Resetting Virtual Absorbers for Vibration Control

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Abstract: A novel class of controllers, called resetting virtual absorbers, is proposed as a means for achieving energy dissipation. A general framework for analyzing resetting virtual absorbers is given, and stability of the closed-loop system is analyzed. Special cases of resetting virtual absorbers, called the virtual trap-door absorber and the virtual one-way absorber, are described, and some illustrative examples are given.

Key Words: Resetting controllers, virtual absorbers, impulsive differential equations

1. INTRODUCTION

For more than 100 years, many different types of vibration absorbers have been used to remove energy from mechanical systems or to block energy from entering a system (Frahm, 1909; Watts, 1883). The design of dynamic vibration absorbers has received considerable attention since the mathematical description of the Den Hartog absorber (Den Hartog, 1956; Ormondroyd and Den Hartog, 1928; Snowdon, 1968), and their use is widespread today (Korenov and Reznikov, 1993).

Dynamic vibration absorbers are passive devices that have the appealing quality that, once properly designed, built, and installed, will generally operate without further attention or energy input. However, for additional flexibility, *virtual absorbers* have recently been introduced (Phan et al., 1993; Juang and Phan, 1992). A virtual absorber emulates the effect of a physical absorber by sensing the motion of the primary structure, utilizing a dynamic compensator to simulate the motion that a physical absorber would undergo, and computing and applying the reaction force that the physical absorber would apply to the primary structure. The computed reaction force is then implemented by means of a suitable actuator. Thus, the implementation of a virtual absorber requires the availability of sensors and actuators as well as processors and power supplies. One advantage of virtual absorbers

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over passive absorbers is that the parameters of the virtual absorber can be adjusted online (Lai and Wang, 1996; Sun, Jolly, and Norris, 1995; Quan and Stech, 1996).

An important feature of physical absorbers is that their components can be associated with some form of energy, usually kinetic or potential. In a mechanical system, positions typically correspond to elastic deformations, which contribute to the potential energy of the system, whereas velocities typically correspond to momenta, which contribute to the kinetic energy of the system. On the other hand, while a virtual absorber has dynamic states that emulate the motion of the physical components, these states only "exist" as numerical representations inside the processor. Consequently, while one can associate an *emulated energy* with these states, this energy is merely a mathematical construct and does not correspond to any physical form of energy.

In vibration control problems, if a plant is at a high energy level, and a physical absorber at a low energy level is attached to it, then energy will generally tend to flow from the plant into the absorber, decreasing the plant energy and increasing the absorber energy (Kishimoto, Bernstein, and Hall, 1995). Conversely, if the plant is at a low energy level and an attached absorber is at a high energy level, then energy will tend to flow from the absorber into the plant. This behavior is also exhibited by a plant with an attached virtual absorber, although in this case, emulated energy, and not physical energy, is accumulated by the virtual absorber. Nonetheless, energy can flow from a virtual absorber to the plant, since a virtual absorber with emulated energy can generate real, physical energy to effect the required energy flow. Therefore, when using a virtual absorber, it may be advantageous to detect when the position and velocity states of the emulated absorber represent a high emulated energy level, and then *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. A virtual absorber whose states are reset is called a *resetting virtual absorber*. The contribution of this paper is the development of resetting virtual absorbers as a novel strategy for vibration suppression.

The concept of a resetting virtual absorber can be viewed as a specialized technique for exploiting the coupling between a structure and a passive controller to remove energy from the structure. This idea provides the motivation for synthesizing positive real controllers for positive real plants using H_2 and H_∞ methods (Bupp et al., 1995; Haddad, Bernstein, and Wang, 1994; Haddad and Chellaboina, 1997; Lozano-Leal and Joshi, 1988; Kishimoto, Bernstein, and Hall, 1995). In addition, in a series of papers (Duquette, Tuer, and Golnaraghi, 1993; Golnaraghi, Tuer, and Wang, 1994, 1995; Tuer, Golnaraghi, and Wang, 1994; Ouceini and Golnaraghi, 1996; Ouceini, Tuer, and Golnaraghi, 1995; Siddiqui and Golnaraghi, 1996), M. F. Golnaraghi and coworkers have developed and demonstrated a modal coupling technique that utilizes energy transfer phenomena and a resetting mechanism to suppress structural vibration. The present paper thus provides a systematic investigation of various realizations of this idea in terms of resetting differential equations.

This paper presents a review in Section 2 of the concept of resetting differential equations, which provide the mathematical foundation for analyzing resetting virtual absorbers. Two kinds of resetting virtual absorbers, namely, time-dependent and state-dependent resetting virtual absorbers, are then described in Section 3, and illustrative examples are provided. Conclusions are presented in Section 4.

2. MATHEMATICAL PRELIMINARIES

In this section, we review some basic results on resetting differential equations (Bainov and Simeonov, 1989; Kulev and Bainov, 1989; Lakshmikantham, Leela, and Martynyuk, 1989; Lakshmikantham, Leela, and Kaul, 1994; Lakshmikantham, Bainov, and Simeonov, 1989; Lakshmikantham and Liu, 1989; Liu, 1988; Simeonov and Bainov, 1985, 1987), which will be used throughout the paper to analyze resetting virtual absorbers. A resetting differential equation consists of three elements:

1. A continuous-time dynamical equation, which governs the motion of the system between resetting events;
2. A difference equation, which governs the way the states are instantaneously changed when a resetting event occurs; and
3. A criterion for determining when the states of the system are to be reset.

A resetting differential equation thus has the form

$$\dot{x}(t) = f(x(t)), \qquad (t, x(t)) \notin \mathcal{S}, \tag{1}$$

$$\Delta x(t) = \rho(x(t)), \qquad (t, x(t)) \in \mathcal{S}, \tag{2}$$

where $t \geq 0$, $x(t) \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f(0) = 0$; $\rho : \mathbb{R}^n \to \mathbb{R}^n$ and satisfies $\rho(0) = 0$, and $\mathcal{S} \subset [0, \infty) \times \mathbb{R}^n$ is the *resetting set*. We refer to the differential equation (1) as the *continuous-time dynamics*, and we refer to the difference equation (2) as the *resetting law*. For convenience, we use the notation $x(t; \tau, \xi)$ to denote the solution of (1) at time $t > \tau$ with initial condition $x(\tau) = \xi$.

For a particular trajectory $x(t)$, we let t_k denote the kth instant of time at which $(t, x(t))$ intersects $\mathcal{S}$, and we call the times t_k the *resetting times*. Thus the trajectory of the system (1), (2) from the initial condition $x(0) = x_0$ is given by $x(t; 0, x_0)$ for $0 < t \leq t_1$. If and when the trajectory reaches a state $x_1 \triangleq x(t_1)$ satisfying $(t_1, x_1) \in \mathcal{S}$, then the state is instantaneously transferred to $x_1^+ \triangleq x_1 + \rho(x_1)$ according to (2). The trajectory $x(t)$, $t_1 < t \leq t_2$, is then given by $x(t; t_1, x_1^+)$, and so on. Note that the solution $x(t)$ of (1), (2) is left continuous; that is, it is continuous everywhere except at the resetting times t_k, and

$$x_k \triangleq x(t_k) = \lim_{\varepsilon \to 0^+} x(t_k - \varepsilon), \tag{3}$$

$$x_k^+ \triangleq x(t_k) + \rho(x(t_k)) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon), \tag{4}$$

for $k = 1, 2, \ldots$.

We make the following additional assumptions:

A1. $(0, x_0) \notin \mathcal{S}$, where $x(0) = x_0$, that is, the initial condition is not in $\mathcal{S}$.

