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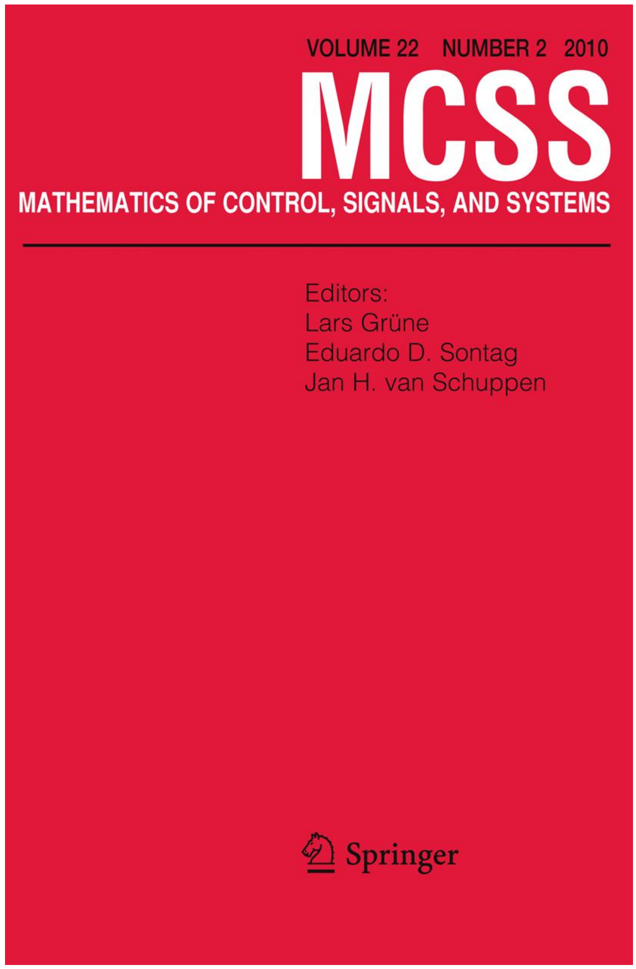

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ORIGINAL ARTICLE

# Arc-length-based Lyapunov tests for convergence and stability with applications to systems having a continuum of equilibria

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**Abstract** In this paper, fundamental relationships are established between convergence of solutions, stability of equilibria, and arc length of orbits. More specifically, it is shown that a system is convergent if all of its orbits have finite arc length, while an equilibrium is Lyapunov stable if the arc length (considered as a function of the initial condition) is continuous at the equilibrium, and semistable if the arc length is continuous in a neighborhood of the equilibrium. Next, arc-length-based Lyapunov tests are derived for convergence and stability. These tests do not require the Lyapunov function to be positive definite. Instead, these results involve an inequality relating the right-hand side of the differential equation and the Lyapunov function derivative. This inequality makes it possible to deduce properties of the arc length function and thus leads to sufficient conditions for convergence and stability. Finally, it is shown that the converses of all the main results hold under additional assumptions. Examples are included to illustrate how our results are particularly suited for analyzing stability of systems having a continuum of equilibria.

**Keywords** Lyapunov stability · Semistability · Convergence · Arc length · Continuum of equilibria · Consensus

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## 1 Introduction

This paper explores fundamental relationships between convergence, semistability, and arc length of orbits. Convergence is the notion that every trajectory of the system converges to a limit point. The limit point, which is necessarily an equilibrium, depends in general on the initial conditions. In a convergent system, the limit points of trajectories may or may not be Lyapunov stable. Semistability is the additional requirement that trajectories converge to limit points that are Lyapunov stable. More precisely, an equilibrium is semistable if it is Lyapunov stable and every trajectory starting in a neighborhood of the equilibrium converges to a (possibly different) Lyapunov stable equilibrium. For linear systems, semistability was originally defined in [1] and applied to matrix second-order systems in [2]. Reference [3] extends the notion of semistability to nonlinear systems and gives Lyapunov-based sufficient conditions for convergence and semistability using the geometric property of nontangency.

Semistability, rather than asymptotic stability, is the appropriate notion of stability in several applications. For instance, an aircraft subject to initial lateral perturbations from level trimmed flight will recover trimmed flight with an offset in the final heading angle that depends on the initial perturbation. The concentrations of reacting species in many chemical reactions converge to limiting values that depend on the initial concentrations [4–7]. The limiting values of the feedback controller gains in an adaptive closed-loop system depend on the initial conditions of the plant states [8–11]. Matrix dynamical systems such as the double bracket equation [12] are isospectral, that is, the evolving matrix state has constant spectrum. Under certain conditions, the matrix trajectories of isospectral matrix dynamical systems such as the double bracket equation converge such that the limit point of each trajectory is a canonical form of the initial matrix state [12]. The set of consensus states of a multi-agent system exhibiting consensus behavior is usually a continuum, and the limiting consensus state is determined by the initial state of the multi-agent system (see [13–15] and references contained therein). The divers dynamics that arise in all the applications mentioned above have two common features. First, the set of equilibria in all these applications is a continuum. Second, unlike the trajectories of an asymptotically stable system, the trajectories in these applications converge to limits that are determined by the initial conditions. Hence semistability, rather than asymptotic stability, is the appropriate notion of stability for all these applications. The notions of convergence and semistability are also relevant to the dynamics of neural networks [16,17] as well as gradient flows and gradient descent algorithms [18].

