overset{\text{def}}{=} \{S \in \mathbf{G}_A : Sx = x \text{ for every } x \in \mathbb{V}_j \text{ and every } j \neq i\}$ and $\mathbf{g}_i \overset{\text{def}}{=} \{B \in \mathbf{g}_A : Bx = 0 \text{ for every } x \in \mathbb{V}_j \text{ and every } j \neq i\}$. Note that $\mathbf{G}_i \cap \mathbf{G}_j = \{I_{2n}\}$ and $\mathbf{g}_i \cap \mathbf{g}_j = \{0\}$ for every $i \neq j$. For each $i \in \{0, \ldots, r\}$, we also let $\mathfrak{G}_i$ and $\mathfrak{g}_i$ denote the center subgroup and center subalgebra of $\mathbf{G}_i$ and $\mathbf{g}_i$, respectively.

Lemma 6.1 *Let* $\mathbf{H}_0 \overset{\text{def}}{=} \mathrm{Sp}(2m_0)$ *and* $\mathbf{h}_0 \overset{\text{def}}{=} \mathrm{sp}(2m_0)$. *Further, for each* $i \in \{1, \ldots, r\}$, *let* $\mathbf{H}_i \overset{\text{def}}{=} \mathrm{Sp}(2m_i) \cap \mathrm{SO}(2m_i)$ *and* $\mathbf{h}_i \overset{\text{def}}{=} \mathrm{sp}(2m_i) \cap \mathrm{so}(2m_i)$. *There exists an invertible matrix* $U \in \mathbb{R}^{2n \times 2n}$ *such that the following statements hold.*

(i) *Suppose* $i \in \{0, \ldots, r\}$. *Then* $S \in \mathbf{G}_i$ *if and only if there exists* $S_i \in \mathbf{H}_i$ *such that* $U^{-1}SU = \mathrm{diag}(S_0, \ldots, S_r)$, *where* $S_j = I_{2m_j}$ *for every* $j \neq i$.
(ii) *Suppose* $i \in \{0, \ldots, r\}$. *Then* $B \in \mathbf{g}_i$ *if and only if there exists* $B_i \in \mathbf{h}_i$ *such that* $U^{-1}BU = \mathrm{diag}(B_0, \ldots, B_r)$, *where* $B_j = 0 \in \mathbb{R}^{2m_j \times 2m_j}$ *for every* $j \neq i$.

Proof Since every eigenvalue of A is semisimple and imaginary, there exists a basis $\mathcal{B} = \{x_1, \ldots, x_{2n}\}$ for $\mathbb{R}^{2n}$ such that, for every $i \in \{0, \ldots, r\}$, $\mathcal{B}_i \overset{\text{def}}{=} \{x_{2m_{i-1}+1}, \ldots, x_{2m_i}\}$

is a symplectic basis for the symplectic subspace $\mathbb{V}_i$, and the matrix representation of the linear map $x \mapsto Ax$ in the basis $\mathcal{B}$ is $\hat{A} = \operatorname{diag}(\hat{A}_1, \ldots, \hat{A}_r)$, where $\hat{A}_i = \beta_i J_{2m_i}$ for every $i \in \{0, \ldots, r\}$ (see Lemma II.C.5 of [14]). Define $U \stackrel{\text{def}}{=} [x_1, \ldots, x_{2n}] \in \mathbb{R}^{2n\times 2n}$. Then it follows that $\hat{A} = U^{-1}AU$. Since the bases $\mathcal{B}_i$, $i \in \{0, \ldots, r\}$, are symplectic bases for mutually skew-orthogonal subspaces, it follows that $\hat{J} \stackrel{\text{def}}{=} U^{\mathrm{T}} J_{2n} U = \operatorname{diag}(J_{2m_0}, \ldots, J_{2m_r})$.

To prove necessity in (i), choose $i \in \{0, \ldots, r\}$ and consider $S \in \mathbf{G}_i$. Since the linear map $x \mapsto Sx$ restricts to the identity map on each subspace $\mathbb{V}_j$, $j \neq i$, and leaves the subspace $\mathbb{V}_i$ invariant, it follows that the matrix representation of the map $x \mapsto Sx$ in the basis $\mathcal{B}$ is given by $\hat{S} \stackrel{\text{def}}{=} U^{-1}SU = \operatorname{diag}(S_0, \ldots, S_r)$, where $S_j = I_{2m_j}$ for each $j \neq i$. $S^{\mathrm{T}}JS = J$ implies that $\hat{S}^{\mathrm{T}}\hat{J}\hat{S} = \hat{J}$, from which it follows that $\hat{S}_i \in \operatorname{Sp}(m_i)$. Thus necessity in (i) follows for the case $i = 0$.

Next, $SA = AS$ implies that $\hat{S}\hat{A} = \hat{A}\hat{S}$, from which it follows that $\beta_i S_i J_{2m_i} = \beta_i J_{2m_i} S_i$. It immediately follows that, if $i \neq 0$, then $S_i J_{2m_i} = J_{2m_i} S_i$, so that $S_i \in \operatorname{Sp}(m_i) \cap \operatorname{SO}(2m_i)$. Thus necessity in (i) follows for the case $i \in \{1, \ldots, r\}$.