A2. If $(t, x(t)) \in \mathbf{cl}\,\mathcal{S} \setminus \mathcal{S}$, where cl $\mathcal{S}$ denotes the closure of the set $\mathcal{S}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$,

$$x(t + \delta; t, x(t)) \notin \mathcal{S}.$$

A3. If $(t, x(t)) \in \mathbf{cl}\,\mathcal{S} \bigcap \mathcal{S}$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$,

$$x(t + \delta; t, x(t) + \rho(x(t))) \notin \mathcal{S}.$$

Assumption A1 ensures that the initial condition for the resetting differential equation (1), (2) is not a point of discontinuity, and this assumption is made for convenience and without loss of generality. If $(0, x_0) \in \mathcal{S}$, then the system initially resets to $x_0^+ = x_0 + \rho(x_0)$, which serves as the initial condition for the continuous-time dynamics (1). It follows from A3 that the trajectory then leaves $\mathcal{S}$. We assume in A2 that if a trajectory reaches the closure of $\mathcal{S}$ at a point that does not belong to $\mathcal{S}$, then the trajectory must be directed away from $\mathcal{S}$, that is, a trajectory cannot enter $\mathcal{S}$ through a point that belongs to the closure of $\mathcal{S}$ but not to $\mathcal{S}$. Finally, A3 ensures that when a trajectory intersects the resetting set $\mathcal{S}$, it instantaneously exits $\mathcal{S}$.

Remark 2.1. It follows from A3 that resetting removes the pair (t_k, x_k) from the resetting set $\mathcal{S}$. Thus, immediately after resetting occurs, the continuous-time dynamics (1), and not the resetting law (2), becomes the active element of the resetting differential equation.

Remark 2.2. It follows from A1–A3 that no trajectory can intersect the interior of $\mathcal{S}$: According to A1, the trajectory $x(t)$ begins outside the set $\mathcal{S}$. Furthermore, it follows from A2 that a trajectory can only reach $\mathcal{S}$ at a point belonging to both $\mathcal{S}$ and its closure. Finally, from A3, it follows that if a trajectory reaches a point in $\mathcal{S}$ that is in the closure of $\mathcal{S}$, then the trajectory is instantaneously removed from $\mathcal{S}$. Since a continuous trajectory starting outside of $\mathcal{S}$ and intersecting the interior of $\mathcal{S}$ must first intersect the closure of $\mathcal{S}$, it follows that no trajectory can reach the interior of $\mathcal{S}$.

Remark 2.3. It follows from A1–A3 and Remark 2.2 that the resetting times t_k are well defined and distinct.

Remark 2.4. Since the resetting times are well defined and distinct, and since solutions of (1) exist and are unique, it follows that the solutions of the resetting differential equation (1), (2) also exist and are unique in forward time.

In Bainov and Simeonov (1989); Hu, Lakshmikantham, and Leela (1989); Sun, Jolly, and Norris (1995); Lakshmikantham, Bainov, and Simeonov (1989); Lakshmikantham, Leela, and Kaul (1994); Lakshikantham and Liu (1989); Liu (1988); and Simeonov and Bainov (1985, 1987), the resetting set $\mathcal{S}$ is defined in terms of a countable number of functions $\tau_k : \mathbb{R}^n \to (0, \infty)$ and is given by

$$\mathcal{S} = \bigcup_k \{(\tau_k(x), x) : x \in \mathbb{R}^n\}. \tag{5}$$

The analysis of resetting differential equations with a resetting set of the form (5) can be quite involved. Here we consider resetting differential equations involving two distinct forms of the resetting set $\mathcal{S}$. In the first case, the resetting set is defined by a prescribed sequence of times that are independent of the state x. These equations are thus called *time-dependent resetting differential equations*. In the second case, the resetting set is defined by a region in the state space that is independent of time. These equations are called *state-dependent resetting differential equations*.

2.1. Time-Dependent Resetting Differential Equations

Time-dependent resetting differential equations, which are studied in Simeonov and Bainov (1985) and Lakshmikantham, Bainov, and Simeonov (1989), can be written as (1), (2) with $\mathcal{S}$ defined as

$$\mathcal{S} \triangleq \mathcal{T} \times \mathbb{R}^n, \tag{6}$$

where

$$\mathcal{T} \triangleq \{t_1, t_2, \ldots\} \tag{7}$$

and $0 < t_1 < t_2 < \cdots$ are prescribed resetting times. When an infinite number of resetting times are used, we assume that $t_k \to \infty$ as $k \to \infty$ so that $\mathcal{S}$ is closed. Now (1), (2) can be rewritten in the form of the *time-dependent resetting differential equation*

$$\dot{x}(t) = f(x(t)), \qquad t \neq t_k, \tag{8}$$

$$\Delta x(t) = \rho(x(t)), \qquad t = t_k. \tag{9}$$

Since $0 \notin \mathcal{T}$ and $t_k < t_{k+1}$, it follows that the assumptions A1–A3 are satisfied. For our purposes, the following stability result is needed. Note that the usual stability definitions are valid.

Theorem 2.1. *Suppose there exists a continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$ satisfying $V(0) = 0$, $V(x) > 0$, $x \neq 0$, and*

$$V'(x) f(x) \leq 0, \qquad x \in \mathbb{R}^n, \tag{10}$$

$$V(x + \rho(x)) \leq V(x), \qquad x \in \mathbb{R}^n. \tag{11}$$

Then the zero solution of (8), (9) is Lyapunov stable. Furthermore, if the inequality in (10) is strict for all $x \neq 0$, then the zero solution of (8), (9) is asymptotically stable. If, in addition,

$$V(x) \to \infty \quad \text{as} \quad ||x|| \to \infty, \tag{12}$$

then the zero solution of (8), (9) is globally asymptotically stable.

Proof. Prior to the first resetting time, we can determine the value of $V(x(t))$ as

$$V(x(t)) = V(x(0)) + \int_0^t V'(x(\tau))f(x(\tau))\mathrm{d}\tau, \quad t \in [0, t_1]. \tag{13}$$

Between consecutive resetting times t_k and t_{k+1}, we can determine the value of $V(x(t))$ as its initial value plus the integral of its rate of change along the trajectory $x(t)$, that is,

$$V(x(t)) = V(x_k + \rho(x_k)) + \int_{t_k}^t V'(x(\tau))f(x(\tau))\mathrm{d}\tau, \quad t \in (t_k, t_{k+1}], \tag{14}$$

for $k = 1, 2, \ldots$. Adding and subtracting $V(x_k)$ to the right-hand side of (14) yields

$$\begin{aligned} V(x(t)) &= V(x_k) + [V(x_k + \rho(x_k)) - V(x_k)] \\ &+ \int_{t_k}^t V'(x(\tau))f(x(\tau))\mathrm{d}\tau, \quad t \in (t_k, t_{k+1}], \end{aligned} \tag{15}$$

and in particular at time t_{k+1},

$$V(x_{k+1}) = V(x_k) + [V(x_k + \rho(x_k)) - V(x_k)] + \int_{t_k}^{t_{k+1}} V'(x(\tau))f(x(\tau))\mathrm{d}\tau. \tag{16}$$

By recursively substituting (16) into (15) and ultimately into (13), we obtain

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t V'(x(\tau))f(x(\tau))\mathrm{d}\tau \\ &+ \sum_{i=1}^k [V(x(t_i) + \rho(x(t_i))) - V(x(t_i))], \quad t \in (t_k, t_{k+1}]. \end{aligned} \tag{17}$$

If we allow $t_0 \triangleq 0$, and $\sum_{i=1}^0 \triangleq 0$, then (17) is valid for $k = 0, 1, 2, \ldots$. From (17) and (11), we obtain

$$V(x(t)) \le V(x(0)) + \int_0^t V'(x(\tau))f(x(\tau))\mathrm{d}\tau, \quad t \ge 0. \tag{18}$$

Furthermore, it follows from (10) that

$$V(x(t)) \le V(x(0)), \quad t \ge 0, \tag{19}$$

so that Lyapunov stability follows from standard arguments.