Convergence as well as semistability imply that, at least locally, the trajectories converge to the set of equilibria. Mere attractivity of the set of equilibria, however, does not imply convergence of trajectories or semistability of equilibria as examples given in [3] show. Even when the set of equilibria is attractive, convergence and semistability depend on the local behavior of the dynamical system near the set of equilibria. Previous sufficient conditions on the local dynamical behavior near an attractive set of equilibria that guarantee convergence of trajectories include hyperbolic transversality [19] and nontangency [3]. In this paper, we give alternative sufficient conditions based on arc length of trajectories. The results that we present complement those of [3,19] in the sense that our results may yield conclusions in situations where the results of [3,19] are not applicable.

In Sect. 3, we introduce the arc length function, which associates with each initial condition the length of its orbit, and show that the arc length function is lower semicontinuous. We also show that if a trajectory converges to a limit, then the continuity properties of the arc length function at every point along the trajectory are determined by the continuity properties of the arc length function at the limit point of the trajectory. These results prove useful for the main results of subsequent sections.

In Sect. 4, we relate arc length to convergence. Specifically, we show that the system is convergent if every orbit has finite arc length. We use this intuitively expected result to obtain a Lyapunov-like sufficient condition for convergence. The sufficient condition requires the Lyapunov function to be only lower semicontinuous, and does not require it to be sign definite. Instead, the sufficient condition involves an inequality that relates the norm of the vector field to the Lyapunov function derivative. This inequality is a sufficient condition for orbits to have a finite arc length. It should be noted, however, that our sufficient condition for convergence does not imply Lyapunov stability of the limits of trajectories.

In Sect. 5, we establish fundamental relationships between the continuity properties of the arc length function and the stability of equilibria. Specifically, we show that if the arc length function is continuous at an equilibrium, then the equilibrium is Lyapunov stable, while if the arc length function is continuous in a neighborhood of the equilibrium, then the equilibrium is semistable. This fact leads to a novel arc-length-based Lyapunov test for Lyapunov stability, semistability, and asymptotic stability. This Lyapunov test requires the Lyapunov function to be continuous and have a local minimum at the equilibrium, that is, the Lyapunov function is required to be locally positive semidefinite with respect to its value at the equilibrium. However, the Lyapunov function is not required to be positive definite. Instead, as in the case of the arc-length-based Lyapunov condition for convergence presented in Sect. 4, the Lyapunov test for stability involves an inequality relating the vector field and the Lyapunov function derivative. The inequality is used along with properties of the Lyapunov function to deduce continuity properties of the arc length function and thus prove stability.

The Lyapunov results of Sects. 4 and 5 are especially suited for analyzing systems having a continuum of equilibria, because they make it possible to draw stability conclusions for a continuum of equilibria using a single Lyapunov function. To illustrate this feature as well as to indicate possible application areas, we apply our results to three examples involving systems having continuum of equilibria in Sect. 6. In the first example, we consider a system introduced in [3] and show how the results of this paper can be used to conclude stability for a larger range of parameter values than possible with the results of [3]. In the second example, we use our results to show that a system of three interacting agents achieves consensus under purely structural conditions on the information flow. Finally, in the third example, we apply our results to the kinetics of the Michaelis–Menten chemical reaction.

In general, a trajectory that converges to a limit may have infinite arc length, as an example in Sect. 7.1 demonstrates. However, it is intuitively clear that such a trajectory would have to curl up upon itself. In other words, a trajectory that does not curl upon itself can converge to a limit only if it has finite arc length. To capture this intuitive idea, we consider the limiting direction set of a vector field introduced in [3]. The limiting direction set of the vector field $f$ at a point $x$ is the set of limit points of the unit vector along $f(z)$ as $z$ approaches $x$. The condition that a convergent trajectory should not curl up upon itself is captured in the condition that no connected component of the limiting direction set at the limit point contains the zero vector in its convex hull. In Sect. 7.1, we show that if this condition is satisfied at every equilibrium, then every convergent trajectory has finite arc length. This fact yields a partial converse to the Lyapunov-based sufficient condition for convergence given in Sect. 4.

In general, converses of the results of Sect. 5 mentioned above may not hold. Thus, the arc length function may not be continuous, bounded, or even defined in a neighborhood of an equilibrium that is Lyapunov stable or semistable. In Sect. 7.2, we give an example of a system having a semistable equilibrium such that the arc length function is defined everywhere but is unbounded in every neighborhood of the equilibrium. However, we show that in the case where no connected component of the limiting direction set at the equilibrium contains the zero vector in its convex hull, the arc length function is continuous at the equilibrium if the equilibrium is Lyapunov stable, and continuous in a neighborhood of the equilibrium if it is semistable. This fact leads to a partial converse of the arc-length-based Lyapunov result given in Sect. 5.