To show sufficiency in (i), let $i \in \{0, \ldots, r\}$ and choose $S_i \in \mathbf{H}_i$. Define $\hat{S} \stackrel{\text{def}}{=} \operatorname{diag}(S_0, \ldots, S_r)$ with $S_j = I_{2m_j}$ for every $j \neq 0$. Then $\hat{S}$ satisfies $\hat{S}^{\mathrm{T}}\hat{J}\hat{S} = \hat{J}$ and $\hat{S}\hat{A} = \hat{A}\hat{S}$, so that $S \stackrel{\text{def}}{=} U\hat{S}U^{-1} \in \mathbf{G}_A$. To show that $S \in \mathbf{G}_i$, let $j \neq i$, and choose $x \in \mathbb{V}_j$. Then $U^{-1}x$ is the representation of x in the basis $\mathcal{B}$. Since $x \in \mathbb{V}_j$, those elements of $U^{-1}x$ which correspond to the components of x in the bases $\mathcal{B}_k$, $k \neq j$, are zero. Consequently, $\hat{S}U^{-1}x = U^{-1}x$. It now follows that $Sx = x$. Since $j \neq i$ and $x \in \mathbb{V}_j$ were chosen arbitrarily, it follows that $S \in \mathbf{G}_i$.

The proof that (ii) holds is similar, and left to the reader. □

Using Lemma 6.1 it is easy to show that, if $i \neq j$ and $S_1 \in \mathbf{G}_i$, $S_2 \in \mathbf{G}_j$, $B_1 \in \mathbf{g}_i$ and $B_2 \in \mathbf{g}_j$, then $S_1S_2 - S_2S_1 = S_1B_2 - B_2S_1 = B_1B_2 - B_2B_1 = 0$.

The next result lists some properties of the Lie groups $\mathbf{G}_i$, $\mathfrak{G}_i$, and Lie algebras $\mathbf{g}_i$, $\mathfrak{g}_i$, $i = 0, \ldots, r$.

Proposition 6.1 *The following statements hold.*

(i) *For every $i \in \{0, \ldots, r\}$, $\mathbf{G}_i$ is a Lie subgroup of $\mathbf{G}_A$ and its Lie algebra is $\mathbf{g}_i$. Moreover, $\mathbf{G}_i$ is a normal subgroup of $\mathbf{G}_A$ while $\mathbf{g}_i$ is an ideal of $\mathbf{g}_A$ for every $i \in \{0, \ldots, r\}$.*

(ii) *$\mathbf{G}_0$ is isomorphic to $\operatorname{Sp}(m_0)$, while, for every $i \in \{1, \ldots, r\}$, $\mathbf{G}_i$ is isomorphic to $\operatorname{Sp}(m_i) \cap \operatorname{SO}(2m_i)$.*

(iii) *The Lie subalgebra $\mathbf{g}_0$ is isomorphic to the $\frac{1}{2}m_0(m_0+1)$-dimensional Lie algebra $\operatorname{sp}(m_0)$, while, for every $i \in \{1, \ldots, r\}$, $\mathbf{g}_i$ is isomorphic to the m_i^2-dimensional Lie algebra $\operatorname{sp}(m_i) \cap \operatorname{so}(2m_i)$.*

(iv) *$\mathfrak{G}_0$ is isomorphic to the discrete group $\{1, -1\}$, while, for every $i \in \{1, \ldots, r\}$, $\mathfrak{G}_i$ is isomorphic to the one-parameter subgroup of $\operatorname{Sp}(m_i)$ generated by J_{2m_i}.*

(v) *For every $i \in \{1, \ldots, r\}$, $\mathfrak{g}_i$ is isomorphic to the one-dimensional Lie algebra $\operatorname{span}\{J_{2m_i}\}$, while $\mathfrak{g}_0 = \{0\}$.*

Proof (i) Each $\mathbf{G}_i$ is a closed subgroup of $\mathbf{G}_A$, and hence a Lie subgroup of $\mathbf{G}_A$ [15, Thm. I.2.3.6]. The proof that the Lie algebra of $\mathbf{G}_i$ is $\mathbf{g}_i$ is straightforward and left to the reader. To show that $\mathbf{G}_i$ is a normal subgroup of $\mathbf{G}_A$, let $j \neq i$ and consider $S \in \mathbf{G}_i$, $T \in \mathbf{G}_A$ and $x \in \mathbb{V}_j$. Since $\mathbb{V}_j$ is invariant under T, it follows that $STx = Tx$, so that $T^{-1}STx = x$. Thus, $T^{-1}ST \in \mathbf{G}_i$, and hence $\mathbf{G}_i$ is a normal subgroup of $\mathbf{G}_A$. It now follows immediately that $\mathbf{g}_i$ is an ideal in $\mathbf{g}_A$.

(ii) The proof of (ii) follows directly from (i) of Lemma 6.1.
(iii) The proof of (iii) follows directly from (ii) of Lemma 6.1.
(iv) The proof that $\mathfrak{G}_0$ is isomorphic to the discrete group $\{1, -1\}$ follows from Proposition 2.3 and (ii) of Proposition 6.1. The proof that $\mathfrak{G}_i$ is isomorphic to $\{e^{J_{2m_i}t} : t \in \mathbb{R}\}$ for every $i \in \{1, \dots, r\}$ follows from Proposition 2.1 and (ii) of Proposition 6.1.
(v) The proof that $\mathfrak{g}_i$ is isomorphic to $\mathrm{span}\{J_{2m_i}\}$ for every $i \in \{1, \dots, r\}$ follows from Proposition 2.2 and (iii) of Proposition 6.1. The proof that $\mathfrak{g}_0 = \{0\}$ follows from Proposition 2.4 and (iii) of Proposition 6.1. $\square$

The next result completely characterizes the groups $\mathbf{G}_A$ and $\mathfrak{G}_A$ along with their respective Lie algebras $\mathbf{g}_A$ and $\mathfrak{g}_A$.