Next, it follows from (11) and (17) that

$$V(x(t)) - V(x(s)) \leq \int_s^t V'(x(\tau)) f(x(\tau)) \mathrm{d}\tau, \quad t > s, \tag{20}$$

and, assuming strict inequality in (10), we obtain

$$V(x(t)) < V(x(s)), \quad t > s, \tag{21}$$

provided $x(s) \neq 0$. Asymptotic stability, and with (12) global asymptotic stability, then follow from standard arguments.

Remark 2.5. In the proof of Theorem 2.1, we note that assuming strict inequality in (10), the inequality (21) is obtained *provided* $x(s) \neq 0$. This proviso is necessary since it may be possible to reset the states to the origin, in which case $x(s) = 0$ for a finite value of s. In this case, for $t > s$, we have $V(x(t)) = V(x(s)) = V(0) = 0$. This situation does not present a problem, however, since reaching the origin in finite time is a stronger condition than reaching the origin as $t \to \infty$.

2.2. State-Dependent Resetting Differential Equations

State-dependent resetting differential equations, which are discussed in Bainov and Simeonov (1989), can be written as (1), (2) with $\mathcal{S}$ defined as

$$\mathcal{S} \triangleq [0, \infty) \times \mathcal{M}, \tag{22}$$

where $\mathcal{M} \subset \mathbb{R}^n$. Therefore, (1), (2) can be rewritten in the form of the *state-dependent resetting differential equation*

$$\dot{x}(t) = f(x(t)), \qquad x(t) \notin \mathcal{M}, \tag{23}$$

$$\Delta x(t) = \rho(x(t)), \qquad x(t) \in \mathcal{M}. \tag{24}$$

We assume that $x_0 \notin \mathcal{M}$, $0 \notin \mathcal{M}$, and that the resetting action removes the state from the set $\mathcal{M}$; that is, if $x \in \mathcal{M}$, then $x + \rho(x) \notin \mathcal{M}$. In addition, we assume that if at time t the trajectory $x(t) \in \mathrm{cl}\,\mathcal{M} \setminus \mathcal{M}$, then there exists $\varepsilon > 0$ such that for $0 < \delta < \varepsilon$, $x(t+\delta) \notin \mathcal{M}$. These assumptions represent the specializations of A1–A3 for the particular resetting set (22). It follows from these assumptions that for a particular initial condition, the resetting times t_k are distinct and well defined.

Remark 2.6. Let $x^* \in \mathbb{R}^n$ satisfy $\rho(x^*) = 0$. Then $x^* \notin \mathcal{M}$. To see this, suppose $x^* \in \mathcal{M}$. Then $x^* + \rho(x^*) = x^* \in \mathcal{M}$, which contradicts the assumption that if $x \in \mathcal{M}$, then $x + \rho(x) \notin \mathcal{M}$.

For our purposes, the following result for the stability of the zero solution is needed.

Theorem 2.2. *Suppose there exists a continuously differentiable function* $V : \mathbb{R}^n \to [0,\infty)$ *satisfying* $V(0) = 0$, $V(x) > 0$, $x \neq 0$, *and*

$$V'(x)f(x) \leq 0, \qquad x \notin \mathcal{M}, \tag{25}$$

$$V(x+\rho(x)) \leq V(x), \qquad x \in \mathcal{M}. \tag{26}$$

Then the zero solution of (23), (24) is Lyapunov stable. Furthermore, if the inequality in (25) is strict for all $x \neq 0$, *then the zero solution of (23), (24) is asymptotically stable. If, in addition, (12) is satisfied, then the zero solution of (23), (24) is globally asymptotically stable.*

Because the resetting times are well defined and distinct for any trajectory of (23), (24), the proof of Theorem 2.2 follows from the proof of Theorem 2.1 given in Subsection 2.1.

3. RESETTING VIRTUAL ABSORBERS

In this section, we apply the theory of resetting differential equations to the analysis of resetting virtual absorbers. We consider a continuous-time plant of the form

$$\dot{x}_p(t) = f_p(x_p(t), u(t)), \tag{27}$$

$$y(t) = h_p(x_p(t)), \tag{28}$$

where $x_p \in \mathbb{R}^{n_c}$, $f_p : \mathbb{R}^{n_p} \times \mathbb{R}^m \to \mathbb{R}^{n_p}$ is continuously differentiable and satisfies $f_p(0,0) = 0$, and $h_p : \mathbb{R}^{n_p} \to \mathbb{R}^p$ is continuous and satisfies $h_p(0) = 0$. We also consider a controller of the form

$$\dot{x}_c(t) = f_c(x_c(t), y(t)), \qquad (t, x_c(t), y(t)) \notin \mathcal{S}_c, \tag{29}$$

$$\Delta x_c(t) = \rho_c(x_c(t), y(t)), \qquad (t, x_c(t), y(t)) \in \mathcal{S}_c, \tag{30}$$

$$u(t) = h_c(x_c(t), y(t)), \tag{31}$$

where $\mathcal{S}_c \subset [0,\infty) \times \mathbb{R}^{n_c} \times \mathbb{R}^p$, $x_c \in \mathbb{R}^{n_c}$, $f_c : \mathbb{R}^{n_c} \times \mathbb{R}^p \to \mathbb{R}^{n_c}$ is continuously differentiable and satisfies $f_c(0,0) = 0$, $h_c : \mathbb{R}^{n_c} \times \mathbb{R}^p \to \mathbb{R}^m$ is continuous and satisfies $h_c(0,0) = 0$, and $\rho : \mathbb{R}^{n_c} \times \mathbb{R}^p \to \mathbb{R}^{n_c}$ satisfies $\rho_c(0,0) = 0$. We assume $x_c(0) = \rho_c(0, h_p(x_p(0)))$, which is generally a nonzero initial condition for the controller.

The equations of motion for the closed-loop system (27) through (31) have the form

$$\dot{x}(t) = f(x(t)), \qquad (t,x(t)) \notin \mathcal{S}, \tag{32}$$

$$\Delta x(t) = \rho(x(t)), \qquad (t,x(t)) \in \mathcal{S}, \tag{33}$$

where

$$f(x) \triangleq \begin{bmatrix} f_p(x_p, h_c(x_c, h_p(x_p))) \\ f_c(x_c, h_p(x_p)) \end{bmatrix}, \qquad x \triangleq \begin{bmatrix} x_p \\ x_c \end{bmatrix} \in \mathbb{R}^{n_p+n_c}, \tag{34}$$

$$\rho(x) \triangleq \begin{bmatrix} 0 \\ \rho_c(x_c, h_p(x_p)) \end{bmatrix} \tag{35}$$

and

$$\mathcal{S} \triangleq \{(t,x) : (t, x_c, h_p(x_p)) \in \mathcal{S}_c\}. \tag{36}$$

Note that $n \triangleq n_p + n_c$ in the notation of Section 2.

Notice that (32), (33) has the form of the resetting differential equation (1), (2). However, while the closed-loop state vector consists of plant states and controller states, it is clear from (35) that only those states associated with the controller are reset.

We associate with the plant a positive-definite, continuously differentiable function $V_p(x_p)$, satisfying $V_p(0) = 0$, which we will refer to as the *plant energy*. We associate with the controller a nonnegative-definite, continuously differentiable function $V_c(x_c, y)$, satisfying $V_c(0,0) = 0$ and $V_c(x_c, 0) > 0$, $x_c \neq 0$, called the *emulated energy*. Finally, we associate with the closed-loop system the function

$$V(x) \triangleq V_p(x_p) + V_c(x_c, y), \tag{37}$$

called the *total energy*.

