Identification of Sensor-Only MIMO Pseudo Transfer Functions*

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***Abstract*— In many applications of system identification, output measurements are available, but measurements of the input are not available. In these cases, sensor-only identification techniques are needed. In the present paper, we define a discrete-time MIMO pseudo transfer function (PTF) between sensor measurements in a sampled-data system with multiple excitation signals. We show that the MIMO PTF does not depend on the unknown initial state or the unknown excitation. We also characterize the order and relative degree of the MIMO PTF. To consistently estimate the Markov parameters of the MIMO PTF in the presence of sensor noise, we use quadratically constrained least squares identification for MIMO systems. We apply this technique to a three-degree-of-freedom mass-spring-damper system to assess the accuracy of the identified PTFs.**

I. INTRODUCTION

In some applications, the excitation may be unknown and thus sensor data may be the only available information for system identification. In this case, it is typically assumed that the excitation is generated by a white random process, and various system identification techniques are used to detect changes in the dynamics of the system [1–4]. Researchers have considered MIMO sensor-to-sensor relationships [5], but correct identification in the presence of multiple excitations is difficult [6]. Authors have noted that parameters identified using only sensor measurements may not correspond to the poles of the system [7], but the connection with system zeros has only recently been established [8].

Pseudo transfer functions (PTFs) are used in [8, 9] to detect system changes under unknown excitation. SISO PTFs for single-input, two-output systems are characterized in [9]. Sampling introduces additional zeros into a discrete-time input-output model if the relative degree of the continuous-time system is greater than 1 [10]. Hence, for a strictly proper continuous-time system, the order of the SISO PTF arising from a sampled-data application is $n-1$, where n is the order of the underlying system [8, 9]. Since a PTF is essentially a ratio of transfer functions, the information in a PTF consists of information about the zeros of the system from the unknown excitation to each of the sensors.

For applications involving structural dynamics such as operational modal analysis (OMA) [11], PTFs are a generalization of transmissibilities, where PTFs do not require that one of the sensors be colocated with the prescribed displacement [2]. To compare a transmissibility [12] and a PTF, consider the system G in Figure 1. The transmissibility from y_1 to y_2 in Figure 1(a) assumes y_1 is colocated with the displacement excitation u so that the transfer function from u to the displacement y_1 is

$$y_1 = u,$$

the transfer function from u to y_2 is

$$y_2 = \frac{N_2}{D}u,$$

and the transmissibility from y_1 to y_2 is

$$y_2 = \frac{N_2}{D}y_1. \tag{1}$$

However, the PTF from y_1 to y_2 in Figure 1(b) does not assume that y_1 and u are colocated, and hence

$$y_1 = \frac{N_1}{D}u.$$

The PTF from y_1 to y_2 is then [8]

$$y_2 = \frac{N_2}{N_1}y_1. \tag{2}$$

The transmissibility (1) contains pole information, while the PTF (2) does not. Pole information appears in the output-to-output relationship only if the excitation and sensor measurement are colocated, and hence a transmissibility is a special type of PTF.

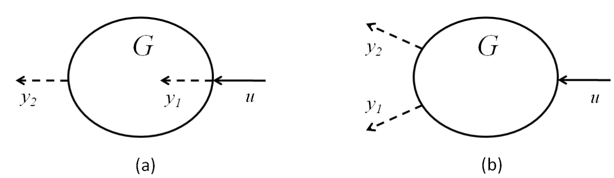

Fig. 1. Difference between the transmissibility from y_1 to y_2 (a) and the pseudo transfer function (PTF) from y_1 to y_2 (b).

Given multiple excitation signals, it is shown in [8] that additional sensors can be used to obtain a MIMO PTF that is independent of the excitation signals. For example, the system shown in Figure 2 is excited by both u_1 and u_2 so that the sensor measurements y_1, y_2, and y_3 contain contributions from both u_1 and u_2. For $j = 1, 2, 3$, as shown in Figure 2(a), $y_{j,2}$ appears as sensor noise corrupting measurement y_j if the SISO PTF from y_1 to y_2 is identified. Furthermore, identification of the SISO PTF from y_1 to y_2 involves estimation in the presence of a disturbance due to u_2. However, as shown in Figure 2(b), identification of the MIMO PTF from $[y_1\ y_2]^{\mathrm{T}}$ to y_3 is noise-free in the sense that, in the absence of additional noise sources, exact identification of the MIMO PTF is possible using finite data.

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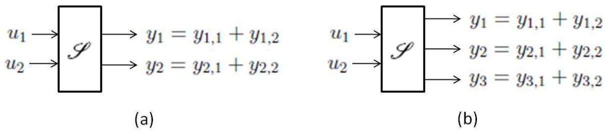

Fig. 2. Illustration of how unknown multiple excitation signals can cause sensor noise. For $i = 1, 2$, $j = 1, 2, 3$, excitation u_i yields measurement $y_{j,i}$. In diagram (a), identification of the SISO PTF from y_1 to y_2 involves estimation in the presence of a disturbance due to u_2. In diagram (b), identification of the MIMO PTF from $[y_1\ y_2]^{\mathrm{T}}$ to y_3 is noise-free.

