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Input Richness and Zero Buffering in Time-Domain Identification

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***Abstract*— We consider the notion of persistency within a deterministic, finite-data context, namely, in terms of the rank and condition number of the regressor matrix, which contains input and output data. The novel contribution of this work is the technique of *zero buffering*, in which the input signal begins with a sequence of zeros. We show that the *degree of persistency* of the input, which is the order of the minimal AR model that can generate the input signal, is increased by zero buffering. We then demonstrate the effectiveness of zero buffering in increasing the degree of persistency of a Schroeder-phased signal, which, without zero buffering, yields a poorly conditioned regressor matrix. We also investigate the feasibility of estimating the dynamic order in terms of the singular values of the regressor matrix by showing that the rank of the regressor matrix is related to the degree of persistency of the input, the order of the model, and the order of the true system. Under reasonable signal to noise ratios, this technique provides a useful estimate of the true system order.**

I. INTRODUCTION

Persistency is a bedrock requirement of system identification. Roughly speaking, persistency guarantees that the inputs to the system and the resulting outputs have sufficient richness in spectral content to ensure that the system dynamics can be determined to a desired level of accuracy. These comments apply to both time-domain and frequency-domain identification objectives.

In the frequency-domain context, necessary and sufficient conditions are established in [1] for the degree of richness of the input to generate an informative experiment. One of these conditions is equivalent to the requirement that the spectral density of the input be nonzero at a specified number of frequencies. These conditions are also extended to closed-loop identification. In [2], signals that maximize persistency as defined by various cost criteria are examined, whereas in [3], persistency in the time domain is based on the informative value of the state. Persistency within a behavioral context is developed in [4].

All of these persistency conditions are defined in terms of either the statistics of the input and output signals, or in terms of the asymptotic nature of these signals, see, for example, [1]. This approach is especially applicable to stochastic analysis in which unbiasedness (zero mean of the error probability distribution) and consistency (convergence with probability 1 to the true value in the limit of infinite data) are desired properties of the estimate.

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In the present paper we reconsider the notion of persistency within a deterministic, finite-data context. Instead of stochastic analysis, we approach persistency in terms of the condition number of the regressor matrix, which plays a fundamental role in the effect of noise on the estimated parameters. In particular, we consider an input model in the form of an autogressive (AR) model, and then analyze the resulting rank and condition number of the regressor matrix. We make no assumption about the input or output of the system prior to the start of the data record, nor do we assume that the system begins at rest. The novel contribution of this work is the technique of *zero buffering*, in which the input signal begins with a sequence of zeros. We show that the *degree of persistency* of the input is increased by zero buffering. This simple technique is shown to be effective in increasing the persistency of both impulsive and sinusoidal inputs. In particular, we demonstrate the effectiveness of zero buffering in increasing the degree of persistency of a Schroeder-phased signal, which minimizes the peak-to-peak amplitude of a multi-sine signal [5], and which, without zero buffering, yields an extremely poorly conditioned regressor matrix. Thus, without zero buffering, the Schroeder-phased signal has limited value in time-domain least squares identification.

This paper also investigates the feasibility of estimating the dynamic order in terms of the rank or singular values of the regressor matrix. In particular, we show that the rank of the regressor matrix is related to the degree of persistency of the input, the order of the model, and the order of the true system, providing an easily implementable technique for estimating the order of the true system. Although noise in the input and output signals corrupts this order estimate, under moderate signal to noise ratios, the order of the true system can be estimated with useful accuracy.

The contents of the paper are as follows. In Section II, we examine several signals, including multi-sines and impulses, and present numerical and analytical results concerning the singular values of the regressor matrix for a finite impulse response (FIR) model with these signals. We also define degree of persistency and prove its relation to the rank of the FIR regressor matrix. In Section III, we examine an auto-regressive model with exogenous inputs (ARX), and we provide conditions on the order of the model and degree of persistency of the input such that the regressor matrix has full rank. In Section IV, we introduce zero buffering and prove that it can increase the persistency of a signal. In Section V, we give a numerical example in which a Schroeder-phased signal is zero buffered, and show that zero buffering can render the regressor matrix well-conditioned. In Section VI,

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we introduce a rank-estimation technique and present two examples. Lastly, in Section VII, we consider the effect of zero buffering in the presence of noise with a numerical example.

