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A double-commutator guaranteed cost bound for robust stability and performance[☆]

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Abstract

In this paper, we present a new guaranteed cost bound motivated by the maximum entropy equation. This bound is of interest due to its sensitivity to skew-symmetric structured uncertainty, which is difficult to address by means of conventional guaranteed cost bounds.

Keywords: Double-commutator; Robust stability; Robust performance

1. Introduction

Although considerable research has been devoted to the robustness problem for linear uncertain systems, there remain important unsolved problems of intense interest. These problems include the determination of nonconservative robust stability and performance bounds for systems involving possibly repeated multiple uncertainty blocks that may be real or complex. A comprehensive review of the multiple complex block case is given in [9], while bounds for systems involving both real and complex blocks are given in [6].

From a Lyapunov function point of view, robust stability and performance can be guaranteed by means of a fixed Lyapunov function whose existence is equivalent to a small-gain condition [8]. As shown in [3], this Lyapunov function is one of a large class of guaranteed cost bounds whose origin can be traced to the original work of Chang and Peng [5]. The best-known members of this class of bounds are the absolute value bound [5], the linear bound [1, 2, 7], and the quadratic bound [8, 10]. A useful benefit of this Lyapunov function framework is its applicability to robust controller synthesis [2, 10].

A valuable test of the effectiveness of a given robustness technique is to examine its ability to predict stability and performance in simple cases in which the robustness is readily apparent. We have in mind, for example, the case of a dissipative matrix A (satisfying $A + A^{\mathrm{T}} < 0$) and a skew-symmetric matrix A_1 (satisfying $A_1 + A_1^{\mathrm{T}} = 0$). In this case it can readily be seen that $A + \sigma_1 A_1$ is asymptotically stable for all real σ_1. Nevertheless, this problem is difficult to address by means of conventional guaranteed cost bounds, which

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often yield conservative estimates of robust stability and performance. This point was the motivation for [4], where a "maximum entropy"-type Lyapunov function was used to (correctly) predict unconditional asymptotic stability.

The goal of the present paper is to improve upon the results of [4] in several ways. Like the results of [4], the Lyapunov bounds proposed herein are motivated by the double-commutator modification of the Lyapunov equation that arises in the maximum entropy control approach as applied to skew-symmetric uncertainty. These modifications have the useful property that in the limit of high uncertainty the solution of the modified Lyapunov equation commutes with the uncertainty structure [4]. However, whereas the construction of the Lyapunov bound in [4] involved both a modified Lyapunov equation and an auxiliary term (called P_0 in [4]), the Lyapunov bound in the present paper involves *only* a modified Lyapunov equation. The elimination of the auxiliary term is desirable since it yields a simpler framework for both analysis and synthesis.

Another advantage of the Lyapunov bound given in this paper is its generality in applying to arbitrary nominal dynamics matrices and arbitrary uncertainty structures. Thus, the results given herein address the problem of dissipative nominal dynamics and skew-symmetric uncertainty as a special case of a much broader class of problems. This extension represents a significant improvement over the results of [4] which were limited to the dissipative/skew-symmetric case.

The contents of the paper are as follows. In Section 2 we state the robust stability and performance problem. Then, using the notation and terminology of [3, 4], we present the new bound (Proposition 2.1) and then apply it to robust stability and robust H_2 performance (Theorem 2.1). Connections between this bound and the absolute value bound of [5] as well the maximum entropy bounds of [4] are discussed. In Section 3 numerical examples are given to illustrate properties of the new bound.

Notation

I_r	$r \times r$ identity matrix,
$\mathcal{N}^n$, $\mathcal{S}^n$	$n \times n$ nonnegative-definite matrices, symmetric matrices,
$\mathbb{R}^{n \times n}$	$n \times n$ real matrices,
$[F, G]$	$FG - GF$,
$\lvert X \rvert$	$\sqrt{X^2}$, where $X \in \mathcal{S}^n$.

2. Robust stability and performance problems

Let $\mathcal{U} \subset \mathbb{R}^{n \times n}$ denote a set of perturbations ΔA of a given nominal dynamics matrix $A \in \mathbb{R}^{n \times n}$. It is assumed that A is asymptotically stable and that $0 \in \mathcal{U}$.

