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Finite-Time Stability of Homogeneous Systems

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Abstract

This paper examines finite-time stability of homogeneous systems. The main result is that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative degree of homogeneity.

1. Introduction

Most of the available techniques for feedback stabilization lead to closed-loop systems with Lipschitzian dynamics. The convergence in such systems can at best be exponential with infinite settling time. Finite-time convergence, however, implies nonuniqueness of solutions (in backward time) which is not possible in the presence Lipschitz continuous dynamics. It is interesting to study systems that exhibit finite-time convergence, not only because of the faster convergence, but also because such systems seem to perform better in the presence of uncertainties and disturbances [2, 7].

Homogeneous systems have attracted attention in recent years as a means of studying the stability or stabilizability of general nonlinear systems [3, 4]. However, homogeneous systems of negative degree as well finite-time stable homogeneous systems have not been treated in the literature. In this paper, we show that there exists a connection between the two.

Our main result is that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has negative degree of homogeneity. This result offers considerable simplification over sufficient Lyapunov conditions that involve differential inequalities [1, 7].

We also show that a finite-time stable homogeneous system has a smooth homogeneous Lyapunov function that satisfies a finite-time differential inequality.

2. Finite-time Stability

Consider the system

$$\dot{y}(t) = f(y(t)), \tag{1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0) = 0$. We will assume that for every initial condition in $\mathbb{R}^n$, (1) possesses a unique solution in forward time which is defined on $[0, \infty)$. In this case, the solutions of (1) define a continuous *global semi-flow* ψ on $\mathbb{R}^n$.

Definition 1. The origin is said to be a *finite-time-stable equilibrium* of (1) if there exists an open neighborhood $\mathcal{N}$ of the origin and a function $T\colon \mathcal{N} \to [0, \infty)$, called the *settling time function* such that the following statements hold:

1. $T(0) = 0$ and $T(x) \to 0$ as $x \to 0$.
2. For every $x \in \mathcal{N}\backslash\{0\}$, $\psi_t(x) \in \mathcal{N}\backslash\{0\}$, $t \in [0, T(x))$, and $\psi_t(x) = 0$ for all $t \geq T(x)$.
3. For every open set $\mathcal{U}_\varepsilon$ such that $0 \in \mathcal{U}_\varepsilon \subseteq \mathcal{N}$, there exists an open subset $\mathcal{U}_\delta$ of $\mathcal{N}$ containing 0, such that for every $x \in \mathcal{U}_\delta\backslash\{0\}$, $\psi_t(x) \in \mathcal{U}_\varepsilon$, $t \geq 0$.

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The origin is said to be a *globally finite-time stable equilibrium* if it is a finite-time stable equilibrium and $\mathcal{N}$ can be chosen to be $\mathbb{R}^n$.

Note that *ii)* implies that T is positive definite on $\mathcal{N}$. It can also be shown that under the various assumptions above, T is continuous on $\mathcal{N}$. It should be pointed out that *ii)* and *iii)* in Definition 1 above do not imply *i)* [7]. The following result appears in [1] and is reproduced here for completeness.

Theorem 1. Suppose there exists an open neighborhood $\mathcal{V}$ of the origin, a C^1 positive-definite function $V : \mathcal{V} \to \mathbb{R}$ and real numbers $k > 0$ and $\alpha \in (0, 1)$, such that $\dot{V} + kV^\alpha$ is negative semidefinite on $\mathcal{V}$, where $\dot{V}(x) = \frac{\partial V}{\partial x}(x)f(x)$. Then the origin is a finite-time stable equilibrium of (1). Moreover, if T is the settling time function, then $T(x) \leq \frac{1}{k(1-\alpha)}V(x)^{1-\alpha}$ for all x in some open neighborhood of the origin.

It can be shown that if the origin is a finite-time equilibrium, then there exists a continuous Lyapunov function satisfying the hypotheses of Theorem 1 [1]. However, there are systems with finite-time stable equilibria for which there exists no continuously differentiable Lyapunov function satisfying the hypotheses of Theorem 1 [7].