3.1. Time-Dependent Resetting Virtual Absorber

Consider the closed-loop system (32), (33) with resetting set defined by (6), (7) and where the prescribed elements of $\mathcal{T}$ satisfy $0 < t_1 < t_2 < \cdots$. Suppose that V_c satisfies

$$V_c(x_c + \rho_c(x_p, h_p(x_p))) \leq V_c(x_c, h_p(x_p)), \tag{38}$$

and $V(x)$ defined in (37) satisfies

$$V'(x) f(x) \leq 0, \qquad x \in \mathbb{R}^n. \tag{39}$$

Then Lyapunov (resp., asymptotic) stability of the closed-loop system follows from Theorem 2.1. The design of time-dependent resetting virtual absorbers is considered in the following examples.

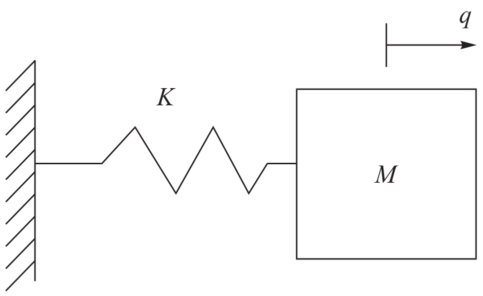

Figure 1. Undamped single-degree-of-freedom oscillator.

3.1.1. Stabilization of the undamped oscillator. For illustrative purposes, we consider the problem of stabilizing the undamped single-degree-of-freedom oscillator shown in Figure 1. For convenience, and without loss of generality, we assume M and K are mass and spring elements with unit value. The displacement of the mass is denoted by q. We assume that an ideal actuator is available to apply a control force u to the mass. We then have

$$x_p = \begin{bmatrix} x_{\mathrm{p}1} \\ x_{\mathrm{p}2} \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \qquad f_p(x_p, u) = \begin{bmatrix} x_{\mathrm{p}2} \\ -x_{\mathrm{p}1} + u \end{bmatrix}, \tag{40}$$

and we set

$$y = h_p(x_p) = x_{\mathrm{p}1}. \tag{41}$$

We define the plant energy V_p to be the energy of the oscillator, that is, the sum of its kinetic and elastic energy. Hence

$$V_p(x_p) = \frac{1}{2} x_p^T x_p, \tag{42}$$

which satisfies

$$V'_p(x_p) f_p(x_p, 0) = 0, \qquad x_p \in \mathbb{R}^{n_p}. \tag{43}$$

For this example, the continuous-time dynamics of the resetting controller emulates a linear spring-mass dynamic vibration absorber attached to the primary mass M. In the absence of resetting, the closed-loop system behaves like the undamped two-mass, two-spring system shown in Figure 2. Thus we define

$$\begin{aligned} x_c &= \begin{bmatrix} x_{\mathrm{c}1} \\ x_{\mathrm{c}2} \end{bmatrix} = \begin{bmatrix} q_c \\ \dot{q}_c \end{bmatrix}, \quad f_c(x_c, y) = \begin{bmatrix} x_{\mathrm{c}2} \\ -\frac{k_c}{m_c}(x_{\mathrm{c}1} - y) \end{bmatrix}, \\ h_c(x_c, y) &= k_c(x_{\mathrm{c}1} - y). \end{aligned} \tag{44}$$

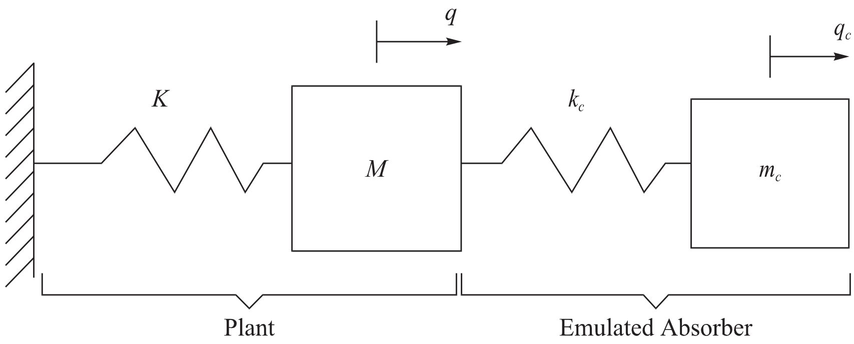

Figure 2. In the absence of resetting, the controller emulates a loss-less dynamic vibration absorber, and the closed-loop system behaves like this undamped two-mass, two-spring system.

We define the emulated energy to be the energy associated with the emulated absorber, so that

$$V_c(x_c, y) = \frac{1}{2}k_c(x_{\mathrm{c}1} - y)^2 + \frac{1}{2}m_c x_{\mathrm{c}2}^2, \tag{45}$$

which represents the amount of energy that the emulated absorber would possess if its spring were elongated by an amount $\pm(x_{\mathrm{c}1} - y)$ and its mass were moving with velocity $\pm x_{\mathrm{c}2}$. Forming the closed-loop system as in (34) and defining the total energy

$$V(x) \triangleq V_p(x_p) + V_c(x_c, h_p(x_p)), \tag{46}$$

we obtain

$$V'(x)f(x) = 0, \qquad x \in \mathbb{R}^n. \tag{47}$$

It is evident from (47) that there is no mechanism for dissipation in the continuous-time dynamics of the closed-loop system. Therefore, the decrease in the total energy can only be accomplished by resetting the controller states. We therefore choose

$$\Delta x_c = \rho_c(x_c, y) = \begin{bmatrix} y - x_{\mathrm{c}1} \\ -x_{\mathrm{c}2} \end{bmatrix}. \tag{48}$$

The effect of the resetting law (48) is to cause both the "elongation" in the emulated spring k_c and the "velocity" of the emulated mass m_c to be instantaneously reset to zero; that is, the resetting law (48) implies $V_c(x_c + \Delta x_c, y) = 0$. The closed-loop resetting law is thus given by

$$\Delta x = \rho(x) = \begin{bmatrix} 0 \\ 0 \\ x_{\mathrm{p}1} - x_{\mathrm{c}1} \\ -x_{\mathrm{c}2} \end{bmatrix}. \tag{49}$$

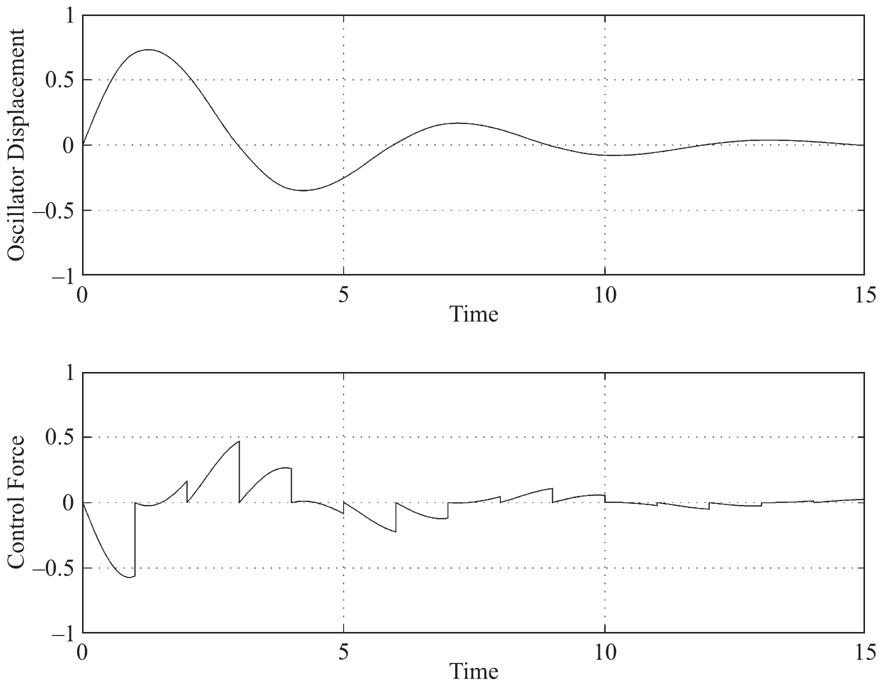

Figure 3. These plots illustrate the removal of energy from the single-degree-of-freedom oscillator by using a time-dependent resetting virtual absorber. Since there are no dashpot elements in the closed-loop system, all of the dissipation is achieved by resetting the controller states. The parameters are $M = m_c = 1$, $K = k_c = 1$, the initial condition is $q(0) = q_c(0) = 0$, $\dot{q}(0) = 1$, $\dot{q}_c(0) = 0$, and the controller states are reset every 1 second. The time history of the displacement of the oscillator mass is shown in the upper plot, while the time history of the control force is given in the lower plot.