The contribution of the present paper is to analyze MIMO PTFs in terms of the conditions on the sensors under which a MIMO PTF can be defined. In particular, we consider the normal rank of the PTF as well its order and relative degree. We also go beyond the results of [8] by considering the case in which the sensors are corrupted by noise that is not due to an excitation signal. Since both sensors may be corrupted by noise, we consider an errors-in-variables identification problem. To address this problem, we apply quadratically constrained least squares [13]. Since the results of [13] are confined to SISO systems with ARMAX model structures, we develop and apply an extension to MIMO systems with μ-Markov model structures. Unlike an ARMAX model structure, a μ-Markov model structure requires only a lower bound on the estimated model order.

II. PROBLEM FORMULATION

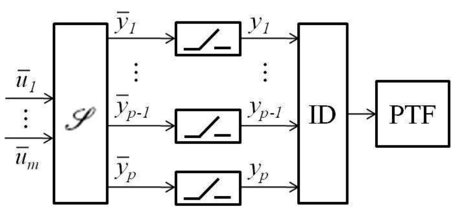

Fig. 3. Pseudo transfer function (PTF) identification problem.

Consider system $\mathscr{S}$ with unknown inputs $\bar{u}_1, \ldots, \bar{u}_m$ and measured outputs $\bar{y}_1, \ldots, \bar{y}_p$ in Figure 3. Define $\bar{u} \triangleq [\bar{u}_1 \cdots \bar{u}_m]^{\mathrm{T}}$ and $\bar{y} \triangleq [\bar{y}_1 \cdots \bar{y}_p]^{\mathrm{T}}$. For $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$, a state space representation of $\mathscr{S}$ is denoted by $(\bar{A}, \bar{B}, C, D)$, where

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t), \quad \bar{x}(0) = x_0, \tag{3}$$

$$\bar{y}(t) = C\bar{x}(t) + D\bar{u}(t). \tag{4}$$

For $k \in \{0, 1, \ldots\}$ and sample interval $h > 0$, we solve (3) for $\bar{x}(t)$ from $t = kh$ to $t = kh + h$, which yields

$$\bar{x}(kh+h)=e^{\bar{A}h}\bar{x}(kh)+\int_{kh}^{kh+h} e^{\bar{A}(kh+h-\tau)}\bar{B}\bar{u}(\tau)d\tau. \tag{5}$$

We assume that $\bar{u}(t)$ changes sufficiently slowly that $\bar{u}(t) \approx \bar{u}(kh)$ for all $t \in [kh, kh + h]$. Hence, we rewrite (5) as

$$x(k + 1) = Ax(k) + Bu(k), \tag{6}$$

where $x(k) \triangleq \bar{x}(kh)$, $u(k) \triangleq \bar{u}(kh)$, $A \triangleq e^{\bar{A}h}$, and

$$B \triangleq \int_0^h e^{\bar{A}\tau} d\tau \bar{B}.$$

We rewrite (6) in terms of the forward-shift operator $\mathbf{q}$ as

$$\delta(\mathbf{q})x = \operatorname{adj}(\mathbf{q}I - A)\, Bu, \tag{7}$$

where

$$\delta(\mathbf{q}) \triangleq \det(\mathbf{q}I - A)$$

and adj$(\cdot)$ denotes the *adjugate* operator. Defining $y(k) \triangleq \bar{y}(kh)$, it follows from (4) that

$$y(k) = Cx(k) + Du(k). \tag{8}$$

For the remainder of this article, we assume $p > m$. Hence, substituting (7) into (8) yields

$$\delta(\mathbf{q})y = N(\mathbf{q})u, \tag{9}$$

where, for $U(\mathbf{q}) \in \mathbb{R}^{m\times m}[\mathbf{q}]$, $L(\mathbf{q}) \in \mathbb{R}^{(p-m)\times m}[\mathbf{q}]$, $C_U \in \mathbb{R}^{m\times n}$, $C_L \in \mathbb{R}^{(p-m)\times n}$, $D_U \in \mathbb{R}^{m\times m}$, and $D_L \in \mathbb{R}^{(p-m)\times m}$,

$$\begin{aligned} N(\mathbf{q}) &\triangleq C\operatorname{adj}(\mathbf{q}I - A)\, B + \delta(\mathbf{q})D \\ &= \begin{bmatrix} U(\mathbf{q}) \\ L(\mathbf{q}) \end{bmatrix} \\ &= \begin{bmatrix} C_U\operatorname{adj}(\mathbf{q}I - A)\, B + \delta(\mathbf{q})D_U \\ C_L\operatorname{adj}(\mathbf{q}I - A)\, B + \delta(\mathbf{q})D_L \end{bmatrix} \in \mathbb{R}^{p\times m}[\mathbf{q}]. \end{aligned} \tag{10}$$