3. Homogeneity

We adopt the intrinsic coordinate-free approach to homogeneity described in [4]. Let ν be a smooth (at least C^1) complete vector field on $\mathbb{R}^n$ such that the origin is a globally exponentially stable equilibrium of the differential equation $\dot{y}(t) = -\nu(y(t))$. Let Φ denote the globally defined flow of ν. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous* of degree $l \in \mathbb{R}$ with respect to ν if

$$V \circ \Phi_t = e^{lt}V \tag{2}$$

for $t \in \mathbb{R}$. The flow Φ_t carries level sets of a homogeneous function to level sets. The vector field f on $\mathbb{R}^n$ is said to be homogeneous of degree m with respect to ν if

$$f \circ \Phi_t(x) = e^{mt}(\Phi_{t*})_x(f(x)) \tag{3}$$

for $t \in \mathbb{R}$, where $(\Phi_{t*})_x$ denotes the *push-forward* at $x \in \mathbb{R}^n$ of the diffeomorphism Φ_t. The flow Φ_t carries integral curves of f to integral curves of f after a reparametrization [4]. More precisely,

$$\Phi_s \circ \psi_t = \psi_{e^{-ms}t} \circ \Phi_s. \tag{4}$$

Often, we shall simply use "homogeneous" to mean "homogeneous with respect to ν"

It is a simple matter to verify using equations (2) and (3) that for V a homogeneous function of degree l and f a homogeneous vector field of degree m, the Lie-derivatives $L_\nu V$ and $L_\nu f$ are defined and satisfy $L_\nu V = lV$, $L_\nu f = mf$ [4] and $L_f V$, if defined, is homogeneous of degree $m + l$ [4].

It can be shown using (2) that there exists no continuous homogeneous function of negative degree. It is perhaps for this reason that homogeneity with negative degree is not usually considered. However, homogeneity with negative degree need not pose such problems in the case of vector fields.

The map $\Delta : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\Delta_k(x) = \Phi_{\ln(k)}(x)$, $k > 0, x \in \mathbb{R}^n$ is called the *homogeneous dilation* associated with the vector field ν while ν is said to be the *Euler vector field* of the dilation Δ. Functions and vector fields homogeneous with respect to ν are said to be homogeneous with respect to the corresponding dilation Δ.

The dilations often considered in the literature [3, 4] are of the form $\Delta_k(x_1, \ldots, x_n) = (k^{r_1}x_1, \ldots, k^{r_n}x_n)$ where $x_1, \ldots, x_n$ are suitable coordinates on $\mathbb{R}^n$ and $r_1, \ldots, r_n$ are positive real numbers. The Euler vector field of such a dilation is given by $\nu = \sum_{i-1}^{n} r_i x_i \frac{\partial}{\partial x_i}$, which is linear and has the flow $\Phi_t(x) = \Delta_{e^t}(x)$. Using (2), it is easy to see that a function V is homogeneous of degree l if and only if $V(k^{r_1}x_1, \ldots, k^{r_n}x_n) = k^l V(x_1, \ldots, x_n)$, $k > 0$. A vector field f is homogeneous of degree m if and only if the ith component f_i is homogeneous of degree $m + r_i$. It follows that a continuous vector field can have negative degree m for $m > -\min_{i=1,\ldots,n} r_i$.

Example 1. Consider the vector field $f = f_1(x_1,x_2)\frac{\partial}{\partial x_1} + f_2(x_1,x_2)\frac{\partial}{\partial x_2}$ on $\mathbb{R}^2$ with $f_1(x_1,x_2) = x_2$ and $f_2(x_1,x_2) = -\frac{1}{m}\text{sign}(x_2)|x_2|^\alpha - \frac{1}{m}\text{sign}(x_1)|x_1|^{\frac{\alpha}{2-\alpha}}$, where $\alpha \in (0,1)$, $m > 0$. We notice that $f_1(k^{2-\alpha}x_1, kx_2) = kx_2 = k^{(2-\alpha)+(\alpha-1)}f_1(x_1,x_2)$, while $f_2(k^{2-\alpha}x_1, kx_2) = k^\alpha f_2(x_1,x_2) = k^{1+(\alpha-1)}f_2(x_1,x_2)$. Hence, we conclude that the vector field f is homogeneous of negative degree $\alpha - 1$ with respect to the dilation, $\Delta_k(x_1,x_2) = (k^{2-\alpha}x_1, kx_2)$.