Theorem 6.1 *The following statements hold.*

(i) $\mathbf{G}_A = \bigotimes_{i=0}^r \mathbf{G}_i$ *and* $\mathbf{g}_A = \bigoplus_{i=0}^r \mathbf{g}_i$. *Consequently, the dimensions of the subspaces* $\mathbf{g}_A$ *and* kernel $\mathcal{L}_A$ *are both equal to* $\frac{1}{2}m_0(m_0+1) + m_1^2 + \cdots + m_r^2$.
(ii) $\mathfrak{G}_A$ *is the center subgroup of* $\mathbf{G}_A$ *and equals* $\bigotimes_{i=0}^r \mathfrak{G}_i$. *Moreover,* $\mathfrak{g}_A = \bigoplus_{i=1}^r \mathfrak{g}_i$. *Consequently, the dimension of the Lie algebra* $\mathfrak{g}_A$ *equals* r, *i.e., half the number of distinct nonzero eigenvalues of* A.

Proof (i) To prove $\mathbf{G}_A = \bigotimes_{i=0}^r \mathbf{G}_i$, it suffices to show that $\mathbf{G}_A \subseteq \bigotimes_{i=0}^r \mathbf{G}_i$. Consider $S \in \mathbf{G}_A$. For every $i \in \{0, \dots, r\}$, there exists a matrix $S_i \in \mathbb{R}^{2n\times 2n}$ such that $S_i x = x$ for every $j \neq i$ and every $x \in \mathbb{V}_j$, while $S_i x = Sx$ for all $x \in \mathbb{V}_i$.

We claim that S_k is symplectic for every k. Let $i, j, k \in \{0, \dots, r\}$ and consider $x \in \mathbb{V}_i$ and $y \in \mathbb{V}_j$. If $i = j = k$, then $x^{\mathrm{T}} S_k^{\mathrm{T}} J_{2n} S_k y = x^{\mathrm{T}} S^{\mathrm{T}} J_{2n} S y = x^{\mathrm{T}} J_{2n} y$. If $i = j \neq k$, then $x^{\mathrm{T}} S_k^{\mathrm{T}} J_{2n} S_k y = x^{\mathrm{T}} J_{2n} y$. If $i \neq j$, then the fact that $\mathbb{V}_i$ and $\mathbb{V}_j$ are skew-orthogonal, invariant subspaces of S_k implies that $x^{\mathrm{T}} S_k^{\mathrm{T}} J_{2n} S_k y$ and $x^{\mathrm{T}} J_{2n} y$ are both zero and hence equal. We have shown that, for every $i, j \in \{0, \dots, r\}$, $x^{\mathrm{T}} S_k^{\mathrm{T}} J_{2n} S_k y = x^{\mathrm{T}} J_{2n} y$ for all $x \in \mathbb{V}_i$ and $y \in \mathbb{V}_j$. Since $\mathbb{R}^{2n}$ is the direct sum of the subspaces $\mathbb{V}_0, \dots, \mathbb{V}_r$, it follows that $x^{\mathrm{T}} S_k^{\mathrm{T}} J_{2n} S_k y = x^{\mathrm{T}} J_{2n} y$ for all $x, y \in \mathbb{R}^{2n}$. We conclude that $S_k^{\mathrm{T}} J_{2n} S_k = J_{2n}$, and $S_k \in \mathrm{Sp}(n)$.

Next, to show that $S_i \in \mathbf{G}_i$, let $j \in \{0, \dots, r\}$, and consider $x \in \mathbb{V}_j$. If $j \neq i$, then $S_i A x = Ax$ and $S_i x = x$, so that $S_i A x = A S_i x$. If $j = i$, then $S_i A x = SAx = ASx = AS_i x$. Since $\mathbb{R}^{2n}$ is a direct sum of the subspaces $\mathbb{V}_j$, $j = 0, \dots r$, it follows that $S_i A = A S_i$, i.e., $S_i \in \mathbf{G}_A$. Our construction of S_i now implies that $S_i \in \mathbf{G}_i$ for every i. It is easy to check that, for every $i \in \{0, \dots, r\}$ and every $x \in \mathbb{V}_i$, $Sx = S_0 S_1 S_2 \dots S_r x$. Since $\mathbb{R}^{2n} = \bigoplus_{i=0}^r \mathbb{V}_i$, it follows that $S = S_0 S_1 S_2 \dots S_r$. Thus $\mathbf{G}_A \subseteq \bigotimes_{i=0}^r \mathbf{G}_i$.

The proof that $\mathbf{g}_A = \bigoplus_{i=0}^r \mathbf{g}_i$ is similar, and left to the reader. The assertion about the dimension of $\mathbf{g}_A$ follows from (iii) of Proposition 6.1.

(ii) We prove the first part of (ii) by showing that $\bigotimes_{i=0}^{r} \mathfrak{G}_i \subseteq \mathcal{C} \subseteq \mathfrak{G}_A \subseteq \bigotimes_{i=0}^{r} \mathfrak{G}_i$, where $\mathcal{C}$ is the center subgroup of $\mathbf{G}_A$.