II. FIR Identification

Consider the n^{th}-order finite impulse response (FIR) model,

$$y(k) = H_0 u(k) + H_1 u(k-1) + \cdots + H_n u(k-n), \quad (1)$$

where $n \geq 0$ and $H_0, \ldots, H_n$ are Markov parameters. Choosing a model of the order $\hat{n}$ and sampling the system $\hat{n}+l$ times, we have the overdetermined system of equations

$$\Theta A = b, \quad (2)$$

where

$$\Theta \triangleq \begin{bmatrix} \hat{H}_0 & \cdots & \hat{H}_{\hat{n}} \end{bmatrix} \in \mathbb{R}^{1\times(\hat{n}+1)}, \quad (3)$$

$$A \triangleq \begin{bmatrix} u(\hat{n}) & \cdots & u(2\hat{n}) & \cdots & u(\hat{n}+l-1) \\ \vdots & \ddots & \vdots & & \vdots \\ u(0) & \cdots & u(\hat{n}) & \cdots & u(l-1) \end{bmatrix} \in \mathbb{R}^{(\hat{n}+1)\times l}, \quad (4)$$

$$b \triangleq \begin{bmatrix} y(\hat{n}) & \cdots & y(\hat{n}+l-1) \end{bmatrix} \in \mathbb{R}^{1\times l}. \quad (5)$$

Note that we do not use $y(0)$, $y(1)$, $\cdots$, $y(\hat{n}-1)$ because we do not assume that $u(k)$ is known for $k < 0$.

A. Sinusoidal Persistency

The persistency of a signal is related to the rank of the regressor matrix (4). Roughly speaking, if the regressor matrix (4) has a moderate condition number for large $\hat{n}$, then $u(k)$ is highly persistent. On the other hand, if the regressor matrix (4) has a large condition number or does not have full row rank for moderate values of $\hat{n}$, then $u(k)$ is weakly persistent.

Consider the multi-sine signals

$$u(k) = \sum_{i=1}^{20} \cos\left(\left[\frac{2\pi i}{T}\right] k\right), \quad (6)$$

$$v(k) = \sum_{i=1}^{20} \cos\left(10\left[\frac{2\pi i}{T}\right] k\right), \quad (7)$$

$$w(k) = \sum_{i=1}^{20} \cos\left(100\left[\frac{2\pi i}{T}\right] k\right), \quad (8)$$

where $T = 1000$ s and $k = 1, \ldots, 1000$. Figures 1-3 display the power spectral densities of (6)-(8). Note that all of the signals have 20 sinusoidal components, although their frequency content is spread out differently.

Figure 4 shows that the regressor matrix (4) is poorly conditioned for $\hat{n} = 39$ and all of the signals (6)-(8). However, note that, in Figure 4, the signal $w(k)$, which has the largest bandwidth of (6)-(8), is also the most persistent. Therefore, Figure 4 and additional examples suggest that multi-sine signals with larger bandwidths are more persistent than those with dense frequency spectra.

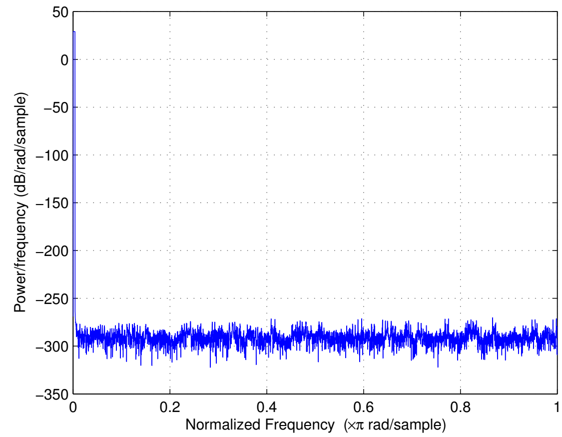

Fig. 1. Power spectral density of $u(k)$ given by (6).