Robust stability problem. Determine whether the linear system

$$\dot{x}(t) = (A + \Delta A)x(t), \quad t \in [0, \infty), \tag{1}$$

is asymptotically stable for all $\Delta A \in \mathcal{U}$.

Robust performance problem. For the disturbed linear system

$$\dot{x}(t) = (A + \Delta A)x(t) + Dw(t), \quad t \in [0, \infty), \tag{2}$$

$$z(t) = Ex(t), \tag{3}$$

where $w(\cdot)$ is a zero-mean d-dimensional white-noise signal with intensity I_d, determine a performance bound β satisfying

$$\mathcal{J}(\mathcal{U}) \triangleq \sup_{\Delta A \in \mathcal{U}} \limsup_{t \to \infty} \boldsymbol{E}\{\|z(t)\|_2^2\} \leqslant \beta. \tag{4}$$

For convenience define the $n \times n$ nonnegative-definite matrices $R \triangleq E^{\mathrm{T}}E$ and $V \triangleq DD^{\mathrm{T}}$. The following result is immediate.

Lemma 2.1. *Suppose $A + \Delta A$ is asymptotically stable for all $\Delta A \in \mathscr{U}$. Then*

$$\mathscr{J}(\mathscr{U}) = \sup_{\Delta A \in \mathscr{U}} \operatorname{tr}(Q_{\Delta A} R) = \sup_{\Delta A \in \mathscr{U}} \operatorname{tr}(P_{\Delta A} V), \tag{5}$$

where $Q_{\Delta A} \in \mathbb{R}^{n \times n}$ and $P_{\Delta A} \in \mathbb{R}^{n \times n}$ are the unique, nonnegative-definite solutions to

$$0 = (A + \Delta A)^{\mathrm{T}} Q_{\Delta A} + Q_{\Delta A}(A + \Delta A) + V \tag{6}$$

and

$$0 = (A + \Delta A)^{\mathrm{T}} P_{\Delta A} + P_{\Delta A}(A + \Delta A) + R. \tag{7}$$

Proof. See [3]. □

In the present paper our approach is to obtain robust stability as a consequence of sufficient conditions for robust performance. Such conditions are given by the following result.

Theorem 2.1. *Let $\Omega: \mathscr{N}^n \to \mathscr{S}^n$ be such that*

$$\Delta A^{\mathrm{T}} P + P \Delta A \leqslant \Omega(P), \quad \Delta A \in \mathscr{U}, \; P \in \mathscr{N}^n, \tag{8}$$

and suppose there exists $P \in \mathscr{N}^n$ satisfying

$$0 = A^{\mathrm{T}} P + PA + \Omega(P) + R. \tag{9}$$

Then

$$(E, A + \Delta A) \text{ is detectable}, \quad \Delta A \in \mathscr{U}, \tag{10}$$

if and only if

$$A + \Delta A \text{ is asymptotically stable}, \quad \Delta A \in \mathscr{U}. \tag{11}$$

In this case,

$$P_{\Delta A} \leqslant P, \qquad \Delta A \in \mathscr{U}, \tag{12}$$

where $P_{\Delta A}$ satisfies (7), *and*

$$\mathscr{J}(\mathscr{U}) \leqslant \operatorname{tr} PV. \tag{13}$$

Proof. See [4]. □

Here we specialize to the case in which $\mathscr{U}$ is given by

$$\mathscr{U} \triangleq \left\{ \Delta A \in \mathbb{R}^{n \times n}: \Delta A = \sum_{i=1}^{r} \sigma_i A_i, \; |\sigma_i| \leqslant \delta_i, \; i = 1, \ldots, r \right\}, \tag{14}$$

where $\delta_i > 0$ and the matrices $A_i \in \mathbb{R}^{n \times n}$ represent the uncertainty structure. Furthermore, for each $i = 1, \ldots, r$, let Z_i be a nonnegative-definite matrix satisfying

$$-Z_i \leqslant A_i + A_i^{\mathrm{T}} \leqslant Z_i. \tag{15}$$

One choice of Z_i is $Z_i = |A_i + A_i^{\mathrm{T}}| = [(A_i + A_i^{\mathrm{T}})^2]^{1/2}$. Thus, if A_i is skew symmetric, i.e., $A_i + A_i^{\mathrm{T}} = 0$, then we set $Z_i = 0$, so that (15) is satisfied as an equality.