Consider $S \in \bigotimes_{i=0}^{r} \mathfrak{G}_i$. We wish to show that S commutes with every element of $\mathbf{G}_A$. In light of (i) of Theorem 6.1, it suffices to prove that, for every i, S commutes with every element of $\mathbf{G}_i$. Hence consider $i \in \{0, \dots, r\}$ and let $T \in \mathbf{G}_i$. T clearly commutes with those factors of S that do not lie in $\mathbf{G}_i$. T also commutes with that factor of S which lies in $\mathfrak{G}_i$, since $\mathfrak{G}_i$ is the center subgroup of $\mathbf{G}_i$. Thus it follows that S and T commute. Since $i \in \{0, \dots, r\}$ and $T \in \mathbf{G}_i$ were chosen arbitrarily, it follows that $S \in \mathcal{C}$.

Next, consider $S \in \mathcal{C}$ and $B \in \mathbf{g}_A$. Since B commutes with A, it follows that, for every t, the symplectic matrix e^{Bt} commutes with A, i.e., $e^{Bt} \in \mathbf{G}_A$. Since $S \in \mathcal{C}$, it follows that $Se^{Bt} = e^{Bt}S$ for all t. Differentiation yields $SB = BS$, i.e., $S \in \mathbf{G}_B$. Since $B \in \mathbf{g}_A$ was chosen arbitrarily, it follows that $S \in \mathfrak{G}_A$.

Now consider $S \in \mathfrak{G}_A$. By (i) of Theorem 6.1, there exist $S_i \in \mathbf{G}_i$, $i \in \{0, \dots, r\}$, such that $S = S_0 \dots S_r$. First consider $i \in \{1, \dots, r\}$. Since $S \in \mathfrak{G}_A$ and $\mathbf{g}_i \subseteq \mathbf{g}_A$, it follows that S commutes with every element of $\mathbf{g}_i$. For every $j \neq i$, the matrix $S_j \in \mathbf{G}_j$ commutes with every matrix in $\mathbf{g}_i$. Hence it follows that S_i commutes with every matrix in $\mathbf{g}_i$. Lemma 6.1 implies that, for each $i \in \{1, \dots, r\}$, there exists a map $\beta_i : \mathbb{R}^{2n\times 2n} \to \mathbb{R}^{2m_i\times 2m_i}$ such that the restriction of β_i to $\mathbf{G}_i$ is a group isomorphism between $\mathbf{G}_i$ and $\mathrm{Sp}(m_i) \cap \mathrm{SO}(2m_i)$, the restriction of β_i to $\mathbf{g}_i$ is a Lie-algebra isomorphism between $\mathbf{g}_i$ and $\mathrm{sp}(m_i) \cap \mathrm{so}(2m_i)$, and $\beta_i(TB) = \beta_i(T)\beta_i(B)$ for every $T, B \in \mathbf{G}_i \cup \mathbf{g}_i$. Since S_i commutes with every matrix in $\mathbf{g}_i$, it follows that $\beta_i(S_i)$ commutes with every matrix in $\mathrm{sp}(m_i) \cap \mathrm{so}(2m_i)$. Proposition 2.1 now implies that $\beta_i(S_i)$ belongs to the center subgroup of $\mathrm{Sp}(m_i) \cap \mathrm{SO}(2m_i)$. Since β_i is a group isomorphism, it follows that $S_i \in \mathfrak{G}_i$. A similar argument along with Proposition 2.3 shows that $S_0 \in \mathfrak{G}_0$. Thus, $S \in \bigotimes_{i=0}^{r} \mathfrak{G}_i$, and the conclusion follows.

Using similar arguments, it can be shown that $\mathfrak{g}_A = \bigoplus_{i=1}^{r} \mathfrak{g}_i$. The assertion about the dimension of $\mathfrak{g}_A$ now follows from (v) of Proposition 6.1. □

The dimension of the space of quadratic integrals of motion of (2) given in Theorem 6.1 above is also given in Proposition 12 of [16] for the special case where $m_0 = 0$. The following corollary, which is given as Corollary 12 in [16], specializes Theorem 6.1 to the case where the matrix A is *simple*, i.e., every eigenvalue of A has algebraic multiplicity equal to unity.

Corollary 6.1 *The following statements are equivalent.*

(i) *A is simple.*
(ii) $\mathfrak{G}_A = \mathbf{G}_A$.
(iii) *$\mathbf{G}_A$ is commutative.*

Furthermore, if A is simple, then $\mathfrak{g}_A = \mathbf{g}_A$, and $\mathbf{g}_A$ and kernel $\mathcal{L}_A$ *are n-dimensional subspaces of* $\mathrm{sp}(n)$ *and* $\mathrm{Sym}(2n)$*, respectively.*

Proof The result follows from Theorem 6.1 by noting that if A is simple, then $r = n$, $m_0 = 0$ and $m_i = 1$ for every $i \in \{1, \dots, n\}$. □

7 Undamped lumped-parameter systems

In this section, we apply Theorem 5.1 to the undamped lumped-parameter mechanical system described by

$$M\ddot{q}(t) + Kq(t) = 0, \tag{6}$$

where $q \in \mathbb{R}^n$, and $M, K \in \mathrm{Sym}(n)$ are the positive-definite mass and stiffness matrices, respectively.

As a first application of Theorem 5.1, we prove the classical virial theorem as applied to (6) which states that the time-averaged potential and kinetic energies of the system (6) are equal. Our proof is novel in that it relies on ideas of symmetry.