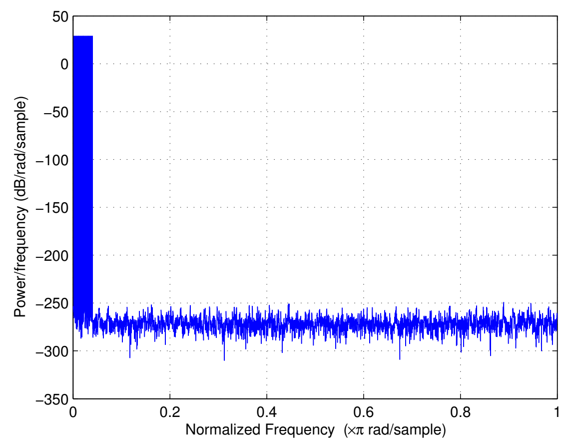

Fig. 2. Power spectral density of $v(k)$ given by (7).

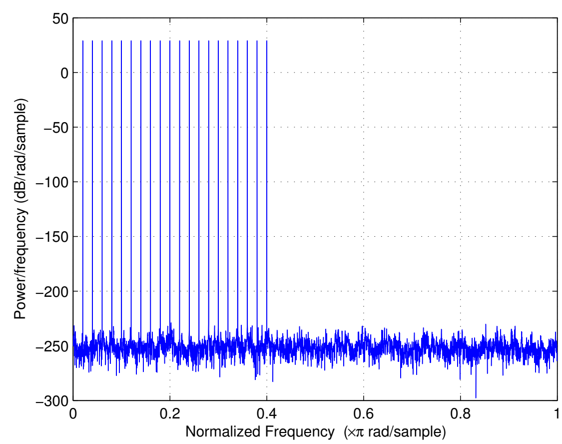

Fig. 3. Power spectral density of $w(k)$ given by (8).

B. Impulse Persistency

Consider the unit impulse for identification, that is,

$$u(k) = \begin{cases} 1, & k = 0, \\ 0, & k > 0, \end{cases} \quad (9)$$

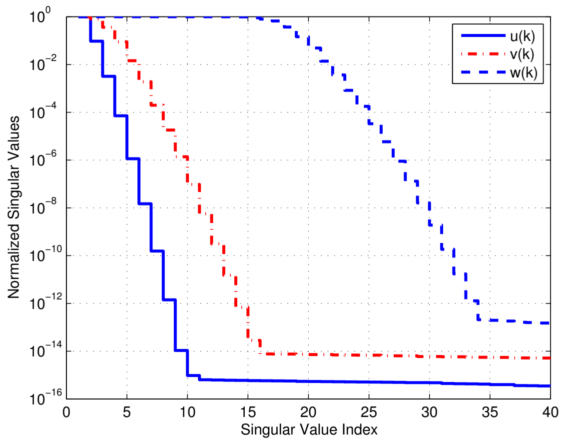

Fig. 4. Normalized singular values of the regressor matrix (4) with $\hat{n} = 39$ and the signals (6)-(8)

which yields the regressor matrix

$$A = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{10}$$

Since rank $A = 1$, the solution Θ of (2) is unique if and only if $\hat{n} = 0$.

Now consider the shifted impulse

$$u_s(k) = \delta(k - \hat{n}), \tag{11}$$

where $\delta(\cdot)$ is the unit impulse function. The input yields the regressor matrix

$$A = \left[\begin{array}{cc} I_{\hat{n}+1} & 0_{(\hat{n}+1)\times(l-\hat{n}-1)} \end{array} \right], \tag{12}$$

where rank $A = \hat{n}+1$, and thus A has full row rank. Hence shifting the impulse $\hat{n}$ time steps yields a unique solution of (2).

C. Degree of Persistency

Definition 2.1: Let $u(0), \ldots, u(l) \in \mathbb{R}$, where $u(0), \ldots, u(l)$ are not all zero. Then the degree of persistency of $u(0), \ldots, u(l)$ is the smallest positive integer m such that there exist $c_1, \ldots, c_m \in \mathbb{R}$ that satisfy the auto-regressive (AR) model

$$u\left(k+m\right) = c_1 u\left(k+m-1\right) + \cdots + c_m u\left(k\right), \tag{13}$$

for $k = 0, \ldots, l-m$. That is, $C(\mathbf{q})u(k) = 0$, where $\mathbf{q}$ denotes the forward shift operator and $C(\mathbf{q}) = \mathbf{q}^m + c_1\mathbf{q}^{m-1} + \cdots + c_m$.

Theorem 2.1: Let m denote the degree of persistency of $u(0), \ldots, u(l)$, and let $\hat{n}$ denote the order of (1). Then

$$\text{rank } A = \min(\hat{n}+1, m), \tag{14}$$

where A is given by (4).

