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Brief paper

Time-domain analysis of sensor-to-sensor transmissibility operators ☆

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ABSTRACT

In some applications, multiple measurements are available, but the driving input that gives rise to those outputs may be unknown. This raises the question as to whether it is possible to model the response of a subset of sensors based on the response of the remaining sensors without knowledge of the driving input. To address this issue, we develop time-domain sensor-to-sensor models that account for nonzero initial conditions. The sensor-to-sensor model is in the form of a transmissibility operator that is a rational function of the differentiation operator. The development is carried out for both SISO and MIMO transmissibility operators. These time-domain sensor-to-sensor models can be used for diagnostics and output prediction.

1. Introduction

The traditional concept of input–output modeling distinguishes between inputs that evoke response and outputs that capture the response. In some applications, multiple measurements are available, but the driving inputs that give rise to those outputs may be unknown. This raises the question as to whether it is possible to model the response of a subset of sensors based on the response of the remaining sensors without knowledge of the driving input. Models of this type, which are called transmissibilities, are widely used in structural modeling and health monitoring (Chesné & Deraemaeker, 2013; Devriendt & Guillaume, 2008; Gajdatsy, Janssens, Desmet, & Van Der Auweraer, 2010; Hrovat, 1997; Johnson & Adams, 2002; Maia, Silva, & Ribeiro, 2001; Urgueira, Almeida, & Maia, 2011; Weijtjens, De Sitter, Devriendt, & Guillaume, 2014; Zhang, Pintelon, & Schoukens, 2013). In structural vibration analysis, a transmissibility is a relation between a pair of sensor measurements of the same type, for example, displacements, accelerations, or forces (Da Silva, 2007).

While the transmissibility literature is extensive, a common feature is that transmissibilities are modeled in the frequency domain. A transmissibility is not a transfer function in the usual sense, however, since neither sensor captures the input driving the system except in the special case that one of the sensors measures the driving input. Consequently, a transmissibility does not have a state space realization with physically meaningful states.

The goal of the present paper is to develop sensor-to-sensor models that account for nonzero initial conditions and thus are necessarily defined in the time domain. These models, which we call transmissibility operators, are rational functions of the differentiation operator. Accordingly, a transmissibility operator defines a differential equation involving the sensor signals. The internal state of the underlying input–output system loses its meaning within the context of a transmissibility operator. What is essential in defining the transmissibility operator, however, is that it must be independent of both the initial condition and inputs of the underlying system, which is assumed to be time-invariant.

Transmissibility operators are developed in the present paper within the context of continuous-time, linear, time-invariant systems. We show that a transmissibility operator that relates sensor signals can be defined independently of the initial condition and inputs. This operator is a rational function of the differential operator, and thus represents a differential equation. However, the transmissibility operator cannot be defined in terms of the Laplace variable "s", due to the nonzero initial condition. This observation is a key conceptual contribution of this paper.

A feature of the transmissibility operator is the presence of a common factor in its numerator and denominator. The main technical contribution of this paper is a proof that this factor can be canceled; without such a proof, such cancellation can potentially exclude solutions of the transmissibility differential equation and render it invalid. Since this proof is lengthy, several technical lemmas are sequestered in the appendices.

The contents of the paper are as follows. In Section 2 we derive a time-domain model for MIMO transmissibility operators.

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In Section 3 we discuss the cancellation of a common factor that appears in the numerator and denominator of the transmissibility operator. SISO and MIMO transmissibility operators are illustrated in Section 4. Finally, we present conclusions in Section 5.

The content of the present paper builds on the precursor paper Brzezinski, Kukreja, Ni, and Bernstein (2011). The present paper goes beyond this paper by providing a significantly more detailed and rigorous treatment of transmissibility operators, including complete proofs.

2. Time-domain transmissibility operator

Consider the MIMO linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$y(t) = Cx(t) + Du(t), \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and $p > m$. No assumptions are made about the controllability of (A, B) or the observability of (A, C). Let

$$C = \begin{bmatrix} C_{\rm i} \\ C_{\rm o} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{\rm i} \\ D_{\rm o} \end{bmatrix}, \tag{4}$$

where $C_{\rm i} \in \mathbb{R}^{m\times n}$, $C_{\rm o} \in \mathbb{R}^{(p-m)\times n}$, $D_{\rm i} \in \mathbb{R}^{m\times m}$, and $D_{\rm o} \in \mathbb{R}^{(p-m)\times m}$. Then,

$$y_{\rm i}(t) \triangleq C_{\rm i}x(t) + D_{\rm i}u(t) \in \mathbb{R}^m, \tag{5}$$

$$y_{\rm o}(t) \triangleq C_{\rm o}x(t) + D_{\rm o}u(t) \in \mathbb{R}^{p-m}, \tag{6}$$

$$y(t) \triangleq \begin{bmatrix} y_{\rm i}(t) \\ y_{\rm o}(t) \end{bmatrix} \in \mathbb{R}^p. \tag{7}$$

The goal is to obtain a transmissibility function relating $y_{\rm i}$ and $y_{\rm o}$ that is independent of both the initial condition x_0 and the input u. As a first attempt at obtaining such a function, assuming $m = 1$ and $p = 2$ and letting $b \in \mathbb{R}^n$, $c_{\rm i}, c_{\rm o} \in \mathbb{R}^{1\times n}$, and $d_{\rm i}, d_{\rm o} \in \mathbb{R}$, we consider the system

$$\dot{x}(t) = Ax(t) + bu(t), \tag{8}$$

$$y_{\rm i}(t) = c_{\rm i}x(t) + d_{\rm i}u(t), \tag{9}$$

$$y_{\rm o}(t) = c_{\rm o}x(t) + d_{\rm o}u(t). \tag{10}$$

Transforming (9) and (10) to the Laplace domain yields

$$\hat{y}_{\rm i}(s) = c_{\rm i}(sI - A)^{-1}x_0 + [c_{\rm i}(sI - A)^{-1}b + d_{\rm i}]\hat{u}(s), \tag{11}$$

$$\hat{y}_{\rm o}(s) = c_{\rm o}(sI - A)^{-1}x_0 + [c_{\rm o}(sI - A)^{-1}b + d_{\rm o}]\hat{u}(s), \tag{12}$$

respectively, and thus

$$\frac{\hat{y}_{\rm o}(s)}{\hat{y}_{\rm i}(s)} = \frac{c_{\rm o}(sI - A)^{-1}x_0 + [c_{\rm o}(sI - A)^{-1}b + d_{\rm o}]\hat{u}(s)}{c_{\rm i}(sI - A)^{-1}x_0 + [c_{\rm i}(sI - A)^{-1}b + d_{\rm i}]\hat{u}(s)}. \tag{13}$$

Note that, if x_0 is zero, then $\hat{u}(s)$ can be canceled in (13), and $\hat{y}_{\rm o}(s)$ and $\hat{y}_{\rm i}(s)$ are related by a transmissibility that is independent of the input. However, if x_0 is not zero, then $\hat{u}(s)$ cannot be canceled in (13).