We can also write (10) as

$$\begin{aligned} y &= G(\mathbf{q})u \\ &\triangleq \left[C(\mathbf{q}I - A)^{-1} B + D\right] u \\ &= \begin{bmatrix} G_U(\mathbf{q}) \\ G_L(\mathbf{q}) \end{bmatrix} u \\ &= \begin{bmatrix} C_U(\mathbf{q}I - A)^{-1} B + D_U \\ C_L(\mathbf{q}I - A)^{-1} B + D_L \end{bmatrix} u. \end{aligned}$$

III. OUTPUT-ONLY MODEL

For $1 \le i \le j \le p$, we define

$$y_{[i:j]} \triangleq \begin{bmatrix} y_i & \cdots & y_j \end{bmatrix}^{\mathrm{T}},$$

so that

$$y = \begin{bmatrix} y_{[1:m]} \\ y_{[m+1:p]} \end{bmatrix}.$$

It follows from (9) that

$$\delta(\mathbf{q})y_{[1:m]} = U(\mathbf{q})u \tag{11}$$

and

$$\delta(\mathbf{q})y_{[m+1:p]} = L(\mathbf{q})u. \tag{12}$$

Multiplying (11) on the left by $\operatorname{adj}(U(\mathbf{q}))$ yields

$$\delta(\mathbf{q})\operatorname{adj}(U(\mathbf{q}))\, y_{[1:m]} = \det(U(\mathbf{q}))\, u. \tag{13}$$

Assuming $\det(U(\mathbf{q}))$ is not the zero polynomial and thus $U(\mathbf{q})$ is invertible, we substitute (13) into (12) to obtain

$$\delta(\mathbf{q})\det(U(\mathbf{q}))y_{[m+1:p]}=\delta(\mathbf{q})L(\mathbf{q})\operatorname{adj}(U(\mathbf{q}))y_{[1:m]}.$$

Hence,

$$y_{[m+1:p]} = \Gamma(\mathbf{q})y_{[1:m]},$$

where

$$\Gamma(\mathbf{q}) \triangleq \frac{\delta(\mathbf{q})}{\delta(\mathbf{q})\det(U(\mathbf{q}))} L(\mathbf{q})\mathrm{adj}(U(\mathbf{q})) \in \mathbb{R}^{(p-m)\times m}[\mathbf{q}] \tag{14}$$

is the MIMO PTF from $y_{[1:m]}$ to $y_{[m+1:p]}$. For the case $m = 1$, the common factor $\delta(\mathbf{q})$ can be cancelled [8].

Note that

$$U(\mathbf{q}) = PQ(\mathbf{q}), \tag{15}$$

where

$$P \triangleq \begin{bmatrix} C_U & D_U \end{bmatrix} \in \mathbb{R}^{m\times(n+m)}$$

and

$$Q(\mathbf{q}) \triangleq \begin{bmatrix} \mathrm{adj}\,(\mathbf{q}I - A)\,B \\ \delta(\mathbf{q})I_m \end{bmatrix} \in \mathbb{R}^{(n+m)\times m}.$$

The following result uses (15) to provide a necessary condition for $\det U(\mathbf{q}) \neq 0$.

Proposition 3.1: If $\det U(\mathbf{q}) \neq 0$, then $\mathrm{rank}\, P = m$.

Proof 3.1:

$$\begin{aligned} \mathrm{normal\,rank}\, U(\mathbf{q}) &= \mathrm{normal\,rank}\, PQ(\mathbf{q}) \\ &= \min\{\mathrm{rank}\, P, \mathrm{normal\,rank}\, Q(\mathbf{q})\} \\ &= \mathrm{rank}\, P = m. \qquad \square \end{aligned}$$

Note that

$$U(\mathbf{q}) = R(\mathbf{q})S, \tag{16}$$

where

$$R(\mathbf{q}) \triangleq \begin{bmatrix} C_U \mathrm{adj}\,(\mathbf{q}I - A) & \delta(\mathbf{q})I_m \end{bmatrix} \in \mathbb{R}^{m\times(n+m)}$$

and

$$S \triangleq \begin{bmatrix} B \\ D_U \end{bmatrix} \in \mathbb{R}^{(n+m)\times m}.$$

The following result uses (16) to provide a necessary condition for $\det U(\mathbf{q}) \neq 0$.

Proposition 3.2: If $\det U(\mathbf{q}) \neq 0$, then $\mathrm{rank}\, S = m$.