Proof 2.1: Let $\text{row}_i(A)$ denote the i^{th} row of A. If $m \leq \hat{n}$, then

$$\text{row}_1(A) = \sum_{i=1}^{m} c_i \text{row}_{i+1}(A),$$

where $c_1, \ldots, c_m$ are given by Definition 2.1. Therefore rank $A = m$. Conversely, let $m \geq \hat{n}+1$ and assume that rank $A < \hat{n}+1$. Then there exist $d_1, \ldots, d_{\hat{n}}$ such that

$$\text{row}_1(A) = \sum_{i=1}^{\hat{n}} d_i \text{row}_{i+1}(A).$$

Therefore, for all $k \in [0, l-m]$,

$$u\left(k+\hat{n}\right) = d_1 u\left(k+\hat{n}-1\right) + \cdots + d_{\hat{n}} u\left(k\right).$$

Hence $m \leq \hat{n}$, which is a contradiction. □

III. ARX Model Identification

Consider the ARX model

$$A(\mathbf{q})y(k) = B(\mathbf{q})u(k), \tag{15}$$

where

$$A(\mathbf{q}) = \mathbf{q}^n + a_1\mathbf{q}^{n-1} + \cdots + a_n\mathbf{q}^0, \tag{16}$$

$$B(\mathbf{q}) = b_0\mathbf{q}^n + b_1\mathbf{q}^{n-1} + \cdots + b_n\mathbf{q}^0. \tag{17}$$

Sampling $u(k)$ and $y(k)$ $\hat{n}+l$ times yields the regressor matrix

$$A = \begin{bmatrix} A_y \\ A_u \end{bmatrix}, \tag{18}$$

where

$$A_y \triangleq \begin{bmatrix} y(\hat{n}-1) & \cdots & y(2\hat{n}-2) & \cdots & y(\hat{n}+l-2) \\ \vdots & \ddots & \vdots & & \vdots \\ y(0) & \cdots & y(\hat{n}-1) & \cdots & y(l-1) \end{bmatrix}$$

$$A_u \triangleq \begin{bmatrix} u(\hat{n}) & \cdots & u(2\hat{n}) & \cdots & u(\hat{n}+l-1) \\ \vdots & \ddots & \vdots & & \vdots \\ u(0) & \cdots & u(\hat{n}) & \cdots & u(l-1) \end{bmatrix}. \tag{19}$$

Proposition 3.1: If A given by (18) has full row rank, then the degree of persistency m of $u(k)$ must be greater than $\hat{n}$.

Proof 3.1: Since A has full row rank, both A_y and A_u have full row rank. Also, from Theorem 2.1, since A_u has full row rank, then $m > \hat{n}$. □

The following fact will be used as a basis for later developments in rank estimation.

Fact 3.1: If $\hat{n} > n$, then

$$\text{row}_1(A) = -\sum_{i=1}^{n} a_i \text{row}_{i+1}(A) + \sum_{i=0}^{n} b_i \text{row}_{i+\hat{n}+1}(A). \tag{20}$$

Thus if A given by (18) has full row rank, then $\hat{n} \leq n$.

IV. Zero Buffering

The structure of the regressor matrix A given by (12) due to the shifted impulse (11) suggests an advantage in starting the input signal with a sequence of zeros. We call this procedure *zero buffering*. With zero buffering, the regressor matrix of the FIR model (1) has the form

$$A = \begin{bmatrix} u(0) & \cdots & \cdots & \cdots & \cdots & u(\hat{n}+l-1) \\ 0 & \ddots & & & & \vdots \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & 0 & u(0) & \cdots & u(l-1) \end{bmatrix}. \quad (21)$$

Note that if there exists $k \in [0, l-1]$ such that $u(k) \neq 0$, then A has full row rank and hence there exists a unique solution of (2).