Alternatively, we consider a time-domain analysis using the differentiation operator $\mathbf{p} = {\rm d}/{\rm d}t$ instead of the Laplace variable s. Multiplying (5), (6) by $\det(\mathbf{p}I - A)$ and using the fact that

$$\det(\mathbf{p}I - A)I_n = \operatorname{adj}(\mathbf{p}I - A)(\mathbf{p}I - A) \tag{14}$$

yields the differential equation

$$\begin{aligned} &\det(\mathbf{p}I - A)y_{\rm i}(t) \\ &\quad = C_{\rm i}\det(\mathbf{p}I - A)I_n x(t) + D_{\rm i}\det(\mathbf{p}I - A)u(t) \\ &\quad = C_{\rm i}\operatorname{adj}(\mathbf{p}I - A)(\mathbf{p}I - A)x(t) + D_{\rm i}\det(\mathbf{p}I - A)u(t) \\ &\quad = C_{\rm i}\operatorname{adj}(\mathbf{p}I - A)(\dot{x}(t) - Ax(t)) + D_{\rm i}\det(\mathbf{p}I - A)u(t) \\ &\quad = [C_{\rm i}\operatorname{adj}(\mathbf{p}I - A)B + D_{\rm i}\det(\mathbf{p}I - A)]u(t). \end{aligned} \tag{15}$$

Similarly,

$$\det(\mathbf{p}I - A)y_{\rm o}(t) = [C_{\rm o}\operatorname{adj}(\mathbf{p}I - A)B + D_{\rm o}\det(\mathbf{p}I - A)]u(t). \tag{16}$$

For convenience, we define

$$G_{\rm i}(\mathbf{p}) \triangleq C_{\rm i}(\mathbf{p}I - A)^{-1}B + D_{\rm i} \in \mathbb{R}^{m\times m}(\mathbf{p}), \tag{17}$$

$$G_{\rm o}(\mathbf{p}) \triangleq C_{\rm o}(\mathbf{p}I - A)^{-1}B + D_{\rm o} \in \mathbb{R}^{(p-m)\times m}(\mathbf{p}), \tag{18}$$

and rewrite (15), (16) as

$$y_{\rm i}(t) = G_{\rm i}(\mathbf{p})u(t), \qquad y_{\rm o}(t) = G_{\rm o}(\mathbf{p})u(t), \tag{19}$$

respectively, which are interpreted as the differential equations (15), (16), respectively. Note that (19) includes both the free response due to x_0 and the forced response due to u. In the subsequent analysis, we omit the argument "t" where no ambiguity can arise.

Defining

$$\Gamma_{\rm i}(\mathbf{p}) \triangleq C_{\rm i}\operatorname{adj}(\mathbf{p}I - A)B + D_{\rm i}\delta(\mathbf{p}) \in \mathbb{R}^{m\times m}[\mathbf{p}], \tag{20}$$

$$\Gamma_{\rm o}(\mathbf{p}) \triangleq C_{\rm o}\operatorname{adj}(\mathbf{p}I - A)B + D_{\rm o}\delta(\mathbf{p}) \in \mathbb{R}^{(p-m)\times m}[\mathbf{p}], \tag{21}$$

$$\delta(\mathbf{p}) \triangleq \det(\mathbf{p}I - A), \tag{22}$$

we can rewrite (15), (16) as

$$\delta(\mathbf{p})y_{\rm i} = \Gamma_{\rm i}(\mathbf{p})u, \tag{23}$$

$$\delta(\mathbf{p})y_{\rm o} = \Gamma_{\rm o}(\mathbf{p})u, \tag{24}$$

respectively. Multiplying (23) by $\operatorname{adj}\Gamma_{\rm i}(\mathbf{p})$ from the left yields

$$\delta(\mathbf{p})\operatorname{adj}\Gamma_{\rm i}(\mathbf{p})y_{\rm i} = [\operatorname{adj}\Gamma_{\rm i}(\mathbf{p})]\,\Gamma_{\rm i}(\mathbf{p})u = \det\Gamma_{\rm i}(\mathbf{p})u. \tag{25}$$

Next, multiplying (24) by $\det\Gamma_{\rm i}(\mathbf{p})$ yields

$$[\det\Gamma_{\rm i}(\mathbf{p})]\,\delta(\mathbf{p})y_{\rm o} = [\det\Gamma_{\rm i}(\mathbf{p})]\,\Gamma_{\rm o}(\mathbf{p})u. \tag{26}$$

Substituting the left hand side of (25) in (26) yields

$$\delta(\mathbf{p})\det\Gamma_{\rm i}(\mathbf{p})y_{\rm o} = \delta(\mathbf{p})\Gamma_{\rm o}(\mathbf{p})\operatorname{adj}\Gamma_{\rm i}(\mathbf{p})y_{\rm i}. \tag{27}$$

In the case $m = 1$ and $p = 2$, (27) becomes

$$\delta(\mathbf{p})\Gamma_{\rm i}(\mathbf{p})y_{\rm o} = \delta(\mathbf{p})\Gamma_{\rm o}(\mathbf{p})y_{\rm i}. \tag{28}$$

Definition 2.1. Assume that $\Gamma_{\rm i}(\mathbf{p})$ is nonsingular. Then, the *transmissibility operator* from $y_{\rm i}$ to $y_{\rm o}$ is the operator

$$\mathcal{T}(\mathbf{p}) \triangleq \frac{\delta(\mathbf{p})}{\delta(\mathbf{p})\det\Gamma_{\rm i}(\mathbf{p})}\Gamma_{\rm o}(\mathbf{p})\operatorname{adj}\Gamma_{\rm i}(\mathbf{p}). \tag{29}$$

Note that (29) is independent of the input u and the initial condition x_0. Using (29), the differential equation (27) can be written as

$$y_{\rm o} = \mathcal{T}(\mathbf{p})y_{\rm i}. \tag{30}$$

Since $\Gamma_{\rm i}(\mathbf{p})$ is nonsingular, (29) can be written as

$$\mathcal{T}(\mathbf{p}) = \frac{\delta(\mathbf{p})}{\delta(\mathbf{p})}\Gamma_{\rm o}(\mathbf{p})\Gamma_{\rm i}^{-1}(\mathbf{p}). \tag{31}$$

Unlike common factors in the complex number s, common factors in the differentiation operator $\mathbf{p}$ cannot always be canceled. In particular, the following examples show that canceling common factors may exclude solutions of the original differential equation.

Example 2.1. Consider the signals $y_{\rm i}(t) = t + 1$ and $y_{\rm o}(t) = t + 5$. Operating on $y_{\rm i}(t)$ and $y_{\rm o}(t)$ with $\mathbf{p}$ yields $\mathbf{p}y_{\rm i}(t) = \dot{y}_{\rm i}(t) = 1 = \dot{y}_{\rm o}(t) = \mathbf{p}y_{\rm o}(t)$. Hence $\mathbf{p}y_{\rm i} = \mathbf{p}y_{\rm o}$. However, $y_{\rm i} \neq y_{\rm o}$. ■

Example 2.2. Consider the signals $y_{\rm i}(t) = 1$ and $y_{\rm o}(t) = 1 + e^{-t}$. Operating on $y_{\rm i}(t)$ and $y_{\rm o}(t)$ with $\mathbf{p} + 1$ yields $(\mathbf{p} + 1)y_{\rm i}(t) = \dot{y}_{\rm i}(t) + y_{\rm i}(t) = 1 = \dot{y}_{\rm o}(t) + y_{\rm o}(t) = (\mathbf{p} + 1)y_{\rm o}(t)$. Hence $(\mathbf{p} + 1)y_{\rm i} = (\mathbf{p} + 1)y_{\rm o}$. However, $y_{\rm i} \neq y_{\rm o}$. ■

Despite Examples 2.1 and 2.2, we show in Section 3 that the common factor $\delta(\mathbf{p})$ in (29) can be canceled without excluding any solutions of (25).

3. Cancellation of the common factor $\delta(\mathbf{p})$

We now show that (27) holds if and only if (27) holds with the factor $\delta(\mathbf{p})$ canceled. Since sufficiency is immediate, the goal of this section is to prove necessity. This result allows us to reduce the order of $\mathcal{T}(\mathbf{p})$ without excluding any solutions of (27).