Note that since

$$x + \Delta x = \begin{bmatrix} q \\ \dot{q} \\ q \\ 0 \end{bmatrix}, \tag{50}$$

it follows that

$$V(x + \Delta x) = \frac{1}{2} x_p^T x_p, \tag{51}$$

and

$$V(x + \Delta x) - V(x) = -V_c(x_c, h_p(x_p)) \leq 0. \tag{52}$$

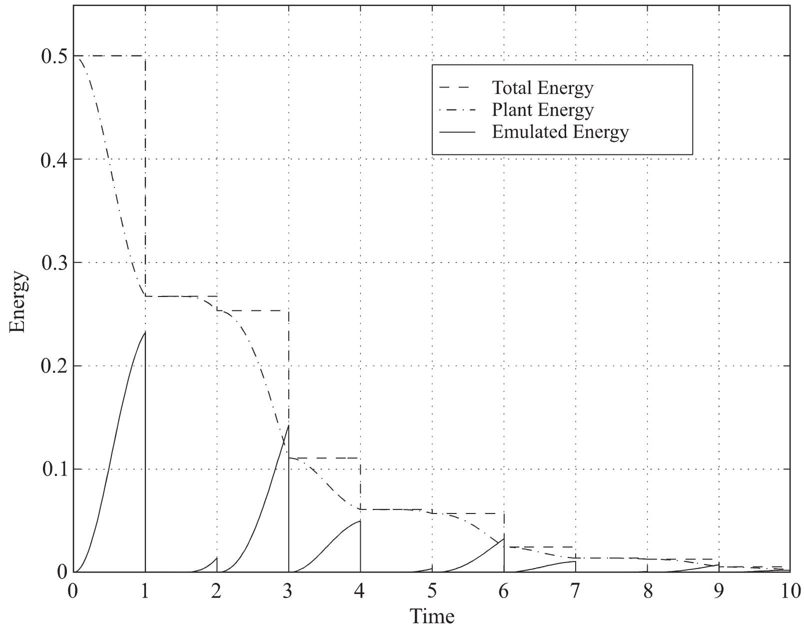

Figure 4. This plot shows the time history of the energy components for the oscillator controlled by a time-dependent resetting virtual absorber. Since there are no dashpot elements in the closed-loop system, the total energy (dashed line) is constant between resetting events, and the increase in the emulated energy (solid line) equals the decrease in plant energy (dash-dot line). When a resetting event occurs, which for this example is every 1 second, the emulated energy is instantaneously reset to zero, while the plant energy is unchanged.

It can be seen in (52) that the resetting law (49) causes the total energy to instantaneously decrease by an amount equal to the accumulated emulated energy.

To illustrate the dynamics of the closed-loop system, let $m_c = 1$, $k_c = 1$, and $\{t_k\} = \{1, 2, 3, \ldots\}$ so that the controller resets periodically with a period of 1 second. The response of the oscillator with the resetting virtual absorber to the initial condition $q(0) = 0, \dot{q}(0) = 1$, $q_c(0) = 0$, $\dot{q}_c(0) = 0$ is shown in Figure 3. The energy in the oscillator is effectively dissipated with an average rate of energy dissipation comparable to a 25% damping ratio. Note that the control force, illustrated in the lower plot of Figure 3, is discontinuous at the resetting times, but not impulsive.

A comparison of the plant energy, emulated energy, and total energy is given in Figure 4. It can be seen that, between resetting events, the total energy is constant, while any increase in the emulated energy is accompanied by a decrease in the plant energy. When a resetting event occurs, the emulated energy is reset to zero, while the plant energy is unchanged.

In this example, the controller parameters and resetting times were chosen arbitrarily. However, a method for choosing the parameters m_c and k_c so as to achieve finite settling time for this single-degree-of-freedom oscillator is described in Bupp et al. (1996). We consider this approach in the following example.

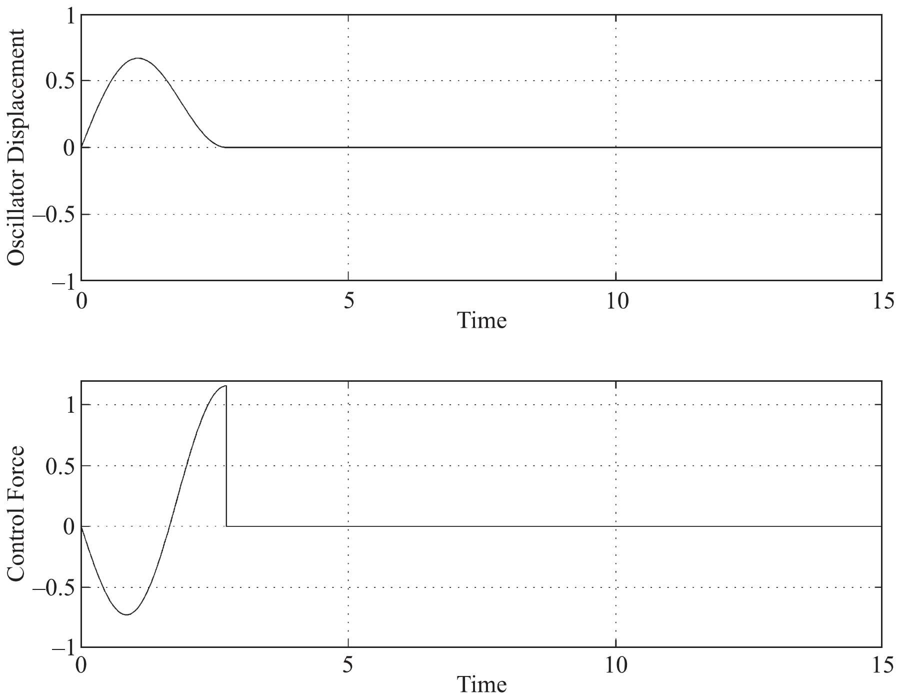

Figure 5. These plots illustrate the finite settling time performance of a suitably tuned time-dependent resetting virtual absorber applied to an undamped oscillator. The parameters are $M = 1$, $K = 1$, $m_c = 4/3$, $k_c = 4/3$, and the initial condition is $q(0) = q_c(0) = 0$, $\dot{q}(0) = 1$, $\dot{q}_c(0) = 0$. The controller states are reset every $0.5\sqrt{3}\pi$ seconds. The time history of the displacement of the oscillator mass is shown in the upper plot, while the time history of the control force is given in the lower plot. The oscillator settles to the origin at the first resetting time and remains there for all future time.