At this point, it is appropriate to mention that the idea of deducing convergence and stability from the arc length of trajectories is not new, and has been used by Łojasiewicz in [20] as far back as 1984 to show that every trajectory of the gradient flow associated with a real analytic function has at most one limit point. Subsequently, Łojasiewicz's idea and technique were used to study stability and convergence properties of gradient flows and gradient descent algorithms in [18,21,22]. Later extensions to nongradient systems and to a nonsmooth setting can be found in [23] and [24,25], respectively. The result that underlies all this work is the Łojasiewicz inequality between an analytic function and the norm of its gradient vector. In all the work mentioned above, the Łojasiewicz inequality is used to construct a Lyapunov function satisfying the differential inequality that we present in Sects. 4 and 5 below. The required stability and convergence conclusions are then deduced by combining the same arguments that we use in this paper. Our treatment clarifies these arguments by explicitly presenting them in the form of results that relate stability and convergence to properties of the arc length function, and results that use Lyapunov functions to deduce properties of the arc length function. Our results do not involve analyticity assumptions and are not restricted to gradient systems. In addition, we also examine conditions under which converses of the main results hold and provide examples to indicate how our results might apply to problems of general interest.

As mentioned earlier, chemical kinetics comprise one of the application areas for semistability theory. Since the kinetic equation for a system of chemical reactions governs concentrations of the reacting species, all solutions of physical interest take values in the nonnegative orthant. For such systems, which evolve on possibly closed, positively invariant subsets of $\mathbb{R}^n$, it is natural to consider relative stability, that is, stability with respect to perturbed initial conditions that belong to the positively invariant subset. Therefore, with applications to nonnegative dynamics in mind, we consider relative stability of dynamical systems that evolve on not-necessarily-open subsets of $\mathbb{R}^n$. Relative stability has been considered previously in [26,27].

Finally, we mention that this paper is a significantly extended version of the conference paper [28] which itself improves upon the results of [29]. Specifically, this paper contains several additional counterexamples, examples and proofs, and provides more elaborate connections between our work and prior literature.

## 2 Preliminaries

Let $\mathcal{G} \subseteq \mathbb{R}^n$ and let $\|\cdot\|$ denote a norm on $\mathbb{R}^n$. A subset $\mathcal{U}$ of $\mathcal{G}$ is *relatively open* in $\mathcal{G}$ if $\mathcal{U}$ is open in the subspace topology induced on $\mathcal{G}$ by the norm $\|\cdot\|$. Given $\mathcal{K} \subseteq \mathcal{G}$, we let int $\mathcal{K}$ and bd $\mathcal{K}$ denote the interior and boundary, respectively, of $\mathcal{K}$ in the subspace topology on $\mathcal{G}$. Thus, int $\mathcal{K}$ is the largest subset of $\mathcal{K}$ that is relatively open in $\mathcal{G}$, while bd $\mathcal{K} = (\overline{\mathcal{K}} \cap \mathcal{G}) \backslash \mathrm{int}\, \mathcal{K}$, where $\overline{\mathcal{K}}$ denotes the closure of $\mathcal{K}$ in $\mathbb{R}^n$. A set $\mathcal{U} \subseteq \mathcal{G}$ is *relatively bounded* in $\mathcal{G}$ if $\overline{\mathcal{U}}$ is compact and contained in $\mathcal{G}$. A point $x \in \mathbb{R}^n$ is a *subsequential limit* of a sequence $\{x_i\}$ in $\mathbb{R}^n$ if there exists a subsequence of $\{x_i\}$ that converges to $x$ in the norm $\|\cdot\|$. Recall that every bounded sequence has at least one subsequential limit. A *divergent sequence* is a sequence having no convergent subsequence. When there is no possibility of confusion, we will use "relatively open (bounded)" instead of "relatively open (bounded) in $\mathcal{G}$". Also, in the case $\mathcal{G} = \mathbb{R}^n$, we will use "open (bounded)" instead of "relatively open (bounded)".

Consider the system of differential equations

$$\dot{y}(t) = f(y(t)), \tag{1}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is continuous on the open set $\mathcal{D} \subseteq \mathbb{R}^n$. We assume that, for every initial condition $y(0) \in \mathcal{D}$, the differential equation (1) possesses a unique right-maximally defined $\mathrm{C}^1$ solution, and this solution is defined on $[0, \infty)$. Letting $\psi(\cdot, x)$ denote the right-maximally defined solution of (1) that satisfies the initial condition $y(0) = x$, the above assumptions imply that the map $\psi : [0, \infty) \times \mathcal{D} \to \mathcal{D}$ is continuous [30, Thm. V.2.1], satisfies $\psi(0, x) = x$ and possesses the semigroup property, that is, $\psi(t, \psi(h, x)) = \psi(t + h, x)$ for all $t, h \geq 0$ and $x \in \mathcal{D}$. Given $t \geq 0$ and $x \in \mathcal{D}$, it will often be convenient to denote the map $\psi(t, \cdot) : \mathcal{D} \to \mathcal{D}$ by $\psi_t$ and the map $\psi(\cdot, x) : [0, \infty) \to \mathcal{D}$ by $\psi^x$. The *orbit* $\mathcal{O}_x$ of a point $x \in \mathcal{D}$ is the set $\psi^x([0, \infty))$.

A set $\mathcal{U} \subseteq \mathbb{R}^n$ is *positively invariant* if $\psi_t(\mathcal{U}) \subseteq \mathcal{U}$ for all $t \geq 0$. The set $\mathcal{U}$ is *invariant* if $\psi_t(\mathcal{U}) = \mathcal{U}$ for all $t \geq 0$.