Theorem 1. *$y_{\rm i}$ and $y_{\rm o}$ satisfy*

$$\det \Gamma_{\rm i}(\mathbf{p})y_{\rm o} = \Gamma_{\rm o}(\mathbf{p})\operatorname{adj}\Gamma_{\rm i}(\mathbf{p})y_{\rm i}. \tag{32}$$

Proof. See Appendix B. □

Theorem 1 implies that we can redefine $\mathcal{T}(\mathbf{p})$ in (30) as

$$\mathcal{T}(\mathbf{p}) \triangleq \Gamma_{\rm o}(\mathbf{p})\Gamma_{\rm i}^{-1}(\mathbf{p}). \tag{33}$$

Note that each entry of $\mathcal{T}(\mathbf{p})$ is a rational operator that is not necessarily proper and whose numerator and denominator are not necessarily coprime.

Consider the case $m = 1$ and $p = 2$. Then, using (33), the SISO transmissibility from $y_{\rm i}$ to $y_{\rm o}$ is

$$\mathcal{T}(\mathbf{p}) = \frac{\Gamma_{\rm o}(\mathbf{p})}{\Gamma_{\rm i}(\mathbf{p})} = \frac{C_{\rm o}\operatorname{adj}(\mathbf{p}I - A)B + D_{\rm o}\delta(\mathbf{p})}{C_{\rm i}\operatorname{adj}(\mathbf{p}I - A)B + D_{\rm i}\delta(\mathbf{p})}, \tag{34}$$

which can be interpreted as the differential equation

$$\Gamma_{\rm i}(\mathbf{p})y_{\rm o} = \Gamma_{\rm o}(\mathbf{p})y_{\rm i}. \tag{35}$$

4. Examples

Example 4.1. Consider the mass–spring system in Fig. 1, where f is the input force, q_1 and q_2 are the displacements of m_1 and m_2, respectively, and (1) holds with

$$x \triangleq \begin{bmatrix} q_1 & q_2 & \dot q_1 & \dot q_2\end{bmatrix}^{\rm T}, \qquad A \triangleq \begin{bmatrix} 0_{2\times 2} & I_2 \\ \Omega & 0_{2\times 2}\end{bmatrix}, \tag{36}$$

$$\Omega \triangleq \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2}\end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 0 & \dfrac{1}{m_1} & 0\end{bmatrix}^{\rm T}. \tag{37}$$

For the transmissibility from $y_{\rm i} = q_1$ to $y_{\rm o} = q_2$, we have

$$C_{\rm i} = \begin{bmatrix} 1 & 0 & 0 & 0\end{bmatrix}, \qquad C_{\rm o} = \begin{bmatrix} 0 & 1 & 0 & 0\end{bmatrix}. \tag{38}$$

Using (20)–(22) it follows that

$$\Gamma_{\rm i}(\mathbf{p}) = C_{\rm i}\operatorname{adj}(\mathbf{p}I_n - A)B = \frac{m_2\mathbf{p}^2 + k_2}{m_1 m_2}, \tag{39}$$

$$\Gamma_{\rm o}(\mathbf{p}) = C_{\rm o}\operatorname{adj}(\mathbf{p}I_n - A)B = \frac{k_2}{m_1 m_2}, \tag{40}$$

$$\delta(\mathbf{p}) = \mathbf{p}^4 + \frac{k_2 m_1 + (k_1+k_2)m_2}{m_1 m_2}\mathbf{p}^2 + \frac{k_1 k_2}{m_1 m_2}, \tag{41}$$

respectively. Therefore, we have

$$\delta(\mathbf{p})q_1 = \Gamma_{\rm i}(\mathbf{p})f, \tag{42}$$

$$\delta(\mathbf{p})q_2 = \Gamma_{\rm o}(\mathbf{p})f. \tag{43}$$

Multiplying (42) and (43) by $\Gamma_{\rm o}(\mathbf{p})$ and $\Gamma_{\rm i}(\mathbf{p})$, respectively, yields

$$\delta(\mathbf{p})\Gamma_{\rm o}(\mathbf{p})q_1 = \Gamma_{\rm i}(\mathbf{p})\Gamma_{\rm o}(\mathbf{p})f, \tag{44}$$

$$\delta(\mathbf{p})\Gamma_{\rm i}(\mathbf{p})q_2 = \Gamma_{\rm i}(\mathbf{p})\Gamma_{\rm o}(\mathbf{p})f. \tag{45}$$

Comparing (44) and (45) yields

$$\delta(\mathbf{p})\Gamma_{\rm o}(\mathbf{p})q_1 = \delta(\mathbf{p})\Gamma_{\rm i}(\mathbf{p})q_2, \tag{46}$$

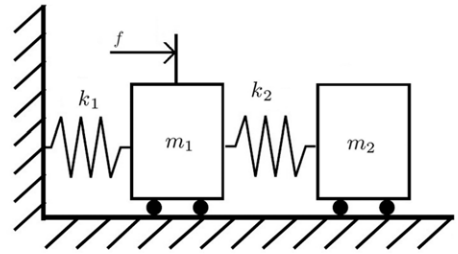

Fig. 1. Mass-spring system for Example 4.1, where f is the input force and the outputs $y_{\rm i}$ and $y_{\rm o}$ are the displacements q_1 and q_2 of m_1 and m_2, respectively.

in accordance with (28). Moreover, Theorem 1 and (35) imply that

$$\Gamma_{\rm o}(\mathbf{p})q_1 = \Gamma_{\rm i}(\mathbf{p})q_2. \tag{47}$$

Alternatively, note that the equation of motion for m_2 is given by

$$m_2\mathbf{p}^2 q_2 + k_2(q_2 - q_1) = 0. \tag{48}$$

Solving (48) for q_1 yields

$$q_1 = \frac{m_2\mathbf{p}^2 + k_2}{k_2}q_2. \tag{49}$$

Hence, (39), (40), and (49) imply

$$\begin{aligned}\Gamma_{\rm o}(\mathbf{p})y_{\rm i} &= \frac{k_2}{m_1 m_2}q_1 = \frac{k_2}{m_1 m_2}\frac{m_2\mathbf{p}^2 + k_2}{k_2}q_2 \\ &= \frac{m_2\mathbf{p}^2 + k_2}{m_1 m_2}q_2 = \Gamma_{\rm i}(\mathbf{p})y_{\rm o},\end{aligned} \tag{50}$$

which confirms (35) directly without using Theorem 1. Thus, $y_{\rm o} = \mathcal{T}(\mathbf{p})y_{\rm i}$ where

$$\mathcal{T}(\mathbf{p}) = \frac{\Gamma_{\rm o}(\mathbf{p})}{\Gamma_{\rm i}(\mathbf{p})} = \frac{k_2}{m_2\mathbf{p}^2 + k_2}. \quad ■$$

Example 4.2. Consider the MIMO system

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1\end{bmatrix}, \tag{51}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1\end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0\end{bmatrix}, \tag{52}$$

$y_{\rm i} = [x_1\ x_2]^{\rm T}$, and $y_{\rm o} = x_3$. Hence, $m = 2$, $p = 3$, and

$$C_{\rm i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}, \qquad C_{\rm o} = \begin{bmatrix} 0 & 0 & 1\end{bmatrix}, \tag{53}$$

$$D_{\rm i} = \begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, \qquad D_{\rm o} = \begin{bmatrix} 0 & 0\end{bmatrix}. \tag{54}$$