Proof 3.2:

$$\begin{aligned} \mathrm{normal\,rank}\, U(\mathbf{q}) &= \mathrm{normal\,rank}\, R(\mathbf{q})S \\ &= \min\{\mathrm{normal\,rank}\, R(\mathbf{q}), \mathrm{rank}\, S\} \\ &= \mathrm{rank}\, S = m. \qquad \square \end{aligned}$$

The following result provides necessary and sufficient conditions for $\det U(\mathbf{q}) \neq 0$.

Proposition 3.3: $\det U(\mathbf{q}) \neq 0$ if and only if $G_U(\mathbf{q})$ has full normal rank.

Proof 3.3:

$$\begin{aligned} \mathrm{normal\,rank}\, G_U(\mathbf{q}) &= \mathrm{normal\,rank}\, C_U\,(\mathbf{q}I - A)^{-1}\,B + D_U \\ &= \mathrm{normal\,rank}\, U(\mathbf{q}) = m. \qquad \square \end{aligned}$$

Propositions 3.1 and 3.2 provide necessary conditions for $\det U(\mathbf{q}) \neq 0$. However, the following example shows that these conditions are not sufficient for $G_U(\mathbf{q})$ to have full normal rank and thus for $\det U(q) \neq 0$.

Example 3.1: Let

$$A = -\frac{1}{4}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$C_U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D_U = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that A is asymptotically stable, B and C_U are full-rank, (A, B, C_U, D_U) is minimal, and $\mathrm{rank}\, P = \mathrm{rank}\, S = m$. However,

$$G_U(\mathbf{q}) = \frac{1}{\mathbf{q}^3 + \mathbf{q}^2 + \frac{9}{16}\mathbf{q} + \frac{5}{32}} \cdot \begin{bmatrix} -(\mathbf{q}+\frac{1}{4})(2\mathbf{q}+1) & -(\mathbf{q}+\frac{1}{2})(2\mathbf{q}+1) \\ \frac{1}{4}(\mathbf{q}+\frac{1}{4})(4\mathbf{q}-3) & \frac{1}{4}(\mathbf{q}+\frac{1}{2})(4\mathbf{q}-3) \end{bmatrix}. \tag{17}$$

IV. MIMO PTF ORDER AND RELATIVE DEGREE

For $i = 1, \ldots, p$, we write

$$y_i(k) = C_i x(k) + D_i u(k),$$

and thus

$$\delta(\mathbf{q})y_i = N_i(\mathbf{q})u,$$

where $N_i(\mathbf{q})$ is the i^{th} row of $N(\mathbf{q})$, C_i is the i^{th} row of C, and D_i is the i^{th} row of D. For all $i \in \{1, \ldots, p\}$ and all $j \in \{1, \ldots, m\}$,

$$\eta_{i,j}(\mathbf{q}) \triangleq (N(\mathbf{q}))_{(i,j)} = C_i \mathrm{adj}\,(\mathbf{q}I - A)\,B_j + D_{i,j}\delta(\mathbf{q}).$$

Proposition 4.1: Let $G_U(\mathbf{q})$ have full normal rank. Then

$$\deg\,(\det\,(U(\mathbf{q}))) \leq nm,$$

and, for all $i, j \in \{1, \ldots, m\}$,

$$\deg\left(\mathrm{adj}\,(U(\mathbf{q}))_{(i,j)}\right) \leq n(m-1).$$

Proof 4.1: For all $r \in \{1, \ldots, p\}$ the degree of $\eta_{r,j}(\mathbf{q})$ is n if $D_{r,j} \neq 0$ and $n-1$ otherwise [8]. Hence, computing $\det U(\mathbf{q})$ using the cofactor expansion yields

$$\begin{aligned} \deg\,(\det\,(U(\mathbf{q}))) &\leq m\left(\max_{i,j} \eta_{i,j}(\mathbf{q})\right) \\ &= mn. \end{aligned}$$

Furthermore,

$$\begin{aligned} \deg\left(\mathrm{adj}\,(U(\mathbf{q}))_{(i,j)}\right) &\leq (m-1)\left(\max_{i,j} \eta_{i,j}(\mathbf{q})\right) \\ &= (m-1)n. \qquad \square \end{aligned}$$

Proposition 4.2: Let $G(\mathbf{q})$ have full normal rank. Then, for all $i \in \{1, \ldots, p-m\}$ and all $j \in \{1, \ldots, m\}$,

$$\deg\,[L(\mathbf{q})\mathrm{adj}\,(U(\mathbf{q}))]_{(i,j)} \leq nm.$$

Proof 4.2: For all $r \in \{1, \ldots, p\}$ the degree of $\eta_{r,j}(\mathbf{q})$ is n if $D_{r,j} \neq 0$ and $n-1$ otherwise [8]. Hence, for all $k \in \{1, \ldots, m\}$,

$$\begin{aligned} \deg\,[L(\mathbf{q})\mathrm{adj}\,(U(\mathbf{q}))]_{(i,j)} &\leq \max_{i,j}\left(\deg\,[L(\mathbf{q})]_{(i,j)}\right) \\ &\quad + \max_{j,k}\left(\deg\,[\mathrm{adj}\,(U(\mathbf{q}))]_{(j,k)}\right) \\ &= n + n(m-1) \\ &= nm. \qquad \square \end{aligned}$$

The following result characterizes the order and relative degree of each of the entries of the MIMO PTF (14).