In the rest of the paper, $\mathcal{G} \subseteq \mathcal{D}$ will denote a positively invariant set so that $\mathcal{O}_x \subseteq \mathcal{G}$ for all $x \in \mathcal{G}$.

An *equilibrium point* of (1) is a point $x \in \mathcal{D}$ satisfying $f(x) = 0$ or, equivalently, $\psi(t, x) = x$ for all $t \geq 0$. We let $\mathcal{E} \overset{\text{def}}{=} f^{-1}(0) \cap \mathcal{G}$, the set of all equilibrium points of (1) in $\mathcal{G}$. An *isolated* equilibrium is an isolated point of $\mathcal{E}$.

**Definition 2.1** The *domain of boundedness* of the system (1) is the set $\mathcal{B}$ of points $z \in \mathcal{G}$ such that $\mathcal{O}_z$ is bounded relative to $\mathcal{G}$. The *domain of convergence* of the system (1) is the set $\mathcal{R}$ of points $z \in \mathcal{G}$ such that $\lim_{t\to\infty} \psi(t, z)$ exists and is contained in $\mathcal{G}$. The system (1) is *convergent* relative to $\mathcal{G}$ if $\mathcal{G} = \mathcal{R}$, that is, for every $x \in \mathcal{G}$, $\lim_{t\to\infty} \psi(t, x)$ exists and is contained in $\mathcal{G}$.

Clearly, $\mathcal{B}$ and $\mathcal{R}$ are positively invariant and $\mathcal{E} \subseteq \mathcal{R} \subseteq \mathcal{B}$. The dynamics (1) give rise to a function $\psi_\infty : \mathcal{R} \to \mathcal{G}$ defined by $\psi_\infty(x) = \lim_{t\to\infty} \psi(t, x),\ x \in \mathcal{R}$. It follows from the continuity and semigroup property of $\psi$ that, for all $h \geq 0$ and all $x \in \mathcal{R}$, $\psi_h(\psi_\infty(x)) = \lim_{t\to\infty} \psi(t+h, x) = \psi_\infty(x)$. Thus $\psi_\infty(x) \in \mathcal{E}$ for all $x \in \mathcal{G}$. Consequently, $\psi_\infty \circ \psi_\infty = \psi_\infty$, $\psi_\infty(\mathcal{R}) = \psi_\infty(\mathcal{E}) = \mathcal{E}$, and $\psi_\infty(x) = x$ if and only if $x \in \mathcal{E}$

**Definition 2.2** An equilibrium point $x \in \mathcal{E}$ is *Lyapunov stable* relative to $\mathcal{G}$ if, for every relatively open neighborhood $\mathcal{U}_\varepsilon \subseteq \mathcal{G}$ of $x$, there exists a relatively open neighborhood $\mathcal{U}_\delta \subseteq \mathcal{G}$ of $x$ such that $\psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$ for all $t \geq 0$.

The following result links the continuity of the function $\psi_\infty$ at a point $x$ to the stability of the equilibrium point $\psi_\infty(x)$.

**Proposition 2.3** *Let $x \in \operatorname{int} \mathcal{R}$. If $\psi_\infty(x)$ is Lyapunov stable relative to $\mathcal{G}$, then $\psi_\infty$ is continuous at $x$.*

*Proof* Suppose $\psi_\infty(x)$ is Lyapunov stable relative to $\mathcal{G}$. Let $\mathcal{U}_\varepsilon \subseteq \mathcal{G}$ be a relatively open neighborhood of $\psi_\infty(x)$. There exist relatively open neighborhoods $\mathcal{U} \subseteq \mathcal{G}$ and $\mathcal{U}_\delta \subseteq \mathcal{G}$ of $\psi_\infty(x)$ such that $\overline{\mathcal{U}} \subset \mathcal{U}_\varepsilon$ and $\psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}$ for all $t \geq 0$. Let $\{x_i\}$ be a sequence in $\mathcal{G}$ converging to $x$. Since $\psi_\infty(x) \in \mathcal{U}_\delta$, there exists $h > 0$ such that $\psi(h, x) \in \mathcal{U}_\delta$. Since $\psi(h, x_i) \to \psi(h, x)$ as $i \to \infty$, there exists $N$ such that, for all $i > N$, $\psi(h, x_i) \in \mathcal{U}_\delta$. Therefore, for all $t \geq 0$ and $i > N$, $\psi(t+h, x_i) \in \psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}$. Consequently, $\psi_\infty(x_i) \in \overline{\mathcal{U}} \subset \mathcal{U}_\varepsilon$ for all $i > N$. Thus $\psi_\infty(x_i) \to \psi_\infty(x)$ as $i \to \infty$ and hence $\psi_\infty$ is continuous at $x$. □

**Definition 2.4** An equilibrium point $x \in \mathcal{G}$ is *semistable* relative to $\mathcal{G}$ if there exists a relatively open neighborhood $\mathcal{U} \subseteq \mathcal{G}$ of $x$ such that $\mathcal{U} \subseteq \mathcal{R}$ and, for every $z \in \mathcal{U}$, $\psi_\infty(z)$ is Lyapunov stable relative to $\mathcal{G}$. An equilibrium point $x \in \mathcal{G}$ is *asymptotically stable* relative to $\mathcal{G}$ if $x$ is Lyapunov stable relative to $\mathcal{G}$ and there exists a relatively open neighborhood $\mathcal{U} \subseteq \mathcal{G}$ of $x$ such that $\mathcal{U} \subseteq \mathcal{R}$ and $\psi_\infty(z) = x$ for every $z \in \mathcal{U}$.