3.1.2. Stabilization of undamped oscillator in finite time. In this example, we consider again the single-degree-of-freedom oscillator with time-dependent resetting virtual absorber. Now, following the procedure in Bupp et al. (1996), we choose $m_c = 4/3$, $k_c = 4/3$, and prescribe the resetting times to be periodic with period $0.5\sqrt{3}\pi$ seconds. This choice of parameters yields finite settling time behavior (Bupp et al., 1996). The time history of the response of the oscillator with this resetting virtual absorber is given in Figure 5, where the initial condition is the same as that used in the previous example. The key to this approach is tuning the resetting virtual absorber so that it absorbs all of the energy of the oscillator at a predetermined time, which is independent of the initial condition of the oscillator. This total absorption of energy is evident in Figure 6, which shows the time histories of the total energy and its components.

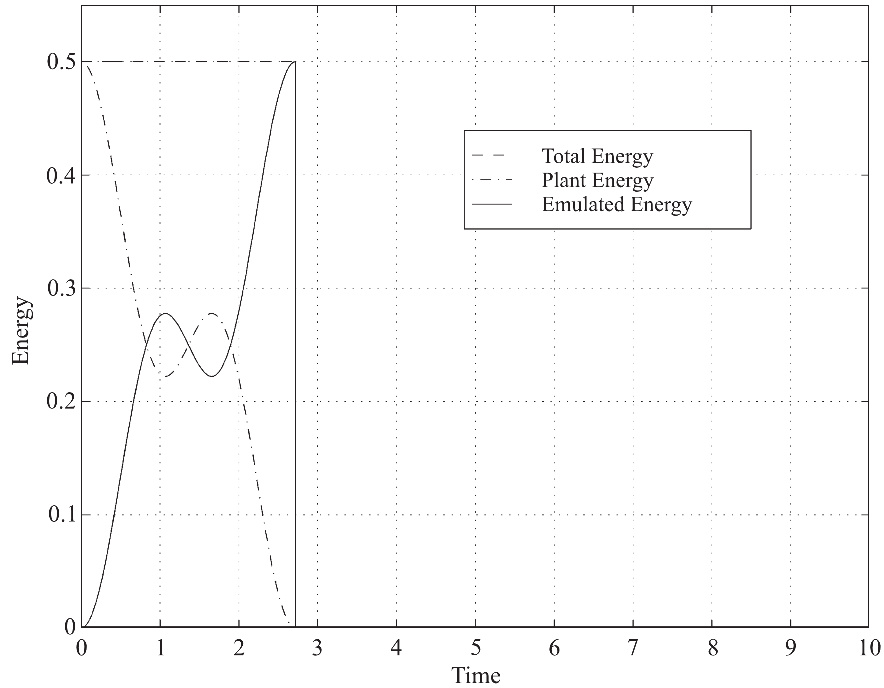

Figure 6. This plot shows the time history of the energy components for the oscillator controlled by a time-dependent resetting virtual absorber tuned so that it absorbs all of the energy of the oscillator (dash-dot line) at a predetermined time. Since there are no dashpot elements in the closed-loop system, the total energy (dashed line) is constant between resetting events. The emulated energy is shown with a solid line.

3.2. A State-Dependent Resetting Virtual Absorber: The Virtual One-Way Absorber

In this subsection, we describe the mathematical setting and design of a state-dependent resetting virtual absorber called the *virtualone-way absorber*. We consider the plant and resetting absorber as described in Section 3, where $\mathcal{S}$ is defined in (22) and $\mathcal{M}$ has the form

$$\mathcal{M} \triangleq \left\{ x = \begin{bmatrix} x_p \\ x_c \end{bmatrix} : \rho_c(x_c, h_p(x_p)) \neq 0 \text{ and } V'_p(x_p) f_p(x_p, h_c(x_c)) \geq 0 \right\}. \tag{53}$$

For practical implementation, knowledge of x_c and y should be sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{M}$. The resetting set $\mathcal{M}$ is thus defined to be the set of all points in the closed-loop state space that represent nondecreasing plant energy, except those that satisfy $\rho(x) = 0$. As mentioned in Remark 2.6, the states x that satisfy $\rho(x) = 0$ are states that do not change under the action of the resetting law, and thus we need to exclude these states from the resetting set to ensure that the assumption A2 is not violated.

By resetting the states, the plant energy can never increase. Also, if the continuous-time dynamics of the closed-loop system are lossless, then a decrease in plant energy is accompanied by a corresponding increase in emulated energy. Hence, this approach allows plant energy to "flow" to the controller, where it increases the emulated energy but does not allow the emulated energy to "flow" back to the plant—hence the name *virtual one-way absorber*.

We assume as before that if $x \in \mathcal{M}$, then $x + \rho(x) \notin \mathcal{M}$. We further assume that if $x \in \mathcal{M}$, then

$$V_c(x_c + \rho_c(x_c, y), y) \leq V_c(x_c, y), \tag{54}$$

and thus

$$V(x + \rho(x)) \leq V(x), \tag{55}$$

so that, if (25) is satisfied, then Lyapunov (resp. asymptotic) stability of the closed-loop system follows from Theorem 2.2.

3.2.1. Stabilization of the RTAC by state-dependent resetting virtual absorber. To illustrate the design of a one-way resetting virtual resetting absorber, we consider the nonlinear Rotational/Translational Actuator (RTAC) system (Wan, Bernstein, and Coppola, 1996) illustrated in Figure 7. The RTAC consists of a translational cart of mass M connected by a spring of stiffness k to a wall. The rotational actuator, which is mounted on the cart, consists of a proof mass of mass m and centroidal moment of inertia I mounted at a fixed distance e from its center of rotation. A control torque N is applied to the rotational proof mass.

Let q denote the translational displacement of the cart from its equilibrium position, and let θ denote the counterclockwise rotational angle of the eccentric mass, where $\theta = 0$ is perpendicular to the direction of translation, as shown in Figure 7.

The equations of motion are thus given by

$$\begin{aligned} (M+m)\ddot{q} + kq &= -me(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), & (56)\\ (I + me^2)\ddot{\theta} &= -me\ddot{q}\cos\theta + N. & (57) \end{aligned}$$

Since this system is not stable, we first Lyapunov stabilize it with the control

$$N = -mge\sin\theta + u, \tag{58}$$

and then design a resetting virtual absorber to apply the control torque u. Note that the control (58) does not introduce any dissipation. The plant equations can now be written as (27), where

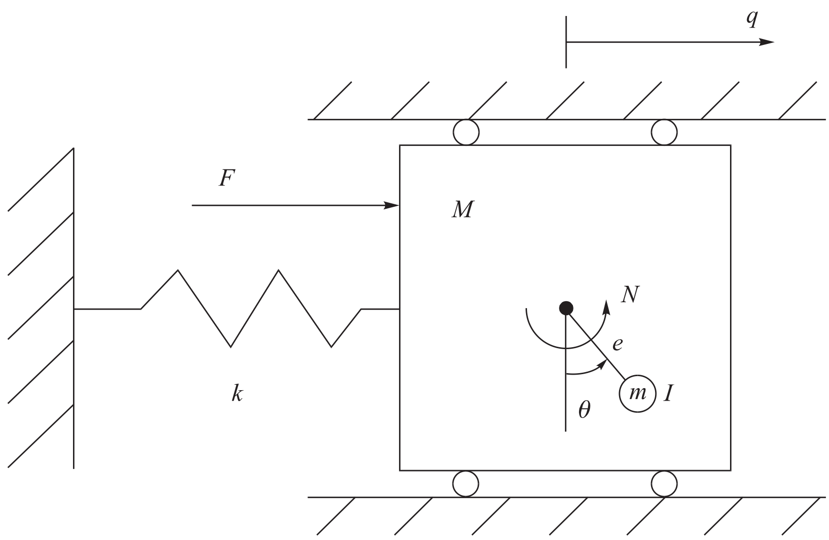

Figure 7. The rotational/translational actuator model.