It follows from (22) that $\delta(\mathbf{p}) = \mathbf{p}^3 + 3\mathbf{p}^2 + 3\mathbf{p} + 1$. Using (20) we have

$$\begin{aligned}\Gamma_{\rm i}(\mathbf{p}) &= C_{\rm i}\operatorname{adj}(\mathbf{p}I - A)B + \delta(\mathbf{p})D_{\rm i} \\ &= \begin{bmatrix} (\mathbf{p}+1)^2(\mathbf{p}+2) + 1 & \mathbf{p}+2 \\ \mathbf{p}+1 & (\mathbf{p}+1)(\mathbf{p}+2)\end{bmatrix}.\end{aligned} \tag{55}$$

Moreover, (21) implies that

$$\begin{aligned}\Gamma_{\rm o}(\mathbf{p}) &= C_{\rm o}\operatorname{adj}(\mathbf{p}I - A)B + \delta(\mathbf{p})D_{\rm o} \\ &= \begin{bmatrix} (\mathbf{p}+1)^2 & (\mathbf{p}+1)^2\end{bmatrix}.\end{aligned} \tag{56}$$

Hence, using (33) we have

$$\mathcal{T}(\mathbf{p}) = \Gamma_{\rm o}(\mathbf{p})\Gamma_{\rm i}^{-1}(\mathbf{p}) = \frac{1}{(\mathbf{p}+1)^3(\mathbf{p}+2)^2}\left[(\mathbf{p}+1)^4 \quad (\mathbf{p}+1)^3(\mathbf{p}^2+3\mathbf{p}+1)\right]. \tag{57}$$

It follows from (30) that

$$(\mathbf{p}+1)^3(\mathbf{p}+2)^2x_3 = (\mathbf{p}+1)^4x_1 + (\mathbf{p}+1)^3(\mathbf{p}^2+3\mathbf{p}+1)x_2, \tag{58}$$

that is,

$$x_3^{(5)} + 7x_3^{(4)} + 19x_3^{(3)} + 25\ddot{x}_3 + 16\dot{x}_3 + 4x_3 = x_1^{(4)} + 4x_1^{(3)} + 6\ddot{x}_1 + 4\dot{x}_1 + x_1 + x_2^{(5)} + 6x_2^{(4)} + 13x_2^{(3)} + 13\ddot{x}_2 + 6\dot{x}_2 + x_2. \tag{59}$$

To confirm (32), substituting x, A, and B from (51) and (52) and $u = [u_1\ u_2]^{\rm T}$ into (1) yields

$$\mathbf{p}x_1 = -x_1 + x_2 + u_1, \tag{60}$$

$$\mathbf{p}x_2 = -x_2 + x_3 + u_2, \tag{61}$$

$$\mathbf{p}x_3 = -x_3 + u_1 + u_2. \tag{62}$$

Using (60)–(62) note that

$$\begin{aligned}\det\Gamma_{\rm i}(\mathbf{p})y_{\rm o} &= (\mathbf{p}+1)^3(\mathbf{p}+2)^2x_3\\ &= (\mathbf{p}+1)^3\big((\mathbf{p}+2)x_3 + (\mathbf{p}+2)(\mathbf{p}+1)x_3\big)\\ &= (\mathbf{p}+1)^3\big((\mathbf{p}+2)x_3 + (\mathbf{p}+2)(u_1+u_2)\big)\\ &= (\mathbf{p}+1)^3\big((\mathbf{p}+2)(x_3+u_2) + (\mathbf{p}+2)u_1\big)\\ &= (\mathbf{p}+1)^3\big((\mathbf{p}+2)(\mathbf{p}+1)x_2 + (\mathbf{p}+2)u_1\big)\\ &= (\mathbf{p}+1)^3\big(x_2 + u_1 + (\mathbf{p}+1)u_1\\ &\quad + ((\mathbf{p}+2)(\mathbf{p}+1)-1)x_2\big)\\ &= (\mathbf{p}+1)^3\big((\mathbf{p}+1)(x_1+u_1)\\ &\quad + (\mathbf{p}^2+3\mathbf{p}+1)x_2\big)\\ &= \Gamma_{\rm o}(\mathbf{p})\operatorname{adj}\Gamma_{\rm i}(\mathbf{p})y_{\rm i}.\end{aligned} \tag{63}$$

Hence, $y_{\rm i}$ and $y_{\rm o}$ satisfy (32) in accordance with Theorem 1. Moreover, multiplying (63) by $\delta(\mathbf{p})$ shows that $y_{\rm i}$ and $y_{\rm o}$ satisfy (27). ■

Example 4.3. Consider the mass–spring system in Fig. 2, where f is the input force, q_1, q_2, q_3 are the displacements of m_1, m_2, m_3, respectively, and (1) holds with

$$x \triangleq \begin{bmatrix} q_1 & q_2 & q_3 & \dot{q}_1 & \dot{q}_2 & \dot{q}_3\end{bmatrix}^{\rm T}, \qquad A \triangleq \begin{bmatrix} 0_{3\times3} & I_3\\ \Omega & 0_{3\times3}\end{bmatrix}, \tag{64}$$

$$\Omega \triangleq \begin{bmatrix} -\dfrac{k_{01}+k_{12}+k_{13}}{m_1} & \dfrac{k_{12}}{m_1} & \dfrac{k_{13}}{m_1}\\ \dfrac{k_{12}}{m_2} & -\dfrac{k_{12}+k_{23}}{m_2} & \dfrac{k_{23}}{m_2}\\ \dfrac{k_{13}}{m_3} & \dfrac{k_{23}}{m_3} & -\dfrac{k_{13}+k_{23}}{m_3},\end{bmatrix}, \tag{65}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & \dfrac{1}{m_1} & 0 & 0\end{bmatrix}^{\rm T}. \tag{66}$$

For $i = 1, 2, 3$, define

$$y_i \triangleq C_i x, \tag{67}$$

where

$$C_1 \triangleq e_{1,6}^{\rm T}, \qquad C_2 \triangleq e_{2,6}^{\rm T}, \qquad C_3 \triangleq e_{3,6}^{\rm T}, \tag{68}$$

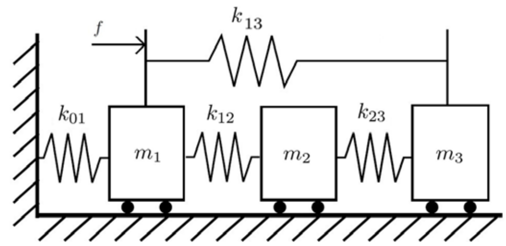

Fig. 2. Mass-spring system for Example 4.3, where f is the input force and the outputs y_1, y_2, and y_3 are the displacements q_1, q_2, and q_3 of m_1, m_2, and m_3, respectively.