Note that if the equilibrium $x \in \mathcal{G}$ is semistable relative to $\mathcal{G}$, then every equilibrium in some relatively open neighborhood of $x$ is Lyapunov stable relative to $\mathcal{G}$. In particular, every equilibrium that is semistable relative to $\mathcal{G}$ is also Lyapunov stable relative to $\mathcal{G}$. An equilibrium that is asymptotically stable relative to $\mathcal{G}$ is an isolated equilibrium and is semistable relative to $\mathcal{G}$. Conversely, if $x \in \mathcal{G}$ is an isolated equilibrium and is semistable relative to $\mathcal{G}$, then all solutions in a sufficiently small relatively open neighborhood of $x$ converge to $x$, and thus $x$ is asymptotically stable relative to $\mathcal{G}$.

Given a function $V : \mathcal{G} \to \mathbb{R}$, a point $x \in \mathcal{G}$ is a *local minimizer* of $V$ relative to $\mathcal{G}$ if there exists a relatively open neighborhood $\mathcal{U} \subseteq \mathcal{G}$ of $x$ such that $V(x) \leq V(z)$ for all $z \in \mathcal{U}$.

Given a function $V : \mathcal{G} \to \mathbb{R}$ and $x \in \mathcal{G}$, we define $\dot{V}(x)$ to be the upper right Dini derivative of the composite function $V \circ \psi^x$ at 0. In other words, $\dot{V}(x) \overset{\text{def}}{=} \limsup_{h\to 0^+} \frac{1}{h}[V(\psi(h, x)) - V(x)]$. It is well known that if $V$ is locally Lipschitz at $x \in \mathcal{D}$ then [31, §5.1], [32, p. 353], [33, p. 3] $\dot{V}(x) = \limsup_{h\to 0^+} \frac{1}{h}[V(x +$

$hf(x)) - V(x)]$. In addition, if $V$ is continuously differentiable on $\mathcal{D}$, then $\dot{V}(x) = \frac{\partial V}{\partial x}(x)f(x),\ x \in \mathcal{D}$.

In the rest of the paper, we will assume that the positively invariant set $\mathcal{G}$ is locally compact, that is, every point in $\mathcal{G}$ is contained in a relatively open and relatively bounded subset of $\mathcal{G}$.

## 3 Arc length of orbits

Let $\mathcal{A}$ denote the set of points in $\mathcal{G}$ that have orbits with finite arc length, that is, $\mathcal{A} \stackrel{\text{def}}{=} \{x \in \mathcal{G} : \int_0^\infty \|f(\psi(t,x))\|\mathrm{d}t < \infty\}$. We define the *arc length function* $S : \mathcal{A} \to [0,\infty)$ by $S(x) \stackrel{\text{def}}{=} \int_0^\infty \|f(\psi(t,x))\|\mathrm{d}t$ for every $x \in \mathcal{A}$. It will be convenient to extend the definition of $S$ to $\mathcal{G}$ by letting $S(x) = \infty$ for every $x \in \mathcal{G}\backslash\mathcal{A}$. The semigroup property of $\psi$ implies that, for every $x \in \mathcal{A}$ and every $t \geq 0$, $S(\psi(t,x))$ is defined and $S(\psi(t,x)) \leq S(x)$. Thus $\mathcal{A}$ is positively invariant. Moreover, $\mathcal{E} \subseteq \mathcal{A}$ and $\mathcal{E} = S^{-1}(0)$.

In this section we consider continuity properties of the arc length function $S$. The following result shows that the arc length function is lower semicontinuous everywhere:

**Proposition 3.1** *The arc length function $S$ is lower semicontinuous at every point in $\mathcal{A}$. Moreover, $\dot{S}$ is defined on $\mathcal{A}$ and, for every $x \in \mathcal{A}$, $\dot{S}(x) = -\|f(x)\|$.*