Defining the state vector $y = \begin{bmatrix} M^{\frac{1}{2}}q \\ M^{\frac{1}{2}}\dot{q} \end{bmatrix} \in \mathbb{R}^{2n}$ yields the state space description (2) of the system (6) with

$$A = \begin{bmatrix} 0 & I_n \\ -M^{-\frac{1}{2}}KM^{-\frac{1}{2}} & 0 \end{bmatrix}.$$

The matrix A is Hamiltonian. Consequently, the system (6) is Hamiltonian with the Hamiltonian function $x \mapsto x^{\mathrm{T}}Hx$, where

$$H = \begin{bmatrix} M^{-\frac{1}{2}}KM^{-\frac{1}{2}} & 0 \\ 0 & I_n \end{bmatrix}.$$

Under our assumptions on the mass and stiffness matrices, the matrix A is Lyapunov stable [6, Cor. 2] and has only nonzero eigenvalues [6, Lem. 4].

On letting

$$Q_1 = \frac{1}{2}\begin{bmatrix} M^{-\frac{1}{2}}KM^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix},$$

it follows that $x^{\mathrm{T}}Q_1x = \frac{1}{2}q^{\mathrm{T}}Kq$ and $x^{\mathrm{T}}Q_2x = \frac{1}{2}\dot{q}^{\mathrm{T}}M\dot{q}$ are the potential and kinetic energies, respectively, of the system (6). It is a simple matter to verify that the matrix

$$S = \begin{bmatrix} 0 & -(M^{-\frac{1}{2}}KM^{-\frac{1}{2}})^{-\frac{1}{2}} \\ (M^{-\frac{1}{2}}KM^{-\frac{1}{2}})^{\frac{1}{2}} & 0 \end{bmatrix}$$

is symplectic and satisfies

(i) $S^{\mathrm{T}}HS = H$, i.e., $S \in \mathbf{G}_A$, and
(ii) $S^{\mathrm{T}}Q_2S = Q_1$.

We claim that $S \in \mathfrak{G}_A$. To show this, consider $T \in \mathbf{G}_A$, so that T satisfies $AT = TA$. If T is partitioned as

$$T = \begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{bmatrix},$$

where $T_1, T_{12}, T_{21}, T_2 \in \mathbb{R}^{n \times n}$, then it follows from $AT = TA$ that $T_1 = T_2$, $T_{21} = -T_{12}(M^{-\frac{1}{2}} K M^{-\frac{1}{2}})$ and that T_1, T_{12}, T_{21}, T_2 commute with $M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$. These properties of the submatrices T_1, T_{12}, T_{21} and T_2 can be used to show that S commutes with T. Since $T \in \mathbf{G}_A$ was chosen arbitrarily, it follows from (ii) of Theorem 6.1 that $S \in \mathfrak{G}_A$. Theorem 5.1 now leads to the virial theorem, which we state as the following corollary.

Corollary 7.1 *Along every solution of* (6)*, the time averages of the kinetic and potential energies over* $[0, \infty)$ *are equal.*

The interested reader may refer to Section 3.7 of [1] for an alternative treatment of the virial theorem in a general setting. It should be emphasized that the traditional proof of the virial theorem is extremely direct and simple, and applies to a larger class of systems than (6) (see, for instance, [11]). However, the interesting aspect of our proof is that it reveals an unexpected connection between the virial theorem and phase-space symmetries.

Next, a *configuration symmetry* of (6) is an orthogonal matrix $R \in \mathbb{R}^{n \times n}$ such that $R^{\mathrm{T}} M R = M$ and $R^{\mathrm{T}} K R = K$. A configuration symmetry thus represents an orthogonal change of position coordinates that leaves the dynamical equation (6) invariant.

Our next result considers symmetries of (2) that arise from configuration symmetries of (6).

Corollary 7.2 *Suppose the system* (6) *has distinct natural frequencies, and let* $R \in \mathbb{R}^{n \times n}$ *be a configuration symmetry. Then, for every matrix* $P \in \mathrm{Sym}(n)$*, the time averages over* $[0, \infty)$ *of* $t \mapsto q^{\mathrm{T}}(t) P q(t)$ *and* $t \mapsto q^{\mathrm{T}}(t) R^{\mathrm{T}} P R q(t)$ *are equal along every solution of* (6)*. Likewise, the time averages over* $[0, \infty)$ *of* $t \mapsto \dot{q}^{\mathrm{T}}(t) P \dot{q}(t)$ *and* $t \mapsto \dot{q}^{\mathrm{T}}(t) R^{\mathrm{T}} P R \dot{q}(t)$ *are equal along every solution of* (6).

Proof First, note that $q^{\mathrm{T}} P q = x^{\mathrm{T}} Q_1 x$ and $\dot{q}^{\mathrm{T}} P \dot{q} = x^{\mathrm{T}} Q_2 x$, where $Q_1, Q_2 \in \mathrm{Sym}(2n)$ are given by

$$Q_1 = \frac{1}{2} \begin{bmatrix} M^{-\frac{1}{2}} P M^{-\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q_2 = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & M^{-\frac{1}{2}} P M^{-\frac{1}{2}} \end{bmatrix}.$$