and $e_{i,n} \in \mathbb{R}^n$ is the ith unit vector. Then,

$$y_1 = C_1x = q_1, \tag{69}$$

$$y_2 = C_2x = q_2, \tag{70}$$

$$y_3 = C_3x = q_3. \tag{71}$$

Define

$$\Gamma_1(\mathbf{p}) \triangleq C_1\operatorname{adj}(\mathbf{p}I_n - A)B = \frac{m_2m_3\mathbf{p}^4 + (m_3(k_{12}+k_{23}) + m_2(k_{13}+k_{23}))\,\mathbf{p}^2 + k}{m_1m_2m_3}, \tag{72}$$

$$\Gamma_2(\mathbf{p}) \triangleq C_2\operatorname{adj}(\mathbf{p}I_n - A)B = \frac{k_{12}m_3\mathbf{p}^2 + k}{m_1m_2m_3}, \tag{73}$$

$$\Gamma_3(\mathbf{p}) \triangleq C_3\operatorname{adj}(\mathbf{p}I_n - A)B = \frac{k_{13}m_2\mathbf{p}^2 + k}{m_1m_2m_3}, \tag{74}$$

where $k \triangleq k_{12}k_{13} + k_{12}k_{23} + k_{13}k_{23}$. Next, let $\mathcal{T}_{j,i}(\mathbf{p})$ be the transmissibility whose input is q_i and whose output is q_j, where $i, j \in \{1, 2, 3\}$. Therefore, using (34)

$$\mathcal{T}_{2,1}(\mathbf{p}) = \frac{\Gamma_2(\mathbf{p})}{\Gamma_1(\mathbf{p})} = \frac{k_{12}m_3\mathbf{p}^2 + k}{m_2m_3\mathbf{p}^4 + (m_3(k_{12}+k_{23}) + m_2(k_{13}+k_{23}))\,\mathbf{p}^2 + k}, \tag{75}$$

$$\mathcal{T}_{3,1}(\mathbf{p}) = \frac{\Gamma_3(\mathbf{p})}{\Gamma_1(\mathbf{p})} = \frac{k_{13}m_2\mathbf{p}^2 + k}{m_2m_3\mathbf{p}^4 + (m_3(k_{12}+k_{23}) + m_2(k_{13}+k_{23}))\,\mathbf{p}^2 + k}, \tag{76}$$

$$\mathcal{T}_{3,2}(\mathbf{p}) = \frac{\Gamma_3(\mathbf{p})}{\Gamma_2(\mathbf{p})} = \frac{k_{13}m_2\mathbf{p}^2 + k}{k_{12}m_3\mathbf{p}^2 + k} \tag{77}$$

are the transmissibilities from q_1 to q_2, q_1 to q_3, and q_2 to q_3, respectively. Note that

$$q_2 = \mathcal{T}_{2,1}(\mathbf{p})q_1, \tag{78}$$

$$q_3 = \mathcal{T}_{3,2}(\mathbf{p})q_2, \tag{79}$$

and thus

$$q_3 = \mathcal{T}_{3,2}(\mathbf{p})\mathcal{T}_{2,1}(\mathbf{p})q_1 = \mathcal{T}_{3,1}(\mathbf{p})q_1, \tag{80}$$

that is,

$$q_3 = \frac{\Gamma_3(\mathbf{p})}{\Gamma_2(\mathbf{p})}\frac{\Gamma_2(\mathbf{p})}{\Gamma_1(\mathbf{p})}q_1 = \frac{\Gamma_3(\mathbf{p})}{\Gamma_1(\mathbf{p})}q_1, \tag{81}$$

which shows that $\Gamma_2(\mathbf{p})$ can be canceled. ■

5. Conclusions and future research

This paper developed a time-domain framework for MIMO transmissibilities that accounts for nonzero initial conditions as well as cancellation of the common factor occurring in the underlying state space model. A natural extension of these models is to the discrete-time case to facilitate system identification (Brzezinski et al., 2011). Finally, connections between transmissibilities and behavioral models (Willems, 2007) is of potential interest.

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Appendix A. Lemmas A.1–A.5

Lemmas A.1–A.5 concern SISO transmissibility operators. Lemma A.1 is used to prove Lemma A.2, which in turn is used to prove Lemmas A.3 and A.4. Lemmas A.3 and A.4 are used to prove Lemma A.5, which in turn is used to prove Theorem 1 in Appendix B.

Assume that $m = 1$ and $p = 2$ and let (20)–(22) be written as

$$\Gamma_{\mathrm{i}}(\mathbf{p}) = \sum_{j=0}^{n} \beta_{\mathrm{i},j}\mathbf{p}^j, \qquad \Gamma_{\mathrm{o}}(\mathbf{p}) = \sum_{j=0}^{n} \beta_{\mathrm{o},j}\mathbf{p}^j,$$

$$\delta(\mathbf{p}) = \mathbf{p}^n + \sum_{j=0}^{n-1} \alpha_j\mathbf{p}^j,$$

respectively, where $\beta_{\mathrm{i},n} = D_{\mathrm{i}}$ and $\beta_{\mathrm{o},n} = D_{\mathrm{o}}$.

Define

$$\alpha \triangleq \begin{bmatrix}\alpha_0 & \alpha_1 & \cdots & \alpha_{n-1}\end{bmatrix}^{\mathrm{T}},$$

$$A_{\mathrm{c}} \triangleq \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \multicolumn{2}{c}{-\alpha^{\mathrm{T}}} \end{bmatrix}, \qquad B_{\mathrm{c}} \triangleq e_n^{\mathrm{T}},$$

$$C_{\mathrm{c,i}} \triangleq \begin{bmatrix}\beta_{\mathrm{i},0} & \beta_{\mathrm{i},1} & \cdots & \beta_{\mathrm{i},n-1}\end{bmatrix} - \beta_{\mathrm{i},n}\alpha^{\mathrm{T}},$$

$$C_{\mathrm{c,o}} \triangleq \begin{bmatrix}\beta_{\mathrm{o},0} & \beta_{\mathrm{o},1} & \cdots & \beta_{\mathrm{o},n-1}\end{bmatrix} - \beta_{\mathrm{o},n}\alpha^{\mathrm{T}},$$

where e_i is the ith column of I_n. Consider the state space representation

$$\dot{x}_{\mathrm{c}} = A_{\mathrm{c}}x_{\mathrm{c}} + B_{\mathrm{c}}u, \tag{A.1}$$

$$y_{\mathrm{i}} = C_{\mathrm{c,i}}x_{\mathrm{c}} + D_{\mathrm{i}}u, \tag{A.2}$$

$$y_{\mathrm{o}} = C_{\mathrm{c,o}}x_{\mathrm{c}} + D_{\mathrm{o}}u. \tag{A.3}$$

Note that

$$\Gamma_{\mathrm{i}}(\mathbf{p}) = C_{\mathrm{c,i}}\mathrm{adj}(\mathbf{p}I - A_{\mathrm{c}})B_c + D_{\mathrm{i}}\delta(\mathbf{p}), \tag{A.4}$$

$$\Gamma_{\mathrm{o}}(\mathbf{p}) = C_{\mathrm{c,o}}\mathrm{adj}(\mathbf{p}I - A_{\mathrm{c}})B_c + D_{\mathrm{o}}\delta(\mathbf{p}), \tag{A.5}$$

$$\delta(\mathbf{p}) = \det(\mathbf{p}I - A_{\mathrm{c}}). \tag{A.6}$$

That is, (23) and (24) can be represented by (A.1), (A.2) and (A.1), (A.3), respectively.

For all $j = 0, \ldots, n$, define

$$\chi_j \triangleq \begin{cases} e_{j+1}^{\mathrm{T}}, & 0 \le j \le n-1, \\ -\alpha^{\mathrm{T}}, & j = n. \end{cases}$$

For all $i, j = 0, \ldots, n$, define $f_{i,j} \triangleq \chi_i A_{\mathrm{c}}^j$.