$$x = \begin{bmatrix} x_{\mathrm{p}1} \\ x_{\mathrm{p}2} \\ x_{\mathrm{p}3} \\ x_{\mathrm{p}4} \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \\ \theta \\ \dot{\theta} \end{bmatrix}, \quad f_p(x_p, u) = \begin{bmatrix} \dot{q} \\ \frac{(I+me^2)(me\dot{\theta}^2 - kq) - me\cos\theta u}{(M+m)(I+me^2)-(me\cos\theta)^2} \\ \dot{\theta} \\ \frac{me\cos\theta(kq - me\dot{\theta}^2\sin\theta + (M+m)u}{(M+m)(I+me^2)-(me\cos\theta)^2} \end{bmatrix}. \tag{59}$$

We will require as output the position and velocity of the rotational degree of freedom,

$$y = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}. \tag{60}$$

The plant energy corresponds to the sum of the kinetic energies of the cart mass and the proof mass, the potential energy stored in the spring k, and the potential energy function associated with (58). Consequently,

$$V_p(x_p) = \frac{1}{2}(M+m)\dot{q}^2 + \frac{1}{2}(I+me^2)\dot{\theta}^2 + me\dot{q}\dot{\theta}\cos\theta + \frac{1}{2}kq^2 + mge(1-\cos\theta), \tag{61}$$

where $x_p = [q, \dot{q}, \theta, \dot{\theta}]^T$. We assume the RTAC parameters as in Bupp, Bernstein, and Coppola (1996) and summarized in Table 1.

Similar to the design of Subsection 3.1, the continuous-time dynamics of the controller emulate a lossless absorber system. However, since the control input is torque, the resetting

Table 1. Summary of RTAC parameters.

Cart Mass	M	65.50	oz
Arm Mass	m	2.28	oz
Spring Constant	k	18.60	oz/in
Eccentricity	e	2.43	in
Arm Inertia	I	0.74	oz-in

virtual absorber subsystem is based on the emulation of rotational spring and inertia elements. Consequently, the continuous-time dynamics of the closed-loop system behave like the system shown in Figure 8.

The resetting virtual absorber controller design is thus given by

$$\begin{aligned} x_c &= \begin{bmatrix} x_{c1} \\ x_{c2} \end{bmatrix} = \begin{bmatrix} q_c \\ \dot{q}_c \end{bmatrix}, \quad f_c(x_c, y) = \begin{bmatrix} x_{c2} \\ -\frac{k_c}{m_c}(x_{c1} - \theta) \end{bmatrix}, \\ h_c(x_c, y) &= k_c(x_{c1} - \theta), \end{aligned} \tag{62}$$

$$V_c(x_c, y) = \frac{1}{2}k_c(x_{c1} - \theta)^2 + \frac{1}{2}m_c x_{c2}^2, \tag{63}$$

$$\Delta x_c = \rho_c(x_c, \theta) = \begin{bmatrix} \theta - x_{c1} \\ -x_{c2} \end{bmatrix}. \tag{64}$$

We assume the initial condition of the controller to be

$$x_c(0) = \begin{bmatrix} \theta(0) \\ 0 \end{bmatrix}. \tag{65}$$

The resetting set (53) becomes

$$M \triangleq \left\{x : \rho_c(x_c, \theta) \neq 0 \text{ and } k_c\dot{\theta}(q_c - \theta) \geq 0\right\}. \tag{66}$$

Since $\rho_c(x_c(0), \theta(0)) = 0$, it is clear from (65) and (66) that assumption A1 is satisfied, that is, $x(0) \notin \mathcal{M}$.

To show that assumption A2 holds in this case, we show that upon reaching a nonequilibrium point $x(t) \notin \mathcal{M}$ that is in the closure of $\mathcal{M}$, the continuous-time dynamics remove $x(t)$ from cl $\mathcal{M}$, and thus necessarily move the trajectory a finite distance away from $\mathcal{M}$. If $x(t) \notin \mathcal{M}$ is an equilibrium point, then

$$x(s) \notin \mathcal{M}, \quad s \geq t, \tag{67}$$

which is also consistent with assumption A2.

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_p(x_p(t)) = k_c\dot{\theta}(t)(q_c(t) - \theta(t)), \tag{68}$$

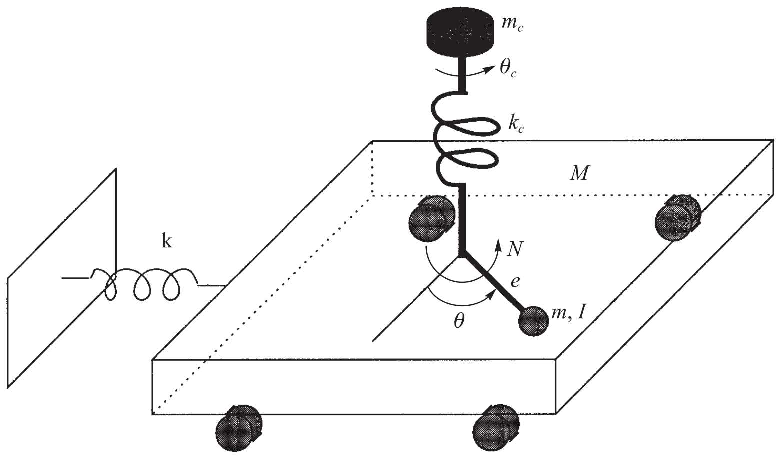

Figure 8. In the absence of resetting, the continuous-time dynamics of the RTAC with the one-way virtual absorber behaves like the above system.

and thus the closure of $\mathcal{M}$ is

$$\mathrm{cl}\,\mathcal{M} = \left\{x : k_c\dot{\theta}\,(q_c - \theta\,) \geq 0\right\}. \tag{69}$$

Furthermore, the points x^* satisfying $\rho(x^*) = 0$ have the form

$$x^* \triangleq \begin{bmatrix} q \\ \dot{q} \\ \theta \\ \dot{\theta} \\ \theta \\ 0 \end{bmatrix}, \tag{70}$$

that is, $q_c = \theta$ and $\dot{q}_c = 0$. It follows that $x^* \notin \mathcal{M}$, although $x^* \in \mathrm{cl}\,\mathcal{M}$.

To show that the continuous-time dynamics remove x^* from $\mathrm{cl}\,\mathcal{M}$, we compute

$$\frac{1}{k_c}\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_p(x_p(t)) = \ddot{\theta}\,(q_c - \theta\,) - \dot{\theta}\,(\dot{q}_c - \dot{\theta}\,), \tag{71}$$

$$\frac{1}{k_c}\frac{\mathrm{d}^3}{\mathrm{d}t^3}V_p(x_p(t)) = \theta^{(3)}(q_c - \theta\,) + 2\ddot{\theta}\,(\dot{q}_c - \dot{\theta}\,) + \dot{\theta}\,(\ddot{q}_c - \ddot{\theta}\,), \tag{72}$$

$$\begin{aligned} \frac{1}{k_c}\frac{\mathrm{d}^4}{\mathrm{d}t^4}V_p(x_p(t)) &= \theta^{(4)}(q_c - \theta\,) + 3\theta^{(3)}(\dot{q}_c - \dot{\theta}\,) \\ &+ 3\ddot{\theta}\,(\ddot{q}_c - \ddot{\theta}\,) + \dot{\theta}\,(q_c^{(3)} - \theta^{(3)}). \end{aligned} \tag{73}$$