We now introduce a specific choice of $\Omega(P)$ which is motivated by the maximum entropy covariance model [4].

Proposition 2.1. *For $i = 1,\dots,r$, let $\delta_i \geqslant 0$, $\beta_i, \gamma_i \in \mathbb{R}$, $\beta_i \neq 0$. Then* (8) *is satisfied with $\Omega(P)$ given by*

$$\Omega(P) = \sum_{i=1}^{r}\left[\beta_i^2(A_i^{\mathrm{T}}P + PA_i)^2 + \frac{\delta_i^2}{4\beta_i^2}I_n + \gamma_i\beta_i(A_i^{2\mathrm{T}}P + 2A_i^{\mathrm{T}}PA_i + PA_i^2) + \gamma_i^2 A_i^{\mathrm{T}}A_i + \frac{\delta_i}{2}|\gamma_i/\beta_i|\,Z_i\right]. \tag{16}$$

Proof. Letting $\Delta A \in \mathcal{U}$ and $P \in \mathcal{N}^n$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{r}\left[\beta_i^2(A_i^{\mathrm{T}}P + PA_i)^2 + \frac{\delta_i^2}{4\beta_i^2}I_n + \gamma_i\beta_i(A_i^{2\mathrm{T}}P + 2A_i^{\mathrm{T}}PA_i + PA_i^2) + \gamma_i^2 A_i^{\mathrm{T}}A_i + \frac{\delta_i}{2}|\gamma_i/\beta_i|\,Z_i\right] \\
&\quad - \sum_{i=1}^{r}\sigma_i(A_i^{\mathrm{T}}P + PA_i) \\
&\quad \geqslant \sum_{i=1}^{r}\left[\beta_i^2(A_i^{\mathrm{T}}P + PA_i)^2 + \frac{\sigma_i^2}{4\beta_i^2}I_n + \gamma_i\beta_i(A_i^{2\mathrm{T}}P + 2A_i^{\mathrm{T}}PA_i + PA_i^2) + \gamma_i^2 A_i^{\mathrm{T}}A_i - \frac{\sigma_i\gamma_i}{2\beta_i}(A_i + A_i^{\mathrm{T}})\right] \\
&\quad - \sum_{i=1}^{r}\sigma_i(A_i^{\mathrm{T}}P + PA_i) \\
&\quad = \sum_{i=1}^{r}\left[\beta_i(A_i^{\mathrm{T}}P + PA_i) + \gamma_i A_i - \frac{\sigma_i}{2\beta_i}I_n\right]^{\mathrm{T}}\left[\beta_i(A_i^{\mathrm{T}}P + PA_i) + \gamma_i A_i - \frac{\sigma_i}{2\beta_i}I_n\right] \\
&\quad \geqslant 0. \qquad \square
\end{aligned}$$

Remark 2.1. As shown in [3] the first two terms of Eq. (16) comprise an upper bound for the absolute value bound proposed by Chang and Peng [5], i.e.

$$\sum_{i=1}^{r}\delta_i|A_i^{\mathrm{T}}P + PA_i| \leqslant \sum_{i=1}^{r}\left[\beta_i^2(A_i^{\mathrm{T}}P + PA_i)^2 + \frac{\delta_i^2}{4\beta_i^2}I_n\right]. \tag{17}$$