Theorem 4.1: Let $m > 0$ denote the degree of persistency of $u(0), \ldots, u(l)$. Then there exist $c_1, \ldots, c_m$ such that $u(k)$ satisfies the AR model

$$u\left(k+m\right) = c_1 u\left(k+m-1\right) + \cdots + c_m u\left(k\right), \quad (22)$$

for $k = 0, \ldots, l-m$. That is, $C(\mathbf{q})u(k) = 0$. Furthermore, let $\bar{m} \leq m$ be the smallest positive integer such that

$$u\left(\bar{m}\right) = c_1 u\left(\bar{m}-1\right) + \cdots + c_{\bar{m}} u\left(0\right). \quad (23)$$

Then, for each nonnegative integer r, the degree of persistency of the zero-buffered signal

$$u_{zb,r}(k) = \begin{cases} 0, & k = 0, \ldots, r-1, \\ u(k-r), & k \geq r. \end{cases} \quad (24)$$

is $r + \bar{m} \geq m$, where $u_{zb,0}(k) = u(k)$ and $u_{zb,r}$ satisfies the AR model

$$\mathbf{q}^{r+\bar{m}-m} C(\mathbf{q}) u_{zb,r}(k) = 0, \quad (25)$$

for $k = 0, \ldots, l+r-m$.

Proof 4.1: Suppose that $u_{zb,r}$ satisfied the $(\bar{m}+r-1)^{th}$ order AR model

$$u_{zb,r}\left(k+\bar{m}+r-1\right) = \alpha_1 u_{zb,r}\left(k+\bar{m}+r-2\right) + \cdots + \alpha_{\bar{m}+r-1} u_{zb,r}\left(k\right).$$

Then for $k = 0$ it follows that

$$u_{zb,r}\left(\bar{m}+r-1\right) = \alpha_1 u_{zb,r}\left(\bar{m}+r-2\right) + \cdots + \alpha_{\bar{m}+r-1} u_{zb,r}\left(0\right),$$

or, equivalently,

$$u\left(\bar{m}-1\right) = \alpha_1 u\left(\bar{m}-2\right) + \cdots + \alpha_{\bar{m}-1} u\left(0\right),$$

which contradicts the assumption that $\bar{m}$ is the smallest positive integer that satisfies (23). □

Note that typically the degree of persistency of $u_{zb,r}$ is $m + r$.

Example 4.1: Let $u(k) = \{1, 1, 0, 0, 1, 1, 0, 0, \ldots\}$. Then $u(k+3) = u(k+2) - u(k+1) + u(k)$, for $k = 0, 1, \ldots,$ and hence $u(k)$ has a degree of persistency of 3. Also, from Theorem 4.1, $\bar{m} = 2$. Thus $u_{zb,2}(k) = \{0, 0, u(k)\}$ has a degree of persistency of 3 as well.

Corollary 4.1: Let $\hat{n}$ denote the order of the FIR model (1). If $u(0), \ldots, u(l)$ are not all zero and $u(0), \ldots, u(l)$ is zero-buffered by $\hat{n}$, then the FIR regressor matrix (4) has full row rank.

Proof 4.2: Since $u(0), \ldots, u(l)$ are not all zero, the degree of persistency of $u(0), \ldots, u(l)$ is $m > 0$. Then by Theorem 4.1, the degree of persistency of the zero-buffered signal

$$u_{zb}(k) = \begin{cases} 0, & k = 0, \ldots, \hat{n}-1, \\ u(k-\hat{n}), & k \geq \hat{n}, \end{cases}$$

is $p = \bar{m} + \hat{n}$, where $\bar{m}$ is the smallest nonnegative integer such that (23) is satisfied. Since $\bar{m} \geq 1$, it follows that $p \geq \hat{n} + 1$, and Theorem 2.1 states that the FIR regressor matrix (4) has full row rank. □

Fact 4.1: If the degree of persistency of $u(k)$ is m_1 and the degree of persistency of $v(k)$ is m_2, then the degree of persistency of $u(k) + v(k)$ is less than or equal to $m_1 + m_2$.