Theorem 4.1: Let $G(\mathbf{q})$ have full normal rank and assume that the common factor $\delta(\mathbf{q})$ in (14) can be cancelled. Then, for all $i \in \{1, \ldots, p-m\}$ and for all $j \in \{1, \ldots, m\}$, the order of $\Gamma(\mathbf{q})_{(i,j)}$ is less than or equal to nm. Furthermore, the relative degree $d_{(i,j)}$ of $\Gamma(\mathbf{q})_{(i,j)}$ is given by

$$d_{(i,j)} \triangleq \deg\left[\det\left(U(\mathbf{q})\right)\right] - \deg\left[L(\mathbf{q})\mathrm{adj}\left(U(\mathbf{q})\right)\right]_{(i,j)}.$$

Proof 4.3: The result follows from Prop. 4.1 and 4.2.

V. THREE-SENSOR, TWO-INPUT CASE

Let $p = 3$ and $m = 1$. From (54) of [8], the PTF from y_1 to y_3 is given by

$$y_3 = \frac{\eta_{3,1}(\mathbf{q})}{\eta_{1,1}(\mathbf{q})} y_1. \tag{18}$$

Next, let $m = 2$. Assuming $\delta(\mathbf{q})$ in (14) can be cancelled, the PTF from $Y_{[1:2]}$ to y_3 is given by

$$y_3 = \Gamma_{(1,1)}(\mathbf{q}) y_1 + \Gamma_{(1,2)} y_2, \tag{19}$$

where

$$\Gamma_{(1,1)}(\mathbf{q}) = \frac{\eta_{3,1}(\mathbf{q})\eta_{2,2}(\mathbf{q}) - \eta_{3,2}(\mathbf{q})\eta_{2,1}(\mathbf{q})}{\eta_{1,1}(\mathbf{q})\eta_{2,2}(\mathbf{q}) - \eta_{2,1}(\mathbf{q})\eta_{1,2}(\mathbf{q})}$$

and

$$\Gamma_{(1,2)}(\mathbf{q}) = \frac{\eta_{3,2}(\mathbf{q})\eta_{1,1}(\mathbf{q}) - \eta_{3,1}(\mathbf{q})\eta_{1,2}(\mathbf{q})}{\eta_{1,1}(\mathbf{q})\eta_{2,2}(\mathbf{q}) - \eta_{2,1}(\mathbf{q})\eta_{1,2}(\mathbf{q})}.$$

It follows from (19) that, if two excitation signals are present, the SISO PTF from y_1 to y_3 given by (18) is incorrect. We see from (19) that both y_1 and y_2 contribute to y_3.

VI. EXAMPLES

Consider the mass-spring-damper structure in Figure 4, which has the equations of motion

$$M\ddot{q}(t) + C_{\mathrm{d}}\dot{q}(t) + Kq(t) = F(t), \tag{20}$$

where

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix}, \quad M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},$$

$$C_{\mathrm{d}} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix},$$

$$F(t) = \underline{B}u(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

We express (20) in state-space form (3), where $\bar{x}(t) = \begin{bmatrix} q_1(t) & q_2(t) & q_3(t) & v_1(t) & v_2(t) & v_3(t) \end{bmatrix}^{\mathrm{T}}$, $\bar{y}(t) = \begin{bmatrix} q_1(t) & q_2(t) & v_2(t) \end{bmatrix}^{\mathrm{T}}$,

$$\bar{A} \triangleq \begin{bmatrix} 0_{3\times 3} & I_3 \\ -M^{-1}K & -M^{-1}C_{\mathrm{d}} \end{bmatrix}, \quad \bar{B} \triangleq \begin{bmatrix} 0_{3\times 2} \\ M^{-1}\underline{B} \end{bmatrix},$$

and where $m_1 = \frac{1}{2}m_2 = \frac{1}{3}m_3 = 1$ kg, $k_1 = \frac{4}{5}k_2 = \frac{2}{3}k_3 = \frac{4}{7}k_4 = 4$ N/m, $c_1 = \frac{1}{2}c_2 = \frac{1}{3}c_3 = \frac{1}{4}c_3 = 0.1$ kg-m/s, and $h = 0.5$ s.