Lemma A.1. *For all $i, j = 0, \ldots, n$, $f_{i,j} = f_{j,i}$.*

Proof. Note that

$$A_{\mathrm{c}}^j = \begin{cases} I_n, & j = 0, \\ E_j, & 1 \le j \le n-1, \\ \Delta_n, & j = n, \end{cases}$$

where,

$$E_j \triangleq \begin{bmatrix} e_{j+1}^{\mathrm{T}} \\ \vdots \\ e_n^{\mathrm{T}} \\ \Delta_j \end{bmatrix} \in \mathbb{R}^{n\times n}, \qquad \Delta_j \triangleq \begin{bmatrix} -\alpha^{\mathrm{T}} \\ \vdots \\ -\alpha^{\mathrm{T}}A_{\mathrm{c}}^{j-1} \end{bmatrix} \in \mathbb{R}^{j\times n}.$$

For all $i = j$, the result holds. For all $0 \le i \le n-1$ and $j = n$, $f_{i,n} = e_{i+1}^{\mathrm{T}}A_{\mathrm{c}}^n = -\alpha^{\mathrm{T}}A_{\mathrm{c}}^i = f_{n,i}$. For all $0 \le i \le n-j-1$ and $0 \le j \le n-1$, $f_{i,j} = e_{i+1}^{\mathrm{T}}E_j = e_{i+j+1}^{\mathrm{T}} = e_{j+1}^{\mathrm{T}}E_i = f_{j,i}$. Finally, for all $n-j \le i \le n-1$ and $0 \le j \le n-1$, $f_{i,j} = e_{i+1}^{\mathrm{T}}E_j = -\alpha^{\mathrm{T}}A_{\mathrm{c}}^{i+j-n} = e_{j+1}^{\mathrm{T}}E_i = f_{j,i}$. $\square$

Define

$$\gamma_{\mathrm{i}}(\mathbf{p}) \triangleq C_{\mathrm{c,i}}\mathrm{adj}(\mathbf{p}I - A_{\mathrm{c}})B_{\mathrm{c}}, \tag{A.7}$$

$$\gamma_{\mathrm{o}}(\mathbf{p}) \triangleq C_{\mathrm{c,o}}\mathrm{adj}(\mathbf{p}I - A_{\mathrm{c}})B_{\mathrm{c}}. \tag{A.8}$$

Then, (20), (21) can be written as

$$\Gamma_{\mathrm{i}}(\mathbf{p}) = \gamma_{\mathrm{i}}(\mathbf{p}) + D_{\mathrm{i}}\delta(\mathbf{p}), \tag{A.9}$$

$$\Gamma_{\mathrm{o}}(\mathbf{p}) = \gamma_{\mathrm{o}}(\mathbf{p}) + D_{\mathrm{o}}\delta(\mathbf{p}). \tag{A.10}$$

Lemma A.2. *For all $t \ge 0$,*

$$\Gamma_{\mathrm{o}}(\mathbf{p})C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} = \Gamma_{\mathrm{i}}(\mathbf{p})C_{\mathrm{c,o}}e^{A_{\mathrm{c}}t}, \tag{A.11}$$

$$\gamma_{\mathrm{o}}(\mathbf{p})C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} = \gamma_{\mathrm{i}}(\mathbf{p})C_{\mathrm{c,o}}e^{A_{\mathrm{c}}t}. \tag{A.12}$$

Proof. Using Lemma A.1 we have

$$\begin{aligned}
\Gamma_{\mathrm{o}}(\mathbf{p})C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} &= \sum_{i=0}^{n} \beta_{\mathrm{o},i}\mathbf{p}^i C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} \\
&= \sum_{i=0}^{n} \beta_{\mathrm{o},i}C_{\mathrm{c,i}}A_{\mathrm{c}}^i e^{A_{\mathrm{c}}t} \\
&= \sum_{i=0}^{n} \beta_{\mathrm{o},i}\left[\sum_{j=0}^{n-1}\left(\beta_{\mathrm{i},j}e_{j+1}^{\mathrm{T}}\right) - \beta_{\mathrm{i},n}\alpha^{\mathrm{T}}\right]A_{\mathrm{c}}^i e^{A_{\mathrm{c}}t} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n} \beta_{\mathrm{o},i}\beta_{\mathrm{i},j}f_{j,i}e^{A_{\mathrm{c}}t} \\
&= \sum_{j=0}^{n}\sum_{i=0}^{n} \beta_{\mathrm{i},j}\beta_{\mathrm{o},i}f_{i,j}e^{A_{\mathrm{c}}t} \\
&= \sum_{j=0}^{n} \beta_{\mathrm{i},j}\left[\sum_{i=0}^{n-1}\left(\beta_{\mathrm{o},i}e_{i+1}^{\mathrm{T}}\right) - \beta_{\mathrm{o},n}\alpha^{\mathrm{T}}\right]A_{\mathrm{c}}^j e^{A_{\mathrm{c}}t} \\
&= \sum_{j=0}^{n} \beta_{\mathrm{i},j}\mathbf{p}^j C_{\mathrm{c,o}}e^{A_{\mathrm{c}}t} \\
&= \Gamma_{\mathrm{i}}(\mathbf{p})C_{\mathrm{c,o}}e^{A_{\mathrm{c}}t},
\end{aligned}$$

which proves (A.11). To prove (A.12) note that

$$\begin{aligned}
\Gamma_{\mathrm{o}}(\mathbf{p})C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} &= (\gamma_{\mathrm{o}}(\mathbf{p}) + D_{\mathrm{o}}\delta(\mathbf{p}))C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} \\
&= \gamma_{\mathrm{o}}(\mathbf{p})C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t} + D_{\mathrm{o}}C_{\mathrm{c,i}}\delta(A_{\mathrm{c}})e^{A_{\mathrm{c}}t} \\
&= \gamma_{\mathrm{o}}(\mathbf{p})C_{\mathrm{c,i}}e^{A_{\mathrm{c}}t},
\end{aligned} \tag{A.13}$$

where δ is the characteristic polynomial of A_{c}, and thus $\delta(A_{\mathrm{c}}) = 0_{n\times n}$. Similarly,

$$\Gamma_{\mathrm{i}}(\mathbf{p})C_{\mathrm{c,o}}e^{A_{\mathrm{c}}t} = \gamma_{\mathrm{i}}(\mathbf{p})C_{\mathrm{c,o}}e^{A_{\mathrm{c}}t}. \tag{A.14}$$

Using (A.11), (A.13) and (A.14) yields (A.12). $\square$

Define

$$y_{\mathrm{i,free}}(t) \triangleq C_{\mathrm{c,i}} e^{A_{\mathrm{c}} t} x_{\mathrm{c}_0}, \qquad y_{\mathrm{o,free}}(t) \triangleq C_{\mathrm{c,o}} e^{A_{\mathrm{c}} t} x_{\mathrm{c}_0}. \tag{A.15}$$

Lemma A.3. *For all* $t \geq 0$,

$$\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,free}}(t) = \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o,free}}(t). \tag{A.16}$$

Proof. Using (A.11) of Lemma A.2 we have

$$\begin{aligned}\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,free}}(t) &= \Gamma_{\mathrm{o}}(\mathbf{p}) C_{\mathrm{c,i}} e^{A_{\mathrm{c}} t} x_{\mathrm{c}_0} \\ &= \Gamma_{\mathrm{i}}(\mathbf{p}) C_{\mathrm{c,o}} e^{A_{\mathrm{c}} t} x_{\mathrm{c}_0} = \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o,free}}(t). \quad \square\end{aligned}$$

Define

$$y_{\mathrm{i,forced}}(t) \triangleq \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) \mathrm{d}\tau + D_{\mathrm{i}} u(t), \tag{A.17}$$

$$y_{\mathrm{o,forced}}(t) \triangleq \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) \mathrm{d}\tau + D_{\mathrm{o}} u(t). \tag{A.18}$$

Lemma A.4. *For all* $t \geq 0$,

$$\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,forced}}(t) = \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o,forced}}(t). \tag{A.19}$$

Proof.