*Proof* Consider $x \in \mathcal{A}$, and let $\{x_k\}$ be a sequence in $\mathcal{G}$ converging to $x$. If $\liminf_{k\to\infty} S(x_k) = \infty$, then clearly $\liminf_{k\to\infty} S(x_k) > S(x)$. Hence suppose $\liminf_{k\to\infty} S(x_k) < \infty$. There exists a subsequence $\{x_{k_i}\}_{i=1}^\infty$ of $\{x_k\}$ in $\mathcal{A}$ such that $\lim_{i\to\infty} S(x_{k_i}) = \liminf_{k\to\infty} S(x_k)$. Let $\varepsilon > 0$. There exists $T > 0$ such that $S(\psi(T,x)) = \int_T^\infty \|f(\psi(\tau,x))\|\mathrm{d}\tau < \varepsilon/2$. By Theorem II.3.2 in [30], the sequence of functions $\{\psi^{x_{k_i}}\}_{i=1}^\infty$ converges to the function $\psi^x$ uniformly on $[0,T]$. Since $\psi^x([0,T]) \subseteq \mathcal{G}$ is compact, and since $\mathcal{G}$ is locally compact, it follows that there exists a relatively bounded neighborhood $\mathcal{U} \subseteq \mathcal{G}$ of the set $\psi^x([0,T])$. Since $f$ is uniformly continuous on the compact set $\overline{\mathcal{U}}$ [34, Thm. 4.47], it follows that the sequence of functions $\{f \circ \psi^{x_{k_i}}\}_{i=1}^\infty$ converges uniformly to the function $f \circ \psi^x$ on $[0,T]$. Next, the triangle inequality implies that the sequence of functions $\{\|f \circ \psi^{x_{k_i}}(\cdot)\|\}$ converges to the function $\{\|f \circ \psi^x(\cdot)\|\}$ uniformly on $[0,T]$. It now follows from standard results on integration [34, Thm. 9.8] that there exists $I > 0$ such that, for every $i > I$, $\int_0^T \|f(\psi(\tau,x_{k_i}))\|\mathrm{d}\tau > \int_0^T \|f(\psi(\tau,x))\|\mathrm{d}\tau - \varepsilon/2$. It therefore follows that $\liminf_{k\to\infty} S(x_k) = \lim_{i\to\infty} S(x_{k_i}) \geq \lim_{i\to\infty} \int_0^T \|f(\psi(\tau,x_{k_i}))\|\mathrm{d}\tau > \int_0^T \|f(\psi(\tau,x))\|\mathrm{d}\tau - \varepsilon/2 = S(x) - S(\psi(T,x)) - \varepsilon/2 > S(x) - \varepsilon$. Since $\varepsilon$ was chosen to be arbitrary, it follows that $\liminf_{k\to\infty} S(x_k) \geq S(x)$.

The arguments above show that $\liminf_{k\to\infty} S(x_k) \geq S(x)$ for every sequence $\{x_k\}$ in $\mathcal{G}$ converging to $x$. It follows that $S$ is lower semicontinuous at $x$. The second part of the proposition follows by direct computation. □

To further investigate the continuity properties of the arc length function, let $\mathcal{C} \subseteq \mathcal{A}$ denote the set of points where the function $S$ is continuous relative to $\mathcal{G}$. It is easy to see that $\mathcal{C} \subseteq \operatorname{int} \mathcal{A}$.

The following result shows that the continuity properties of the arc length function at every point along a trajectory in the domain of convergence of (1) are determined by the continuity properties of the arc length function at the limit point of the trajectory.

**Proposition 3.2** *Suppose $z \in \mathcal{R}$. Then the following statements hold:*

(i) *If $\psi_\infty(z) \in \operatorname{int} \mathcal{A}$, then $z \in \operatorname{int} \mathcal{A}$.*
(ii) *If $\psi_\infty(z) \in \mathcal{C}$, then $z \in \mathcal{C}$.*
(iii) *If $\psi_\infty(z) \in \operatorname{int} \mathcal{C}$, then $z \in \operatorname{int} \mathcal{C}$.*

*Proof* (i) Suppose $\psi_\infty(z) \in \operatorname{int} \mathcal{A}$. Let $\mathcal{U} \subseteq \mathcal{G}$ be a relatively open neighborhood of $\psi_\infty(z)$ such that $\mathcal{U} \subseteq \operatorname{int} \mathcal{A}$. Since $\psi_\infty(z) = \lim_{t\to\infty} \psi(t, z)$, there exists $T > 0$ such that $\psi(t, x) \in \mathcal{U}$ for every $t \geq T$. By continuity of $\psi$, $\mathcal{V} \stackrel{\text{def}}{=} \psi_T^{-1}(\mathcal{U})$ is a relatively open neighborhood of $z$. Consider $w \in \mathcal{V}$. For every $h > 0$, we have $\int_0^{T+h} \|f(\psi(\tau, w))\|\mathrm{d}\tau = \int_0^T \|f(\psi(\tau, w))\|\mathrm{d}\tau + \int_T^{T+h} \|f(\psi(\tau, w))\|\mathrm{d}\tau \leq \int_0^T \|f(\psi(\tau, w))\|\mathrm{d}\tau + \int_0^\infty \|f(\psi(T+\tau, w))\|\mathrm{d}\tau$. The first integral on the right-hand side in the last inequality is clearly defined, while the second integral is defined and equals $S(\psi(T, w))$, since $\psi(T, w) \in \mathcal{U} \subseteq \mathcal{A}$. Thus, for every $h > 0$, it follows that $\int_0^{T+h} \|f(\psi(\tau, w))\|\mathrm{d}\tau \leq \int_0^T \|f(\psi(\tau, w))\|\mathrm{d}\tau + S(\psi(T, w))$. Taking the limit as $h \to \infty$, we conclude that $S(w)$ is defined and hence $w \in \mathcal{A}$. Thus $\mathcal{V}$ is a relatively open neighborhood of $z$ such that $\mathcal{V} \subseteq \mathcal{A}$. Hence it follows that $z \in \operatorname{int} \mathcal{A}$.