Proof 4.3: Let $C(\mathbf{q})u(k) = 0$ and $D(\mathbf{q})v(k) = 0$, where $C(\mathbf{q})$ and $D(\mathbf{q})$ are of order m_1 and m_2, respectively. Then $C(\mathbf{q})D(\mathbf{q})\left[u(k) + v(k)\right] = 0$. Hence the degree of persistency of $u(k) + v(k)$ must be less than or equal to $m_1 + m_2$. □

V. Schroeder-Phased Signals

Schroeder-phased signals minimize the peak-to-peak amplitude of multi-sine signals through judicious phasing [5, 6]. A Schroeder-phased signal with flat power spectrum has the form

$$u_S(k) = \sum_{i=1}^{N} \cos\left(\left[\frac{2\pi i}{T}\right] k - \frac{\pi i^2}{N}\right). \quad (26)$$

Consider a Schroeder-phased multi-sine signal with $N = 20$ for $k = 0, \ldots, 999$, and the zero-buffered signal

$$u_{zb,40}(k) = \begin{cases} 0, & k = 0, \ldots, 39, \\ u_S(k-39), & k \geq 39. \end{cases} \quad (27)$$

Now consider the normalized singular values of a 39^{th} order FIR regressor matrix with the Schroeder-phased multi-sine and its zero-buffered form. Figure 5 shows that the regressor matrix with the Schroeder-phased signal is poorly conditioned, whereas with the zero-buffered signal, the regressor matrix has a good condition number and thus full row rank.

VI. Rank Estimation

Rank estimation is often performed by using the eigensystem realization algorithm (ERA), where the rank estimate is taken to be the rank of the Markov block Hankel matrix [7]. However, this approach pre-supposes knowledge of the system's Markov parameters. Here we show that a rank estimate can be obtained directly from the ARX regressor matrix (18).

Theorem 6.1: Consider an ARX system of the form (15). Let the degree of persistency of $u(k)$ be m, the order

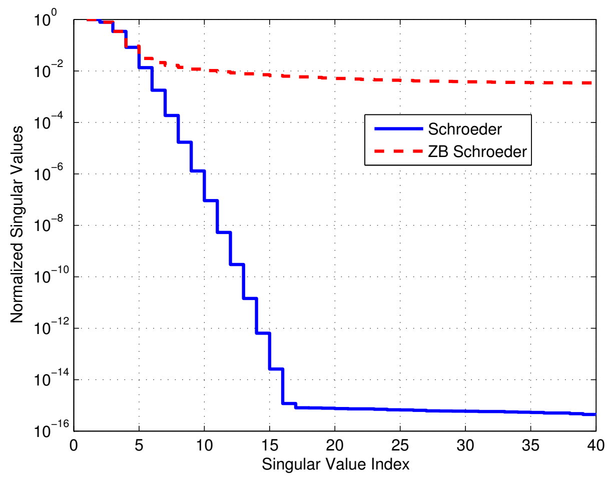

Fig. 5. Comparison of the normalized singular values of a 39^{th} order FIR regressor matrix with a Schroeder-phased and zero-buffered Schroeder-phased signal.

of the true system be n, and the order of the model (15) be $\hat{n}$. Then

$$\min(\hat{n}+1,m) \leq \text{rank } A \leq \min(\hat{n},n)+\min(\hat{n}+1,m), \tag{28}$$

where A is the ARX regressor matrix (18).

Proof 6.1: Note that rank $A_u \leq$ rank A. Hence, from Theorem 2.1, $\min(\hat{n}+1,m) \leq$ rank A. Next, suppose that $n < \hat{n}$. Then

$$\text{row}_1(A) = -\sum_{i=1}^{n} a_i \text{row}_{i+1}(A) + \sum_{i=0}^{n} b_i \text{row}_{i+\hat{n}+1}(A),$$

and rank $A_u = \min(\hat{n}+1,m)$. Therefore, at most n rows of A_y are linearly independent of A_u and it follows that rank $A \leq n + \min(\hat{n}+1,m)$. If $\hat{n} \leq n$, then all of the rows of A_y may be independent of A_u and it follows that rank $A \leq \hat{n} + \min(\hat{n}+1,m)$. □

Numerical testing suggests that rank $A = \min(\hat{n},n) + \min(\hat{n}+1,m)$ for almost all initial conditions of $y(k)$, although for some initial conditions of $y(k)$, rank $A < \min(\hat{n},n)+\min(\hat{n}+1,m)$. The following example demonstrates a specific case in which rank $A < \min(\hat{n},n) + \min(\hat{n}+1,m)$.