$$\begin{aligned}&\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,forced}}(t) \\ &= \Gamma_{\mathrm{o}}(\mathbf{p}) \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau + D_{\mathrm{i}} \Gamma_{\mathrm{o}}(\mathbf{p}) u(t) \\ &= \Gamma_{\mathrm{o}}(\mathbf{p}) \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau + D_{\mathrm{i}} \delta(\mathbf{p}) y_{\mathrm{o,forced}}(t) \\ &= \gamma_{\mathrm{o}}(\mathbf{p}) \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau \\ &\quad + D_{\mathrm{o}} \delta(\mathbf{p}) \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B u(\tau) d\tau \\ &\quad + D_{\mathrm{i}} \delta(\mathbf{p}) \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau + D_{\mathrm{i}} D_{\mathrm{o}} \delta(\mathbf{p}) u(t).\end{aligned} \tag{A.20}$$

Using (A.12) of Lemma A.2 we have

$$\begin{aligned}&\gamma_{\mathrm{o}}(\mathbf{p}) \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau \\ &= \gamma_{\mathrm{o}}(\mathbf{p}) C_{\mathrm{c,i}} e^{A_{\mathrm{c}} t} \int_0^t e^{-A_{\mathrm{c}} \tau} B_{\mathrm{c}} u(\tau) d\tau \\ &= \gamma_{\mathrm{i}}(\mathbf{p}) C_{\mathrm{c,o}} e^{A_{\mathrm{c}} t} \int_0^t e^{-A_{\mathrm{c}} \tau} B_{\mathrm{c}} u(\tau) d\tau \\ &= \gamma_{\mathrm{i}}(\mathbf{p}) \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau.\end{aligned} \tag{A.21}$$

Using (A.17), (A.18), and (A.21), (A.20) can be written as

$$\begin{aligned}&\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,forced}}(t) = \gamma_{\mathrm{i}}(\mathbf{p}) \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau \\ &\quad + D_{\mathrm{o}} \delta(\mathbf{p}) \int_0^t C_{\mathrm{c,i}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau \\ &\quad + D_{\mathrm{i}} \delta(\mathbf{p}) \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau + D_{\mathrm{i}} D_{\mathrm{o}} \delta(\mathbf{p}) u(t) \\ &= \Gamma_{\mathrm{i}}(\mathbf{p}) \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau + D_{\mathrm{o}} \delta(\mathbf{p}) y_{\mathrm{i,forced}}(t) \\ &= \Gamma_{\mathrm{i}}(\mathbf{p}) \int_0^t C_{\mathrm{c,o}} e^{A_{\mathrm{c}}(t-\tau)} B_{\mathrm{c}} u(\tau) d\tau + D_{\mathrm{o}} \Gamma_{\mathrm{i}}(\mathbf{p}) u(t) \\ &= \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o,forced}}(t). \quad \square\end{aligned}$$

Lemma A.5. *For all* $t \geq 0$,

$$\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i}}(t) = \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o}}(t). \tag{A.22}$$

Proof. Using Lemmas A.3 and A.4

$$\begin{aligned}\Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i}}(t) &= \Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,free}}(t) + \Gamma_{\mathrm{o}}(\mathbf{p}) y_{\mathrm{i,forced}}(t) \\ &= \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o,free}}(t) + \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o,forced}}(t) \\ &= \Gamma_{\mathrm{i}}(\mathbf{p}) y_{\mathrm{o}}(t). \quad \square\end{aligned}$$

Appendix B. Proof of Theorem 1

Let

$$B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}, \qquad C_{\mathrm{i}} = \begin{bmatrix} c_{\mathrm{i},1} \\ \vdots \\ c_{\mathrm{i},m} \end{bmatrix}, \qquad C_{\mathrm{o}} = \begin{bmatrix} c_{\mathrm{o},1} \\ \vdots \\ c_{\mathrm{o},p-m} \end{bmatrix},$$

where, for all $i \in \{1, \ldots, m\}$, $b_i \in \mathbb{R}^n$ and $c_{\mathrm{i},i} \in \mathbb{R}^{1\times n}$, and, for all $j \in \{1, \ldots, p-m\}$, $c_{\mathrm{o},j} \in \mathbb{R}^{1\times n}$. Moreover, for all $i, j \in \{1, \ldots, m\}$, let

$$c_{\mathrm{i},i} \mathrm{adj}(\mathbf{p} I_n - A) b_j + D_{\mathrm{i},i,j} \delta(\mathbf{p}) = \sum_{k=0}^{n} \mu_{i,j,k} \mathbf{p}^k,$$

where $D_{\mathrm{i},i,j}$ is the (i, j) entry of D_{i}. Then, we can write

$$\begin{aligned}\Gamma_{\mathrm{i}}(\mathbf{p}) &= \begin{bmatrix} \sum_{i=0}^{n} \mu_{1,1,i} \mathbf{p}^i & \cdots & \sum_{i=0}^{n} \mu_{1,m,i} \mathbf{p}^i \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{n} \mu_{m,1,i} \mathbf{p}^i & \cdots & \sum_{i=0}^{n} \mu_{m,m,i} \mathbf{p}^i \end{bmatrix} \\ &= \begin{bmatrix} \mu_{1,1}(\mathbf{p}) & \cdots & \mu_{1,m}(\mathbf{p}) \\ \vdots & \ddots & \vdots \\ \mu_{m,1}(\mathbf{p}) & \cdots & \mu_{m,m}(\mathbf{p}) \end{bmatrix},\end{aligned} \tag{B.1}$$

where, for all $i, j \in \{1, \ldots, m\}$, $\mu_{i,j}(\mathbf{p}) \triangleq \sum_{k=0}^{n} \mu_{i,j,k} \mathbf{p}^k$. Then, it follows from (B.1) that

$$\mathrm{adj}\, \Gamma_{\mathrm{i}}(\mathbf{p}) = \begin{bmatrix} T_{1,1}(\mathbf{p}) & \cdots & T_{m,1}(\mathbf{p}) \\ \vdots & \cdots & \vdots \\ T_{1,m}(\mathbf{p}) & \cdots & T_{m,m}(\mathbf{p}) \end{bmatrix}, \tag{B.2}$$

where

$$T_{i,j}(\mathbf{p}) \triangleq (-1)^{i+j} \det \Gamma_{\mathrm{i}_{[i,j]}}(\mathbf{p}),$$

and $\Gamma_{\mathrm{i}_{[i,j]}}(\mathbf{p}) \in \mathbb{R}^{(m-1)\times(m-1)}[\mathbf{p}]$ denotes $\Gamma_{\mathrm{i}}(\mathbf{p})$ with the ith row and jth column removed.

For all $i \in \{1, \ldots, p-m\}$ and $j \in \{1, \ldots, m\}$, let

$$c_{\mathrm{o},i} \mathrm{adj}(\mathbf{p} I_n - A) b_j + D_{\mathrm{o},i,j} \delta(\mathbf{p}) = \sum_{k=0}^{n} \nu_{i,j,k} \mathbf{p}^k,$$

where $D_{\mathrm{o},i,j}$ is the (i, j) entry of D_{o}. Then, we can write

$$\begin{aligned}\Gamma_{\mathrm{o}}(\mathbf{p}) &= \begin{bmatrix} \sum_{i=0}^{n} \nu_{1,1,i} \mathbf{p}^i & \cdots & \sum_{i=0}^{n} \nu_{1,m,i} \mathbf{p}^i \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{n} \nu_{p-m,1,i} \mathbf{p}^i & \cdots & \sum_{i=0}^{n} \nu_{p-m,m,i} \mathbf{p}^i \end{bmatrix} \\ &= \begin{bmatrix} \nu_{1,1}(\mathbf{p}) & \cdots & \nu_{1,m}(\mathbf{p}) \\ \vdots & \cdots & \vdots \\ \nu_{p-m,1}(\mathbf{p}) & \cdots & \nu_{p-m,m}(\mathbf{p}) \end{bmatrix},\end{aligned} \tag{B.3}$$

where, for all $i \in \{1,\ldots,p-m\}$ and $j \in \{1,\ldots,m\}$, $\nu_{i,j}(\mathbf{p}) \triangleq \sum_{k=0}^{n} \nu_{i,j,k}\mathbf{p}^k$.