(ii) Suppose $\psi_\infty(z) \in \mathcal{C}$. Since $\mathcal{C} \subseteq \operatorname{int} \mathcal{A}$, it follows from (i) that $z \in \operatorname{int} \mathcal{A}$. Thus, there exists a relatively open neighborhood $\mathcal{Q} \subseteq \mathcal{G}$ of $z$ such that $\mathcal{Q} \subseteq \mathcal{A}$. Since $\mathcal{G}$ is locally compact, we may assume that $\mathcal{Q}$ is relatively bounded in $\mathcal{G}$. Let $\{z_k\}$ be a sequence in $\mathcal{Q}$ converging to $z$, and choose $\varepsilon > 0$. Since $S$ is continuous at $\psi_\infty(z) \in \mathcal{E} = S^{-1}(0)$, there exists a relatively open neighborhood $\mathcal{W}_\varepsilon \subseteq \mathcal{G}$ of $\psi_\infty(z)$ such that $S(w) < \varepsilon/3$ for all $w \in \mathcal{W}_\varepsilon$. Since $\lim_{t\to\infty} \psi(t, z) = \psi_\infty(z) \in \mathcal{W}_\varepsilon$, there exists $T > 0$ such that $\psi(T, z) \in \mathcal{W}_\varepsilon$. By Theorem II.3.2 in [30], the sequence of functions $\{\psi^{z_k}\}$ converges to the function $\psi^z$ uniformly on $[0, T]$. Since $f$ is uniformly continuous on the compact set $\psi([0, T] \times \overline{\mathcal{Q}})$, the sequence of functions $\{f \circ \psi^{z_k}\}$ converges to the function $f \circ \psi^z$ uniformly on $[0, T]$. The triangle inequality implies that the sequence of functions $\{\|f \circ \psi^{z_k}(\cdot)\|\}$ converges to the function $\{\|f \circ \psi^z(\cdot)\|\}$ uniformly on $[0, T]$. Hence it follows from standard results on integration [34, Thm. 9.8] that there exists $K_1 > 0$ such that, for every $k > K_1$, $\left|\int_0^T (\|f(\psi(\tau, z_k))\| - \|f(\psi(\tau, z))\|)\mathrm{d}\tau\right| < \varepsilon/3$. Also, it follows from continuity of $\psi$ that there exists $K_2$ such that $\psi(T, z_k) \in \mathcal{W}_\varepsilon$ for all $k > K_2$. Therefore, for every $k > K_1, K_2$, we have $|S(z_k) - S(z)| = \left|\int_0^\infty (\|f(\psi(\tau, z_k))\| - \|f(\psi(\tau, z))\|)\mathrm{d}\tau\right| \leq \left|\int_0^T (\|f(\psi(\tau, z_k))\| - \|f(\psi(\tau, z))\|)\mathrm{d}\tau\right| + \left|\int_T^\infty (\|f(\psi(\tau, z_k))\| - \|f(\psi(\tau, z))\|)\mathrm{d}\tau\right| < \int_T^\infty \|f(\psi(\tau, z_k))\|\mathrm{d}\tau + \int_T^\infty \|f(\psi(\tau, z))\|\mathrm{d}\tau + \varepsilon/3 \leq S(\psi(T, z_k)) + S(\psi(T, z)) + \varepsilon/3 < \varepsilon$. Thus $|S(z) - S(z_k)| < \varepsilon$ for every $k > K_1, K_2$. It follows that $S(z_k) \to S(z)$ as $k \to \infty$. Hence $S$ is continuous at $z$ and $z \in \mathcal{C}$.

(iii) Suppose $\psi_\infty(z) \in \operatorname{int} \mathcal{C}$. There exists $\varepsilon > 0$ such that the relatively open and relatively bounded set $\mathcal{V} \stackrel{\text{def}}{=} \{x \in \mathcal{G} : \|x - \psi_\infty(z)\| < \varepsilon\}$ satisfies $\overline{\mathcal{V}} \subseteq \mathcal{C}$. Let $\mathcal{U} \subseteq \mathcal{G}$ be a relatively open neighborhood of $\psi_\infty(z)$ such that $\mathcal{U} \subseteq \mathcal{C}$ and every $x \in \mathcal{U}$ satisfies $S(x) < \varepsilon/2$ and $\|x - \psi_\infty(z)\| < \varepsilon/2$. There exists $T > 0$ such that $\psi(T, z) \in \mathcal{U}$.