Example 6.1: Consider the system

$$y(k) = ay(k-1) + u(k), \tag{29}$$

for $k \geq 1$, where $y(0) = y_0$ and $u(k) = r^k$. Note that since $u(k)$ satisfies $u(k) = ru(k-1)$ its degree of persistency is 1. Then letting $\hat{n} \geq 1$ and y_0 be given by

$$y_0 = \frac{r}{r-a}, \tag{30}$$

it follows that $y(k) = y_0 u(k)$ and hence rank $A = 1 < \min(\hat{n},n)+\min(\hat{n}+1,m)$. However, for all other values of y_0, rank $A = 2 = \min(\hat{n},n) + \min(\hat{n}+1,m)$.

The usefulness of Theorem 6.1 is due to the fact that the degree of persistency m of the input can be computed separately from the rank of the regressor matrix A given by (18). Hence when $\hat{n} > n$ and the rank equality holds, that is, rank $A = \min(\hat{n},n) + \min(\hat{n}+1,m)$, then

$$n = \text{rank } A - \min(\hat{n}+1,m). \tag{31}$$

The following examples demonstrate this technique.

Example 6.2: Consider the system

$$G(z) = \frac{z-1}{(z^2-1.5z+0.8)^3}, \tag{32}$$

the input

$$u(k) = \cos(k/10), \quad k = 0,\ldots,999, \tag{33}$$

and model order $\hat{n} = 10$. Hence $n = 6$ and $m = 2$. Then by (28), we expect

$$\begin{aligned}\text{rank } A &= \min(\hat{n},n) + \min(\hat{n}+1,m)\\ &= 6+2 = 8,\end{aligned}$$

which is verified in Figure 6 by the normalized singular values of the regressor matrix A given by (18).

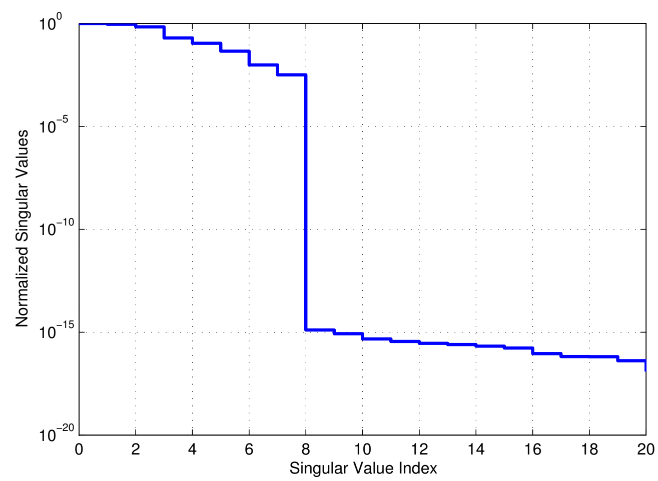

Fig. 6. Normalized singular values of the regressor matrix A given by (18) for the system (32), input (33), and order of the model $\hat{n} = 10$.

Example 6.3: Consider the system (32), the model order $\hat{n} = 10$, and the zero-buffered input

$$u_{zb,11}(k) = \begin{cases} 0, & k = 0,\ldots,10,\\ \cos([k-11]/10), & k \geq 11.\end{cases} \tag{34}$$

Then $n = 6$, $\hat{n} = 10$, and $m = 13$. Hence by (28) we expect

$$\begin{aligned}\text{rank } A &= \min(\hat{n},n) + \min(\hat{n}+1,m)\\ &= 6+11 = 17,\end{aligned}$$

which is verified in Figure 7 by the normalized singular values of the regressor matrix A given by (18).

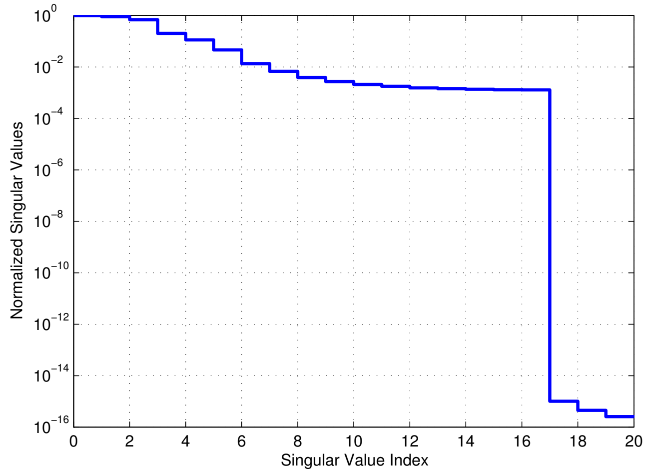

Fig. 7. Normalized singular values of the regressor matrix A given by (18) for the system (32), input (34), and order of the model $\hat{n} = 10$.