Let $u = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}^{\mathrm{T}}$. Define

$$y_{\mathrm{i}} \triangleq \begin{bmatrix} y_{\mathrm{i},1} & \cdots & y_{\mathrm{i},m} \end{bmatrix}^{\mathrm{T}}, \qquad y_{\mathrm{o}} \triangleq \begin{bmatrix} y_{\mathrm{o},1} & \cdots & y_{\mathrm{o},p-m} \end{bmatrix}^{\mathrm{T}}.$$

Multiplying (23) by adj $\Gamma_{\mathrm{i}}(\mathbf{p})$ yields

$$\delta(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{i}} = \det \Gamma_{\mathrm{i}}(\mathbf{p})u.$$

Therefore, for all $i \in \{1,\ldots,m\}$, we have

$$\delta(\mathbf{p}) \sum_{j=1}^{m} T_{j,i}(\mathbf{p})y_{\mathrm{i},j} = \det \Gamma_{\mathrm{i}}(\mathbf{p})u_i. \tag{B.4}$$

Using (B.3), for all $k \in \{1,\ldots,p-m\}$, (24) implies that

$$\delta(\mathbf{p})y_{\mathrm{o},k} = \sum_{i=1}^{m} \nu_{k,i}(\mathbf{p})u_i. \tag{B.5}$$

Note that, for all $k \in \{1,\ldots,p-m\}$ and all $t \geq 0$,

$$y_{\mathrm{o},k,\mathrm{forced}}(t) = \sum_{i=1}^{m} y_{\mathrm{o},k,i,\mathrm{forced}}(t), \tag{B.6}$$

where, for all $k \in \{1,\ldots,p-m\}$ and all $i \in \{1,\ldots,m\}$,

$$y_{\mathrm{o},k,i,\mathrm{forced}}(t) \triangleq \int_0^t c_{\mathrm{o},k}e^{A(t-\tau)}b_i u_i(\tau)\mathrm{d}\tau + D_{\mathrm{o},k,i}u_i(t).$$

Moreover, note that, for all $t \geq 0$,

$$y_{\mathrm{o},k,\mathrm{free}}(t) = c_{\mathrm{o},k}e^{At}x_0 = \sum_{i=1}^{m} y_{\mathrm{o},k,i,\mathrm{free}}(t), \tag{B.7}$$

where

$$y_{\mathrm{o},k,i,\mathrm{free}}(t) \triangleq \frac{1}{m}c_{\mathrm{o},k}e^{At}x_0. \tag{B.8}$$

For all $k \in \{1,\ldots,p-m\}$ and all $i \in \{1,\ldots,m\}$, define

$$y_{\mathrm{o},k,i} \triangleq y_{\mathrm{o},k,i,\mathrm{free}} + y_{\mathrm{o},k,i,\mathrm{forced}}.$$

Then, $y_{\mathrm{o},k,i}$ satisfies

$$\delta(\mathbf{p})y_{\mathrm{o},k,i} = \nu_{k,i}(\mathbf{p})u_i. \tag{B.9}$$

Since

$$y_{\mathrm{o},k} = y_{\mathrm{o},k,\mathrm{free}} + y_{\mathrm{o},k,\mathrm{forced}}, \tag{B.10}$$

it follows from (B.6), (B.7), and (B.10) that

$$y_{\mathrm{o},k} = \sum_{i=1}^{m} y_{\mathrm{o},k,i}. \tag{B.11}$$

Multiplying (B.4) by $\nu_{k,i}(\mathbf{p})$ and multiplying (B.9) by $\det \Gamma_{\mathrm{i}}(\mathbf{p})$ yields

$$\delta(\mathbf{p})\nu_{k,i}(\mathbf{p}) \sum_{j=1}^{m} T_{j,i}(\mathbf{p})y_{\mathrm{i},j} = \nu_{k,i}(\mathbf{p}) \det \Gamma_{\mathrm{i}}(\mathbf{p})u_i, \tag{B.12}$$

$$\delta(\mathbf{p}) \det \Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o},k,i} = \nu_{k,i}(\mathbf{p}) \det \Gamma_{\mathrm{i}}(\mathbf{p})u_i. \tag{B.13}$$

Comparing (B.12) and (B.13) yields

$$\delta(\mathbf{p})\nu_{k,i}(\mathbf{p}) \sum_{j=1}^{m} T_{j,i}(\mathbf{p})y_{\mathrm{i},j} = \delta(\mathbf{p}) \det \Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o},k,i}, \tag{B.14}$$

which represents a SISO relationship between $y_{\mathrm{o},k,i}$ and $\sum_{j=1}^{m} T_{j,i}(\mathbf{p})y_{\mathrm{i},j}$ due to the input u_i with the free response given by (B.8). Therefore, Lemma A.5 implies that

$$\nu_{k,i}(\mathbf{p}) \sum_{j=1}^{m} T_{j,i}(\mathbf{p})y_{\mathrm{i},j} = \det \Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o},k,i}, \tag{B.15}$$

which indicates that $\delta(\mathbf{p})$ can be canceled from (B.14) without excluding any solutions.

Using (B.2) and (B.3) we have

$$\Gamma_{\mathrm{o}}(\mathbf{p})\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p}) = \begin{bmatrix} \sum_{i=1}^{m} \nu_{1,i}(\mathbf{p})\,T_{1,i}(\mathbf{p}) & \cdots & \sum_{i=1}^{m} \nu_{1,i}(\mathbf{p})\,T_{m,i}(\mathbf{p}) \\ \vdots & \cdots & \vdots \\ \sum_{i=1}^{m} \nu_{p-m,i}(\mathbf{p})\,T_{1,i}(\mathbf{p}) & \cdots & \sum_{i=1}^{m} \nu_{p-m,i}(\mathbf{p})\,T_{m,i}(\mathbf{p}) \end{bmatrix}. \tag{B.16}$$

Using (B.11), (B.15), and (B.16) yields

$$\begin{aligned} \Gamma_{\mathrm{o}}(\mathbf{p})\,[\mathrm{adj}\,\Gamma_{\mathrm{i}}(\mathbf{p})]\,y_{\mathrm{i}} &= \begin{bmatrix} \sum_{i=1}^{m}\sum_{j=1}^{m} \nu_{1,i}(\mathbf{p})T_{j,i}(\mathbf{p})y_{\mathrm{i},j} \\ \vdots \\ \sum_{i=1}^{m}\sum_{j=1}^{m} \nu_{p-m,i}(\mathbf{p})T_{j,i}(\mathbf{p})y_{\mathrm{i},j} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{m} \det \Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o},1,i} \\ \vdots \\ \sum_{i=1}^{m} \det \Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o},p-m,i} \end{bmatrix} \\ &= \det \Gamma_{\mathrm{i}}(\mathbf{p})y_{\mathrm{o}}. \quad \square \end{aligned}$$

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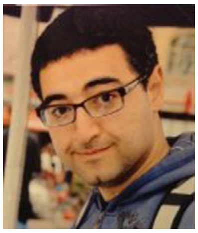

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