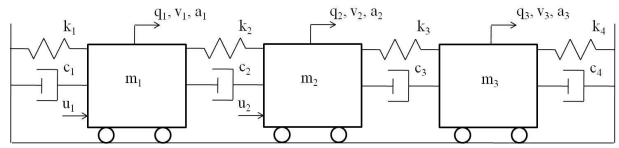

Fig. 4. 3 DOF mass-spring-damper structure.

We use Markov parameter matrices to characterize the PTF estimate. To do this, we use the μ-Markov model structure

$$\begin{aligned} A_0 y_2(k) = &- A_\mu y_2(k-\mu) - \cdots - A_{\mu+n_{\mathrm{mod}}-1} y_2(k-\mu-n_{\mathrm{mod}}+1) \\ &+ H_0 y_1(k) + \cdots + H_{\mu-1} y_1(k-\mu+1) \\ &+ B_\mu y_1(k-\mu) + \cdots + B_{\mu+n_{\mathrm{mod}}-1} y_1(k-\mu-n_{\mathrm{mod}}+1) \end{aligned} \tag{21}$$

of order n_{mod}. The absence of terms involving $y_2(k-1), \ldots, y_2(k-\mu+1)$ is responsible for the explicit presence of the Markov parameter matrices $H_0, \ldots, H_{\mu-1}$.

The μ-Markov model structure has two principal advantages over the traditional ARMAX structure. First, within the context of least squares identification with white input, it is shown in [14] that the μ-Markov model provides consistent estimates of the Markov parameters in the presence of arbitrary output noise. Furthermore, unlike parameter coefficients in an ARMAX model structure, the estimates of the Markov parameters are insensitive to the assumed model order n_{mod} as long as n_{mod} is larger than the true model order n. Consequently, only an upper bound on the true model order is needed.

For MIMO PTF identification, neither of the sensor signals is white, and thus we use quadratically constrained least squares (QCLS) with the μ-Markov model structure (21). MIMO QCLS with a μ-Markov model structure is developed in the Appendix. QCLS yields consistent parameter estimates in the presence of both input and output noise as long as the noise autocorrelation matrices are known to within a scalar multiple [13]. When this assumption is not satisfied, instrumental variables methods can be used [15].

To quantify the difference between the estimated and actual Markov parameter matrices, we define

$$\varepsilon_{\mathcal{T}} \triangleq \frac{\|\mathcal{T} - \hat{\mathcal{T}}\|_2}{\|\mathcal{T}\|_2},$$

where

$$\mathcal{T} \triangleq \begin{bmatrix} \mathrm{vec}\left(H_0\right) \\ \vdots \\ \mathrm{vec}\left(H_3\right) \end{bmatrix}$$

and $\hat{\mathcal{T}}$ is the estimate of $\mathcal{T}$ obtained from QCLS.

A. Effect of model order with noise-free measurements

We investigate the effect of the μ-Markov model order n_{mod} on the accuracy of the estimated Markov parameter matrices of the PTF. We simulate (20) with $x_0 \neq 0$ to obtain the position q_1 of the first mass, the position q_2 of the second mass, and the velocity v_2 of the second mass, where u_1 and

u_2 are realizations of white Gaussian processes with mean 0 and variance 1.

First we consider the SISO PTF from q_1 to v_2 and use LS with the μ-Markov model structure (21) of relative degree 0 to estimate the first 4 (scalar) Markov parameters of the PTF from q_1 to v_2. For 10 realizations of 1000 samples of each u, we construct $\varepsilon_{\mathcal{T}}$ and compute the average estimation error $\bar{\varepsilon}_{\mathcal{T}}$ over all u. Plotting $\bar{\varepsilon}_{\mathcal{T}}$ as a function of the μ-Markov model order $n_{\rm mod}$ in Figure 5 shows that the Markov parameters are not correctly estimated for all $n_{\rm mod}$ from 1 to 15. This is expected because q_1 and v_2 are corrupted by contributions from both u_1 and u_2.

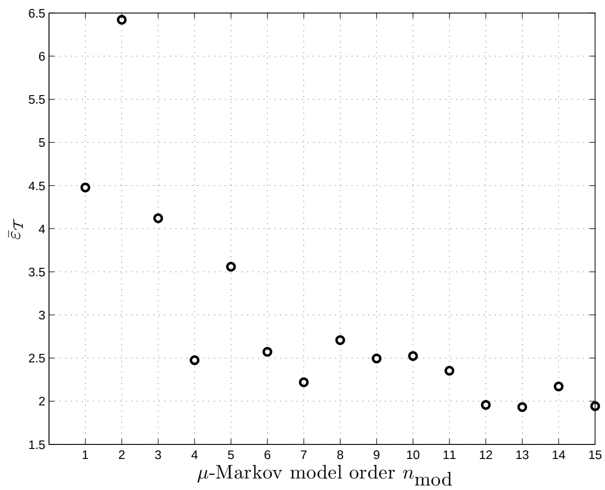

Fig. 5. Error in the Markov parameters of the SISO PTF from q_1 to v_2, estimated using SISO LS with the μ-Markov model structure (21), as the μ-Markov model order increases. The error in estimated Markov parameters is nonzero for all values of $n_{\rm mod}$ because the sensor measurements arise from two excitation signals.