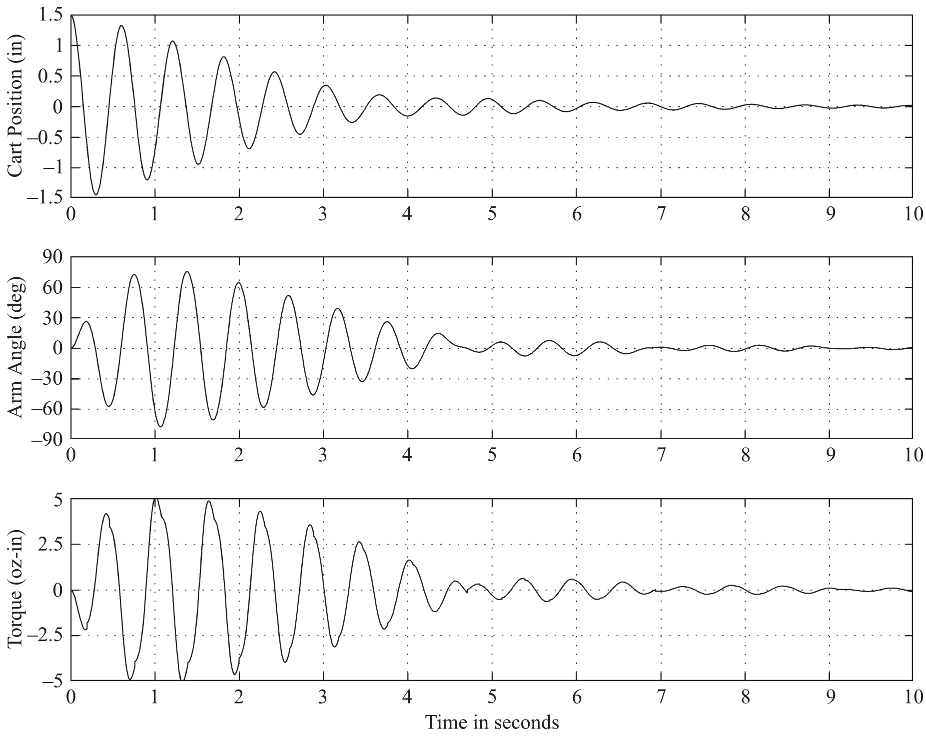

Figure 9. These plots illustrate the performance of a virtual resetting absorber controller applied to the RTAC. All of the dissipation is achieved by resetting the controller states whenever the plant energy becomes nondecreasing. The initial condition is $q(0) = 1.5$ inch, $\dot{q}(0) = 0$, $\theta(0) = q_c(0) = 0$, $\dot{\theta}(0) = 0$, $\dot{q}_c(0) = 0$. The time history of the displacement of the oscillator mass is shown in the top plot, the time history of the rotor angle is given in the middle plot, and the control torque is given in the bottom plot.

Since

$$\frac{1}{k_c}\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_p(x_p(t))\bigg|_{x^*} = -\dot{\theta}^{\,2}, \tag{74}$$

it follows that if $\dot{\theta} \neq 0$, then the continuous-time dynamics remove x^* from $\mathrm{cl}\,\mathcal{M}$. If $\dot{\theta} = 0$, then it follows from (71) through (73) that

$$\frac{1}{k_c}\frac{\mathrm{d}^2}{\mathrm{d}t^2}V_p(x_p(t))\bigg|_{x^*,\dot{\theta}=0} = 0, \tag{75}$$

$$\frac{1}{k_c}\frac{\mathrm{d}^3}{\mathrm{d}t^3}V_p(x_p(t))\bigg|_{x^*,\dot{\theta}=0} = 0, \tag{76}$$

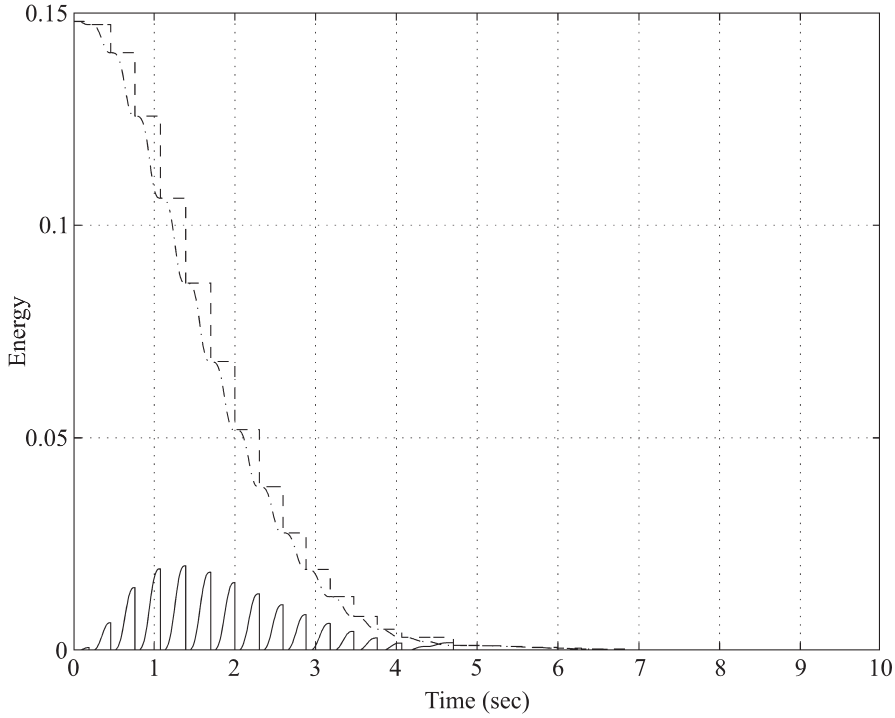

Figure 10. This plot shows the time history of the energy components for the RTAC controlled by a state-dependent resetting virtual absorber. All of the reduction in the total energy (dashed line) is achieved through resetting the controller states. The plant energy is given by the dash-dot line, and the emulated energy is given by the solid line.

$$\frac{1}{k_c}\frac{\mathrm{d}^4}{\mathrm{d}t^4}V_p(x_p(t))\bigg|_{x^*,\dot{\theta}=0} = -3\ddot{\theta}^{\,2}, \tag{77}$$

where in the evaluation of (77) we use the fact that if $q_c = \theta$ and $\dot{q}_c = 0$, then $\ddot{q}_c = 0$, which follows immediately from the continuous-time dynamics (62). Since if $\dot{\theta} = 0$ and $\ddot{\theta} \neq 0$, then the lowest-order nonzero time derivative of $\dot{V}_p$ is negative, it follows that the continuous-time dynamics remove x^* from cl $\mathcal{M}$. However, if $\dot{\theta} = 0$ and $\ddot{\theta} = 0$, then it follows from the continuous-time dynamics that x^* is necessarily an equilibrium position, in which case the trajectory never again enters $\mathcal{M}$. We can conclude therefore that assumption A2 is indeed valid for this system. Also, since $\rho(x + \rho(x)) = 0$, it follows that if $x \in \mathcal{M}$, then $x + \rho(x) \notin \mathcal{M}$, and thus assumption A3 holds.

To illustrate this approach numerically, we choose $m_c = 9.23e - 5$ and $k_c = 8.45e - 3$. The states of the controller are reset whenever $x \in \mathcal{M}$. The response of the RTAC with this state-dependent resetting virtual absorber is given in Figure 9. A plot of the time histories of the total energy, plant energy, and emulated energy is given in Figure 10.

4. CONCLUSIONS

In this paper, we have developed a general framework for describing resetting virtual absorbers for control. Two types of resetting virtual absorbers—time-dependent resetting virtual absorbers and state-dependent resetting virtual absorbers—are fully developed, and examples are given to illustrate the ability of these controllers to remove energy from linear and nonlinear vibrating systems. A remaining theoretical issue is the proof of asymptotic stability for systems whose continuous time dynamics are Lyapunov stable and whose reset law causes V to strictly decrease at each resetting time, that is, (25) is satisfied with equality and (26) is satisfied with strict inequality. Finally, while the paper focused on unforced systems, analysis of virtual resetting absorbers for systems with forcing remains a topic for future research.

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