Next, consider the matrix

$$S = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}.$$

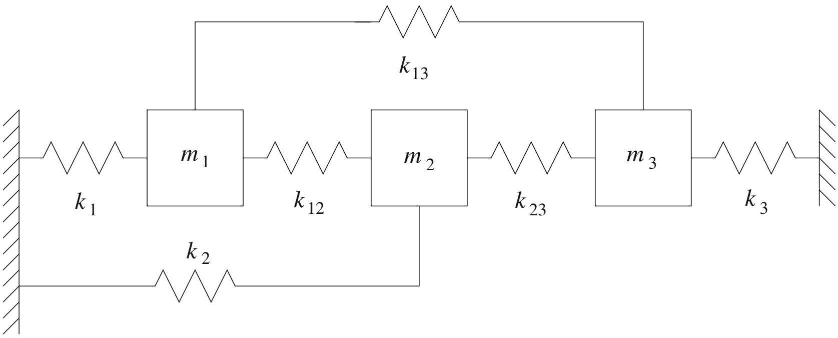
It is easy to verify that S is symplectic and satisfies

i) $S^{\mathrm{T}} H S = H$, i.e., $S \in \mathbf{G}_A$, and
ii) $q^{\mathrm{T}} R^{\mathrm{T}} P R q = x^{\mathrm{T}} S^{\mathrm{T}} Q_1 S x$ and $\dot{q}^{\mathrm{T}} R^{\mathrm{T}} P R \dot{q} = x^{\mathrm{T}} S^{\mathrm{T}} Q_2 S x$.

Since the natural frequencies of the system are distinct, Corollary 6.1 implies that $S \in \mathfrak{G}_A$. The result now follows from Theorem 5.1 by first letting $Q = Q_1$ and then letting $Q = Q_2$. □

If P and $R^{\mathrm{T}} P R$ represent the stiffness (mass) matrices of two subsystems of (6), with R a configuration symmetry of (6), then Corollary 7.2 implies that the two subsystems have the same potential (kinetic) energies on average. Thus Corollary 7.2 allows us to assert that symmetrically related subsystems of the system (6) have the same energy on average. Proposition 3.7.26 of [1] gives a general result on the equality of time averages of functions in a nonlinear Hamiltonian setting, which is then applied to conclude equality of average kinetic energies of a symmetric second-order system. However, the treatment in [1] relies on the assumption of ergodicity and thus differs significantly from our approach.

The following example illustrates an application of Corollary 7.2.



Example 7.1 The undamped three-degree-of-freedom system depicted in the figure above has the mass and stiffness matrices

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_{12} + k_{13} & -k_{12} & -k_{13} \\ -k_{12} & k_{12} + k_2 + k_{23} & -k_{23} \\ -k_{13} & -k_{23} & k_{13} + k_{23} + k_3 \end{bmatrix}.$$

Let $m_1 = m_2$, $k_{13} = k_{23}$ and $k_1 = k_2$. Then, the orthogonal permutation matrix

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

satisfies $R^{\mathrm{T}} M R = M$ and $R^{\mathrm{T}} K R = K$. In other words, (6) remains unchanged when q_1 and q_2 are interchanged.

The potential energies in the springs k_1 and k_2 are equal to $\frac{1}{2}q^{\mathrm{T}}P_1q$ and $\frac{1}{2}q^{\mathrm{T}}P_2q$, where

$$P_1 = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that, under the assumptions $k_{13} = k_{23}$ and $k_1 = k_2$, $R^{\mathrm{T}}P_1R = P_2$. An elementary analysis of the characteristic polynomial of the system shown in the figure reveals that the system has distinct natural frequencies for values of the parameters m_1, m_3, k_{12}, k_{13}, k_1 and k_3 that lie in an open dense subset of $\mathbb{R}^6$. Hence it follows from Corollary 7.2 that, generically, the average potential energy in the spring k_1 equals the average potential energy in the spring k_2 along any given solution of (6).

Our next example shows that the assertion in Corollary 7.2 may not hold when the parameter values are such that the system (6) has repeated natural frequencies.

Example 7.2 Consider the undamped three-degree-of-freedom system introduced in Example 7.1, with all masses and spring stiffnesses set to unity. In this case, the mass matrix is the identity, while the stiffness matrix is given by

$$K = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

It is easy to verify that (6) remains unchanged under orthogonal permutations that interchange q_1 with q_2, q_2 with q_3, and q_3 with q_1. Example 7.1 may thus lead one to expect the potential energies in the springs k_1, k_2 and k_3, which transform into one another under these permutations, to be equal. However, the orthogonal permutations that interchange q_1 with q_2, q_2 with q_3, and q_3 with q_1 do not commute and thus the group of symmetries of the system is not commutative. Therefore, Corollary 6.1 implies that the system has a repeated natural frequency. Indeed, for our choice of the masses and stiffnesses, the system has only two distinct natural frequencies, namely 2 rad/s and 1 rad/s, with the larger natural frequency being repeated twice. However, since the structure is stable, the eigenvalue associated with the repeated natural frequency is semisimple (see Lemma 3 in [6]), and the system has three linearly independent eigenvectors given by

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

with v_1 and v_2 corresponding to the repeated natural frequency.

A general solution of (6) is given by

$$q(t) = a_1 \sin(2t + \phi_1)v_1 + a_2 \sin(2t + \phi_2)v_2 + a_3 \sin(t + \phi_3)v_3, \tag{8}$$

where the amplitudes a_1, a_2 and a_3 as well as the phases ϕ_1, ϕ_2 and ϕ_3 can be assigned arbitrary values by an appropriate choice of initial conditions on q and $\dot{q}$.