Next we consider the two-input, one-output PTF from $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^{\rm T}$ to v_2 and use LS with the μ-Markov model structure (21) of relative degree 0 to estimate the first 4 (2×2) Markov parameters of the PTF from $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^{\rm T}$ to v_2. For 10 realizations of 1000 samples of each u, we construct $\varepsilon_{\mathcal{T}}$ and compute the average estimation error $\bar{\varepsilon}_{\mathcal{T}}$ over all u. Plotting $\bar{\varepsilon}_{\mathcal{T}}$ as a function of the μ-Markov model order $n_{\rm mod}$ in Figure 6 shows that the Markov parameters are correctly estimated for $n_{\rm mod} \geq 4$.

B. Consistency of the estimated MIMO PTF

We now investigate the effect of sensor noise on the accuracy of the identified PTF by adding noise to the sensor measurements. We assume that measurement y_i is corrupted by white, zero-mean Gaussian noise w_i and we assume that each measurement y_j is corrupted by white, zero-mean Gaussian noise v_{j-m}, where $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, p\}$. Hence, the measured pseudo-inputs $\xi_{[1:m]}$ are

$$\xi_{[1:m]} = y_{[1:m]} + v_{[1:m]},$$

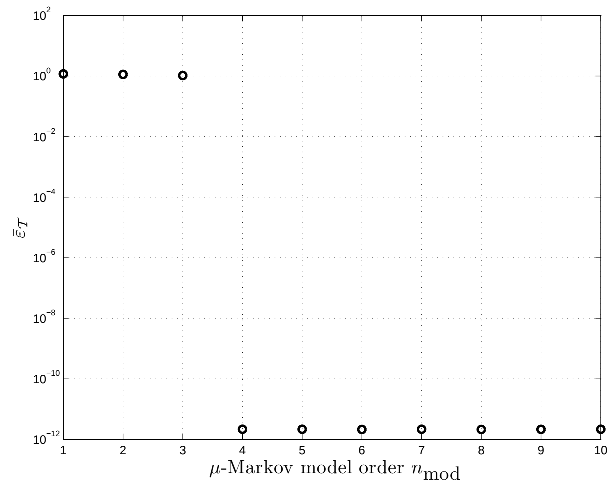

Fig. 6. Error in the Markov parameters of the MIMO PTF from $[q_1 \;\; q_2]^{\rm T}$ to v_2, estimated using MIMO LS with the μ-Markov model structure (21), as the μ-Markov model order increases. For $n_{\rm mod} \geq 4$ the Markov parameter estimates equal the true values.

and the measured pseudo-outputs $\xi_{[m+1:p]}$ are

$$\xi_{[m+1:p]} = y_{[m+1:p]} + w_{[1:p-m]}.$$

Since the pseudo inputs are not realizations of white random processes, LS does not yield consistent estimates of the Markov parameters. Hence, for comparison, we use both MIMO LS and MIMO QCLS with the μ-Markov model structure (21) with $n_{\rm mod} = 4$ and relative degree 0 to estimate the first $\mu = 4$ Markov parameter matrices of the PTF from $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^{\rm T}$ to v_2. To obtain consistent estimates, QCLS requires knowledge of the noise autocorrelation to within a scalar multiple, as discussed in the Appendix and in [13].

We simulate (20) with $x_0 \neq 0$ to obtain the position q_1 of the first mass, the position q_2 of the second mass, and the velocity v_2 of the second mass, where u_1 and u_2 are realizations of white Gaussian processes with mean 0 and variance 1. We choose v_1, v_2, and w_1 so that the signal-to-noise ratio (SNR) of each measurement is 10. For LS and QCLS, we let $\mu = 4$ in (21) and estimate the Markov parameter matrices of the PTF from $\begin{bmatrix} q_1 & q_2 \end{bmatrix}^{\rm T}$ to v_2. We use the estimated and true Markov parameter matrices to compute $\varepsilon_{\mathcal{T}}$, which we average over 10 noise sequences and 10 input sequences u to obtain $\bar{\varepsilon}_{\mathcal{T}}$. The estimates provided by QCLS in Figure 7 appear consistent, while the estimates provided by LS do not. This result is expected since the pseudo-inputs q_1 and q_2 have colored spectra.