Remark 2.2. If $A_i + A_i^{\mathrm{T}} = 0$, $i = 1,\dots,r$, so that $Z_i = 0$, then $\Omega(P)$ can be written as

$$\Omega(P) = \sum_{i=1}^{r}\left[\beta_i^2[A_i^{\mathrm{T}}, P]^2 + \gamma_i\beta_i[A_i^{\mathrm{T}}, [A_i^{\mathrm{T}}, P]] + \gamma_i^2 A_i^{\mathrm{T}}A_i + \frac{\delta_i^2}{4\beta_i^2}I_n\right], \tag{18}$$

which exhibits the double commutator term $[A_i^{\mathrm{T}}, [A_i^{\mathrm{T}}, P]]$ discussed in [4]. For skew-symmetric uncertainty this term is indefinite and has zero trace. In addition, it is shown in [4] that $[A_i^{\mathrm{T}}, P] \to 0$ as $\gamma_i \to \infty$, so that $[A_i^{\mathrm{T}}, P]^2 \to 0$. Consequently, (9) with $\Omega(P)$ given by (16) provides a finite performance bound for large uncertainty levels δ_i. Thus for the case of skew-symmetric uncertainty (9) is nonconservative with respect to robust stability.

3. Numerical examples

Example 3.1. We consider a lightly damped modal system with uncertainty in the damped natural frequency represented by

$$A = \begin{bmatrix} -0.005 & 1 \\ -1 & -0.005 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where A_1 is a skew symmetric. Furthermore, choose

$$R = \begin{bmatrix} 0.25 & 0.12 \\ 0.12 & 2.5 \end{bmatrix}, \qquad V = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

For this example, $A + \sigma_1 A_1$ is asymptotically stable for all $\sigma_1 \in (-\infty, \infty)$, while Fig. 1 shows the worst case H_2 performance for $0 \leqslant \delta_1 \leqslant 2$. Applying the guaranteed cost bound of [5], robust stability is predicted only

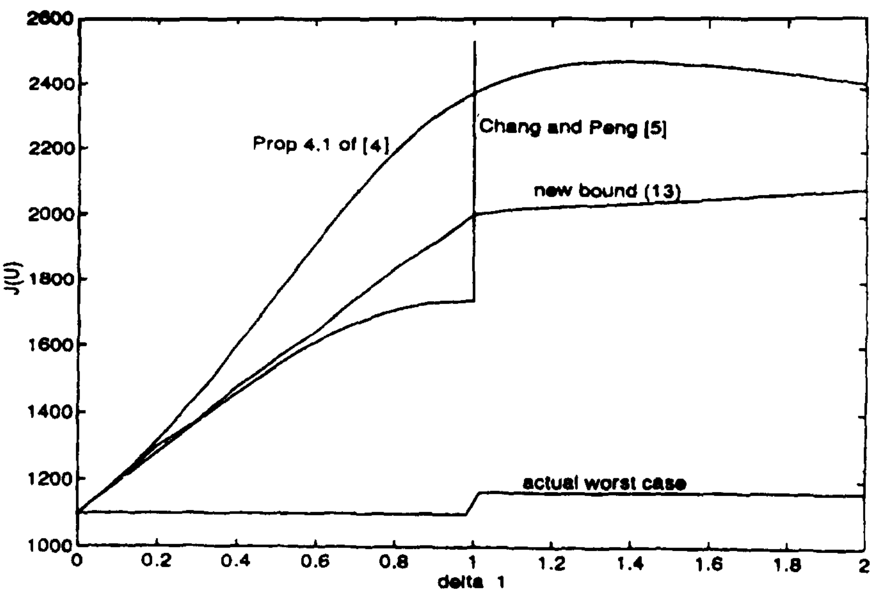

Fig. 1. Comparison of new bound (13), Chang and Peng bound [5], Proposition 4.1 of [4] and actual worst case for a modal matrix A with skew-symmetric uncertainty A_1.

for $\sigma_1 \in (-1, 1)$. By choosing suitable values of β_1 and γ_1, the new bound (13), however, predicts stability for $\sigma_1 \in (-2, 2)$. Although this stability guarantee is also obtainable from Proposition 4.1 of [4], it can be seen from Fig. 1 that the new bound (13) provides a less conservative estimate of robust performance.

4. Conclusions

We presented a new guaranteed cost bound involving a double commutator in the spirit of the maximum entropy-type Lyapunov equation. This bound is sensitive to skew-symmetric uncertainty which is difficult to handle using conventional guaranteed cost bounds.

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