The average energies in the springs k_1, k_2 and k_3 along the solution (8) can easily be calculated to be $\frac{1}{4}[a_1^2+a_2^2+a_3^2+2a_1a_2\cos(\phi_1-\phi_2)]$, $\frac{1}{4}(a_1^2+a_3^2)$ and $\frac{1}{4}(a_2^2+a_3^2)$, respectively. It is clear that the average energies in the springs k_1, k_2 and k_3 are not equal along every solution of (6). Note that unlike in Example 7.1, the average energies in the springs k_1 and k_2 are not equal along every solution, even though the configuration symmetry that transforms the energy in k_1 to the energy in k_2 is the same as in Example 7.1. This case illustrates that the assertion in Corollary 7.2 does not generally hold for systems having repeated natural frequencies.

Remark 7.1 Examples 7.1 and 7.2 indicate that, for values of the parameters k_{12} and k_3 that yield arbitrarily close but distinct natural frequencies, the average energies in the springs k_1 and k_2 are equal. Yet, the average energies in the same two springs are unequal for values of the parameters k_{12} and k_3 that lead to repeated natural frequencies. The examples thus imply that the average energies in k_1 and k_2 are discontinuous functions of the parameters k_{12} and k_3. On the other hand, standard results on differential equations such as [12, Thm. V.2.1] state that solutions of linear differential equations depend continuously on parameters. This apparent contradiction is resolved by noting that the averaging operation in (3) involves integration on $[0, \infty)$ along solutions that are guaranteed to depend continuously on parameters uniformly in time over bounded time intervals but not necessarily over $[0, \infty)$.

8 Conclusion

We have explored the connections between the group of phase space symmetries of a stable linear Hamiltonian system, the set of its quadratic integrals of motion, and the operation of taking the average of a quadratic function along the trajectories of the system. Our main result is that elements of the center subgroup of the group of symmetries preserve the time averages of quadratic functions along the trajectories of the system. In the special case where the system has unrepeated eigenvalues, the group of symmetries of the system is commutative, and therefore every symmetry preserves averages of quadratic functions. These results are used to provide a novel symmetry-based proof of the virial theorem, and obtain a equipartition result for undamped linear lumped-parameter systems.

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A Appendix

The proof of Proposition 2.1 requires the following lemma.

Lemma A.1 *Suppose* $R \in \mathbb{R}^{n\times n}$ *commutes with all matrices in* $\mathrm{Sym}(n)$ *and* $\mathrm{so}(n)$. *Then there exists* $\alpha \in \mathbb{R}$ *such that* $R = \alpha I_n$.

Proof For each $i \in \{1, \ldots, n\}$, let $e_i \in \mathbb{R}^n$ denote the ith column of I_n. Choose $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Since R commutes with all real, symmetric and skew-symmetric matrices, it follows that R commutes with all matrices in $\mathbb{R}^{n \times n}$. In particular,

$$Re_i e_j^{\mathrm{T}} = e_i e_j^{\mathrm{T}} R. \tag{9}$$

Pre-multiplying and post-multiplying (9) by e_i^{T} and e_i, respectively, yields $e_j^{\mathrm{T}} Re_i = 0$, i.e., $R_{ji} = 0$. Next, pre-multiplying and post-multiplying (9) by e_i^{T} and e_j, respectively, yields $e_i^{\mathrm{T}} Re_i = e_j^{\mathrm{T}} Re_j$, i.e., $R_{ii} = R_{jj}$. Since i and j were chosen arbitrarily, we conclude that R is diagonal and all its diagonal entries are equal. The result now follows. □

Proof of Proposition 2.1. To show that (i) implies (ii), suppose S belongs to the center subgroup of $\mathrm{Sp}(n) \cap \mathrm{SO}(2n)$, and consider $B \in \mathrm{sp}(n) \cap \mathrm{so}(2n)$. Then $e^{Bt} \in \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ for all $t \in \mathbb{R}$. Hence $Se^{Bt} - e^{Bt}S = 0$ for all $t \in \mathbb{R}$. Differentiating and letting $t = 0$ yields $SB - BS = 0$.

To show that (ii) implies (iii), suppose $S \in \mathrm{Sp}(n)$ satisfies $SB = BS$ for all $B \in \mathrm{sp}(n) \cap \mathrm{so}(2n)$. Since $J_{2n} \in \mathrm{sp}(n) \cap \mathrm{so}(n)$, it follows that $SJ_{2n} = J_{2n}S$, so that $S \in \mathrm{SO}(n)$. There exist $R_1, R_2 \in \mathbb{R}^{n \times n}$ such that $R_1^{\mathrm{T}} R_1 + R_2^{\mathrm{T}} R_2 = I_n$, $R_2^{\mathrm{T}} R_1 - R_1^{\mathrm{T}} R_2 = 0$, and S is given by (1). Choose $G \in \mathrm{so}(n)$ and $P \in \mathrm{Sym}(n)$. Then, the matrices

$$B_1 \overset{\mathrm{def}}{=} \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}, \quad B_2 \overset{\mathrm{def}}{=} \begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix}$$

are contained in $\mathrm{sp}(n) \cap \mathrm{so}(2n)$, so that $SB_1 = B_1 S$ and $SB_2 = B_2 S$. It can be easily verified that $SB_1 = B_1 S$ and $SB_2 = B_2 S$ imply that $R_1 G - GR_1 = R_1 P - PR_1 = R_2 G - GR_2 = R_2 P - PR_2 = 0$. Since the matrices $G \in \mathrm{so}(n)$ and $P \in \mathrm{Sym}(n)$ were chosen to be arbitrary, it follows that R_1 and R_2 commute with all skew-symmetric and symmetric matrices. Lemma A.1 now implies that there exist $\alpha_i \in \mathbb{R}$, $i \in \{1, 2\}$, such that $R_i = \alpha_i I_n$, $i \in \{1, 2\}$. Since $R_1^{\mathrm{T}} R_1 + R_2^{\mathrm{T}} R_2 = I_n$, it follows that there exists $t \in \mathbb{R}$ such that $\alpha_1 = \cos t$ and $\alpha_2 = \sin t$. It is now a simple matter to verify that $S = e^{J_{2n} t}$. Thus (ii) implies (iii).