VII. ARX Estimation in the Presence of Noise

In this section, we consider a periodic Schroeder-phased signal and a periodic, zero-buffered, Schroeder-phased signal for identifying an ARX system in the presence of output measurement noise. Specifically, we consider zero-mean Gaussian white noise superimposed on the output with a specified signal-to-noise ratio (SNRs). The SNR is taken to be the RMS value of the true signal divided by the RMS value of the noise superimposed on that signal.

Consider the Schroeder-phased multi-sine signal

$$u_S(k) = \sum_{i=1}^{50} \cos\left(\left[\frac{2\pi i}{200}\right] k - \frac{\pi i^2}{50}\right), \tag{35}$$

where $k = 0, \ldots, 999$ and the periodic, zero-buffered signal

$$u_{zb,10}(k + 210l) = \begin{cases} 0, & k = 0, \ldots, 9, \\ u_S(k-10), & k \geq 10, \end{cases} \tag{36}$$

where $l \in [0, 4]$ and $k = 0, \ldots, 209$, although only $u_{zb,10}(0), \ldots, u_{zb,10}(999)$ is considered so that u_S and $u_{zb,10}$ have the same length.

Also, consider the system

$$G(z) = \frac{(z-0.75)(z-0.85)(z^2-1.6z+0.6425)}{(z-0.8)(z^2+0.01)(z^2+0.04)(z^2+0.9025)}, \tag{37}$$

and the performance metric

$$\varepsilon(k) \triangleq \frac{\|H - \hat{H}(k)\|}{\mu\|H\|}, \tag{38}$$

$$H = [H_0 \quad \cdots \quad H_{\mu-1}], \tag{39}$$

$$\hat{H}(k) = \left[\hat{H}_0(k) \quad \cdots \quad \hat{H}_{\mu-1}(k)\right], \tag{40}$$

where $H_0, \ldots, H_{\mu-1}$ are the true Markov parameters and $\hat{H}_0, \ldots, \hat{H}_{\mu-1}$ are the estimated Markov parameters, which are obtained by computing the impulse response of the estimated system. Choosing $\mu = 10$ and $\hat{n} = 10$, 1000 trials are run in which (37) is estimated with standard least-squares. Figure 8 shows that the Schroeder-phased signal is worse than the zero-buffered signal for identifying the ARX model (37) according to the performance metric (38). Note that although the regressor matrix A_u given by (19) for the Schroeder-phased signal has full row rank, it is poorly conditioned and hence poor estimates are obtained.

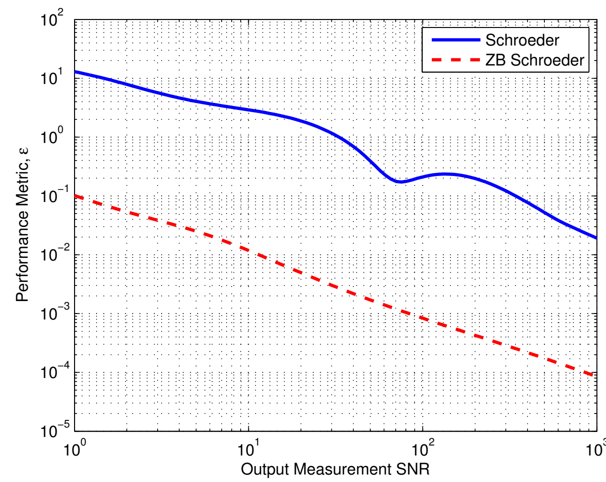

Fig. 8. Comparison of the performance metric (38) averaged over 1000 trials when the Schroeder-phased signal (35) and the zero-buffered Schroeder-phased signal (36) are used to estimate the ARX model (37) with standard least-squares.

VIII. Conclusion

We have considered a new notion of persistency within a deterministic, finite-data context. Furthermore, we have introduced the technique of zero buffering, where the input signal begins with a sequence of zeros. We showed that this technique increases the richness of the input, the condition number of the regressor matrix, and the accuracy of the least-squares estimates of an ARX system. Conditions for rank estimation were also presented and demonstrated.

IX. ACKNOWLEDGMENTS

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