VII. CONCLUSIONS

In applications, the number of excitation signals to a system, and their spectra, may be unknown. However, by using at least one more sensor than the number of excitation signals, MIMO pseudo transfer functions (PTFs) can be used to characterize sensor-to-sensor relationships. In this article, we provide conditions under which MIMO PTFs can be

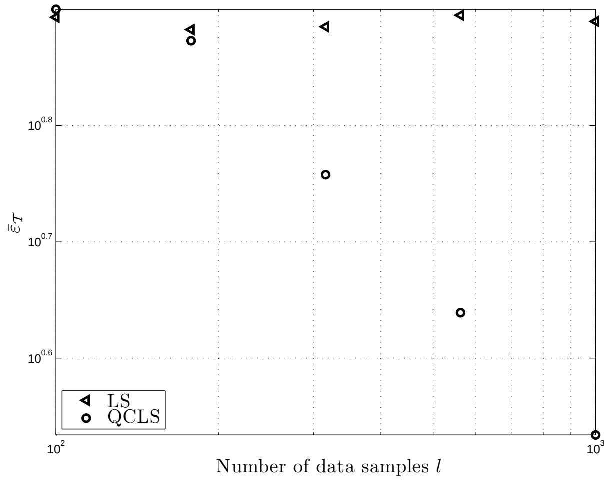

Fig. 7. Error in the PTF from $[q_1\ q_2]^{\mathrm{T}}$ to v_2, estimated using the μ-Markov model structure (21), where white, uncorrelated noise is added to all sensor measurements. The estimates of the PTF Markov parameters obtained using LS are not consistent, but the estimates obtained using QCLS appear consistent. The SNR is 10, u_1 and u_2 are white, $x_0 \neq 0$, and $n_{\mathrm{mod}} = 4$.

defined, as well as results on the order and relative degree of each entry of a MIMO PTF.

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Appendix

We consider QCLS identification of a MIMO system with input $\tilde{u}(k)$ and output $\tilde{y}(k)$ that satisfy

$$\tilde{A}(\mathbf{q})\tilde{y} = \tilde{B}(\mathbf{q})\tilde{u}, \tag{22}$$

where

$$\tilde{A}(\mathbf{q}) \triangleq \tilde{A}_0\mathbf{q}^n + \cdots + \tilde{A}_n$$

and

$$\tilde{B}(\mathbf{q}) \triangleq \tilde{B}_0\mathbf{q}^n + \cdots + \tilde{B}_n.$$

Assuming $\tilde{A}(\mathbf{q})$ has full normal rank, we can express the identification problem (22) as the QCLS problem

$$\min_{\theta\in\mathcal{D}(N)} \frac{1}{l}\theta^{\mathrm{T}}\Phi^{\mathrm{T}}\Phi\theta, \tag{23}$$

where $l > 0$, $N = N^{\mathrm{T}} \in \mathbb{R}^{[pm(n+\mu)+n+1]\times[pm(n+\mu)+n+1]}$,

$$\mathcal{D}(N) \triangleq \left\{\hat{\theta} \in \mathbb{R}^{pm(n+\mu)+n+1} : \hat{\theta}^{\mathrm{T}}N\hat{\theta} = 1\right\},$$

$$\Phi \triangleq \begin{bmatrix} \Phi_y & -\Phi_u \end{bmatrix},$$

$$\Phi_y \triangleq \begin{bmatrix} y(\mu+n-1) & y(n-1) & \cdots & y(0) \\ \vdots & \vdots & \ddots & \vdots \\ y(l) & y(l-\mu) & \cdots & y(l-\mu-n+1) \end{bmatrix},$$

$$\Phi_u \triangleq \begin{bmatrix} u^{\mathrm{T}}(\mu+n-1)\otimes I_p & \cdots & u^{\mathrm{T}}(0)\otimes I_p \\ \vdots & \ddots & \vdots \\ u^{\mathrm{T}}(l)\otimes I_p & \cdots & u^{\mathrm{T}}(l-\mu-n+1)\otimes I_p \end{bmatrix},$$

and

$$\theta \triangleq \begin{bmatrix} \alpha_0 \\ \alpha_\mu \\ \vdots \\ \alpha_{\mu+n-1} \\ \mathrm{vec}(H_0') \\ \vdots \\ \mathrm{vec}(H_{\mu-1}') \\ \mathrm{vec}(B_\mu') \\ \vdots \\ \mathrm{vec}(B_{\mu+n-1}') \end{bmatrix} \in \mathbb{R}^{pm(n+\mu)+n+1}.$$

The solution θ of the QCLS problem (23) corresponds to the generalized eigenvector associated with the smallest positive generalized eigenvalue of $(\frac{1}{l}\Phi^{\mathrm{T}}\Phi, N)$. Note that the solution of the QCLS problem (23) is unbiased and consistent under the persistency conditions given in [13].