Finally, since J_{2n} commutes with all $T \in \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$, it follows that $e^{J_{2n} t}$ commutes with all $T \in \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$ for all $t \in \mathbb{R}$. Thus (iii) implies (i). □

Proof of Proposition 2.2. As noted earlier, J_{2n} commutes with every matrix in $\mathrm{sp}(n) \cap \mathrm{so}(2n)$. Hence $\mathrm{span}\{J_{2n}\}$ is contained in the center subalgebra of $\mathrm{sp}(n) \cap \mathrm{so}(2n)$. To show the reverse inclusion, suppose B is contained in the center subalgebra of $\mathrm{sp}(n) \cap \mathrm{so}(2n)$. There exist $G \in \mathrm{so}(n)$ and $P \in \mathrm{Sym}(n)$ such that B is given by (1). Choose $G_1 \in \mathrm{so}(n)$ and $P_1 \in \mathrm{Sym}(n)$. The matrices

$$B_1 \overset{\mathrm{def}}{=} \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix}, \; B_2 \overset{\mathrm{def}}{=} \begin{bmatrix} 0 & P_1 \\ -P_1 & 0 \end{bmatrix}$$

are contained in $\mathrm{sp}(n) \cap \mathrm{so}(2n)$, so that $BB_1 = B_1 B$ and $BB_2 = B_2 B$. It can be easily verified that $BB_1 = B_1 B$ and $BB_2 = B_2 B$ imply that $GG_1 - G_1 G = PG_1 - G_1 P =$

$GP_1 - P_1G = PP_1 - P_1P = 0$. Since the matrices $G_1 \in \mathrm{so}(n)$ and $P_1 \in \mathrm{Sym}(n)$ were chosen to be arbitrary, it follows that G and P commute with all skew-symmetric and symmetric matrices. Lemma A.1 now implies that there exist $\alpha_i \in \mathbb{R}$, $i \in \{1, 2\}$, such that $G = \alpha_1 I$ and $P = \alpha_2 I$. However, $G \in \mathrm{so}(n)$ implies that $\alpha_1 = 0$, so that $B \in \mathrm{span}\{J_{2n}\}$. The result now follows. □

Proof of Proposition 2.3. The proof that (i) implies (ii) is the same as the proof that (i) implies (ii) in Proposition 2.1.

To show that (ii) implies (iii), suppose $S \in \mathrm{Sp}(n)$ satisfies $SB = BS$ for all $B \in \mathrm{sp}(n)$. Since $J_{2n} \in \mathrm{sp}(n)$, it follows that $SJ_{2n} = J_{2n}S$, Thus $S \in \mathrm{Sp}(n) \cap \mathrm{SO}(2n)$. Moreover, S clearly commutes with all $B \in \mathrm{sp}(n) \cap \mathrm{so}(n) \subseteq \mathrm{sp}(n)$. Hence Proposition 2.1 implies that there exists $t \in \mathbb{R}$ such that

$$S = e^{J_{2n}t} = \begin{bmatrix} (\cos t)I_n & (\sin t)I_n \\ -(\sin t)I_n & (\cos t)I_n \end{bmatrix}.$$

It is easy to verify that the matrix

$$B \stackrel{\text{def}}{=} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

is contained in $\mathrm{sp}(n)$, so that $SB = BS$. The equality $SB - BS = 0$ yields $\sin t = 0$, so that $\cos t \in \{1, -1\}$. Hence it follows that $S \in \{I_n, -I_n\}$.

The implication (iii) implies (i) is obvious. □

Proof of Proposition 2.4. Choose $G_1 \in \mathrm{so}(n)$ and $P_1 \in \mathrm{Sym}(n)$, and suppose B is contained in the center subalgebra of $\mathrm{sp}(n)$. Then B commutes with the Hamiltonian matrices

$$B_1 \stackrel{\text{def}}{=} J_{2n}, \quad B_2 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad B_3 \stackrel{\text{def}}{=} \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix}, \quad B_4 \stackrel{\text{def}}{=} \begin{bmatrix} 0 & P_1 \\ -P_1 & 0 \end{bmatrix}.$$

$BB_1 = B_1B$ implies that $B \in \mathrm{sp}(n) \cap \mathrm{so}(n)$, so that there exist $G \in \mathrm{so}(n)$ and $P \in \mathrm{Sym}(n)$ such that B is given by (1). It is easy to verify that $BB_2 - B_2B = 0$ implies that $P = 0$, while $BB_3 - B_3B = BB_4 - B_4B = 0$ implies that $GG_1 - G_1G = GP_1 - P_1G = 0$. Since $G_1 \in \mathrm{so}(n)$ and $P_1 \in \mathrm{Sym}(n)$ were chosen arbitrarily, it follows that G commutes with all matrices in $\mathrm{so}(n)$ and $\mathrm{Sym}(n)$. By Lemma A.1, there exists $\alpha \in \mathbb{R}$ such that $G = \alpha I_n$. However, $G \in \mathrm{so}(n)$ implies that $\alpha = 0$. Thus $B = 0$, and the result follows. □

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