4. Finite-time Stability and Homogeneity

In this section we consider the finite-time stability of (1) under the assumption that f is homogeneous of degree m with respect to a vector field ν having the flow Φ. It will be instructive to first study the finite-time stablity of scalar homogeneous systems. The continuous scalar system

$$\dot{y}(t) = -c\,\text{sign}(y(t))|y(t)|^\alpha \tag{5}$$

is homogeneous of degree $1-\alpha$, $\alpha > 0$, with respect to the dilation $\Delta_k(x) = kx$. The following observations can easily be made by directly integrating (5).

1. If the origin is finite-time stable, the degree of homogeneity is negative (that is, $\alpha < 1$).
2. The origin is finite-time stable if and only if it is asymptotically stable ($c > 0$) and the degree of homogeneity is negative ($\alpha < 1$).

The following proposition is a generalization of the first observation above. The proof uses the property that Φ carries integral curves of f to integral curves of f.

Proposition 1. Suppose the origin is a finite-time stable equilibrium of f. Then the origin is globally finite-time stable, the settling time T is homogeneous of degree $-m$, and $m < 0$.

The following lemma is needed in the proofs of subsequent results.

Lemma 1. If V is a continuous positive-definite function homogeneous of degree l such that $L_f V$ is continuous, then for all $x \in \mathbb{R}^n$

$$L_f V(x) \leq \left(\max_{\{z:V(z)=1\}} L_f V(z) \right) [V(x)]^{\frac{l+m}{l}}. \tag{6}$$

The following proposition provides a converse to Theorem 1. The proof uses Proposition 1, Lemma 1 and the converse Lyapunov result given in [3].

Proposition 2. Suppose the origin is a finite-time stable equilibrium of f. Then there exists $k > 0$, $\alpha \in (0,1)$ and a C^∞ homogeneous Lyapunov function V such that

$$L_f V(x) \leq -k[V(x)]^\alpha. \tag{7}$$

The following theorem, which is a generalization of observation 2 above, is the main result of this paper.

Theorem 2. The origin is a finite-time stable equilibrium of f if and only if the origin is an asymptotically stable equilibrium of f and $m < 0$.

Example 2. As an application of Theorem 1, we prove that for $\alpha \in (0,1)$, the feedback law $u = \phi(x_1,x_2) = -\text{sign}(x_2)|x_2|^\alpha - \text{sign}(x_1)|x_1|^{\frac{\alpha}{2-\alpha}}$ renders the origin finite-time stable for the double integrator

$$\dot{x_1} = x_2, \ \dot{x_2} = \frac{u}{m}. \tag{8}$$

The closed-loop system is given by the vector-field f in Example 1 where we have seen that for $\alpha < 1$, f is homogeneous of negative degree. To show asymptotic stability, consider the Lyapunov function $V(x_1,x_2) = \frac{1}{2}mx_2^2 + \frac{2-\alpha}{2}|x_1|^{\frac{2}{2-\alpha}}$ which is C^1 for $\alpha \in (0,1)$. We compute $L_f V(x_1,x_2) = |x_2|^{1+\alpha}$. The only invariant set in $\{(x_1,x_2) : L_f V(x_1,x_2) = 0\}$ is the origin and by LaSalle's theorem, the closed-loop system is asymptotically stable. Theorem 2 above now guarantees finite-time stability. Note that the controller ϕ does not depend on the mass m.

The simplification provided by Theorem 2 over Theorem 1 can be seen by comparing the proof of finite-time stability in Example 2 above to that given for the controllers proposed in [5, 6].

5. Conclusions

This paper establishes a connection between the rate of convergence in a homogeneous system and the degree of homogeneity. Our main result reduces checking for finite-time stability to checking for asymptotic stability along with a simple algebraic computation of the degree of homogeneity and thus offers considerable simplification over Lyapunov conditions involving differential inequalities.

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