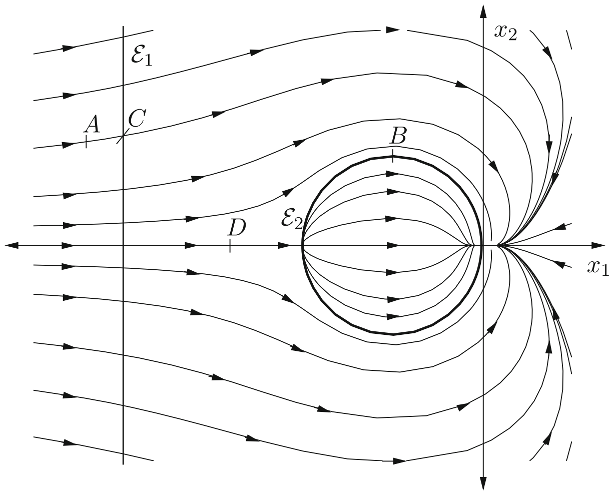


**Fig. 1** Phase portrait of a convergent system

Let $\mathcal{W} = \psi_T^{-1}(\mathcal{U})$. By continuity of $\psi$, $\mathcal{W}$ is a relatively open neighborhood of $z$. Consider $w \in \mathcal{W}$. For every $h > 0$, we have $\|\psi(T+h, w) - \psi_\infty(z)\| \le \|\psi(T + h, w) - \psi(T, w)\| + \|\psi(T, w) - \psi_\infty(z)\| < \| \int_0^h f(\psi(\tau, \psi(T, w)))\mathrm{d}\tau\| + \varepsilon/2 \le \int_0^\infty \|f(\psi(\tau, \psi(T, w)))\|\mathrm{d}\tau + \varepsilon/2 = S(\psi(T, w)) + \varepsilon/2 < \varepsilon$. Thus, for every $h > 0$, $\psi(T + h, w) \in \mathcal{V}$. Since $\mathcal{V}$ is relatively bounded, it follows that $\psi_\infty(w)$ exists and is contained in $\overline{\mathcal{V}} \subseteq \mathcal{C}$. It now follows from (ii) that $w \in \mathcal{C}$. $\mathcal{W}$ is thus a relatively open neighborhood of $z$ that is contained in $\mathcal{C}$. It follows that $z \in \operatorname{int} \mathcal{C}$. $\square$

The following example illustrates the results of this section, and shows that, in general, the assertion of lower semicontinuity in Proposition 3.1 cannot be strengthened, and the converses of statements in Proposition 3.2 do not hold:

*Example 3.3* Figure 1 shows the phase portrait of the system (1) with $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x) = |(x_1 + 2)(W(x) - 0.25)| \left( |W(x) - 0.25| \begin{bmatrix} -x_1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} x_2 \\ -(x_1 + 0.5) \end{bmatrix} \right),$$

where $W : \mathbb{R}^2 \to \mathbb{R}^2$ is the function given by $W(x) = (x_1 + 0.5)^2 + x_2^2$. The set of equilibria is the union of the straight line $\mathcal{E}_1 = \{x \in \mathbb{R}^2 : x_1 = -2\}$ and the circle $\mathcal{E}_2 = \{x \in \mathbb{R}^2 : W(x) = 0.25\}$. The only Lyapunov stable equilibrium is the origin $x = 0$. Since every neighborhood of the origin contains unstable equilibria in $\mathcal{E}_2$, the origin is not semistable.

The phase portrait clearly shows that the system is convergent. Consequently, the function $\psi_\infty$ introduced in Sect. 2 is defined everywhere. It can be seen from the phase portrait that the function $\psi_\infty$ is continuous everywhere except at equilibrium points such as $B$ and $C$ that are unstable, and nonequilibrium points such as $D$ that lie on the segment $\mathcal{I} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 : -2 < x_1 < -1, x_2 = 0\}$. In particular, $\psi_\infty$ is continuous at every initial condition whose solution converges to the Lyapunov stable equilibrium 0, thus illustrating Proposition 2.3. However, $\psi_\infty$ is also continuous at points such as $A$, whose limits points are unstable equilibria, thus showing that the converse of Proposition 2.3 does not hold.

It is clear from the phase portrait that every orbit has finite arc length. Consequently, the arc length function $S$ is defined everywhere. Moreover, $S$ is continuous everywhere except at equilibrium points such as $B$ and $C$ that are unstable, and nonequilibrium points such as $D$ that lie on the segment $\mathcal{I}$. Thus $\mathcal{C} = \{0\} \cup \{\mathbb{R}^2 \backslash (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{I})\}$. This demonstrates that in general, the assertion of lower semicontinuity in Proposition 3.1 cannot be strengthened to continuity. The interior of $\mathcal{C}$ is the set $\mathbb{R}^2 \backslash (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{I})$. Thus a point such as $A$ lies in int $\mathcal{C}$, while its limit point $C$ in $\mathcal{E}_1$ does not even lie in $\mathcal{C}$. This shows that in general, the converses of statements (ii) and (iii) in Proposition 3.2 do not hold. In contrast, neither the point $D$ nor its limit point lie in $\mathcal{C}$.

Finally, we note that the origin, which is the only Lyapunov stable equilibrium, is also the only equilibrium at which the arc length function is continuous. We will explore the relation between continuity of the arc length function and stability in Sect. 5.

Before proceeding to the next section, we observe that the arc length function $S$ depends on our choice of the norm. For example, $S(x)$ may not equal the Euclidean length of the orbit of $x \in \mathcal{G}$ unless $\|\cdot\|$ is the Euclidean norm. The results that we state, however, hold for any arbitrary choice of the norm. Theoretically, this is only to be expected since all norms on $\mathbb{R}^n$ are equivalent. However, in applications, the freedom available in choosing the norm can be usefully exploited as examples that we present in Sect. 6 demonstrate.

## 4 Arc length and convergence

In this section, we relate arc length to convergence. Specifically, we show that the system (1) is convergent if every orbit has finite arc length. This fact leads to a Lyapunov-based sufficient condition for convergence. The results of this section are based on the following lemma, which implies that if the image of a function of time has finite arc length, then the function converges asymptotically to a limit. Though the result appears to be widely known, we could not find a specific reference. Hence, we provide a proof in the appendix for the sake of completeness.

**Lemma 4.1** *Let $y : [0, \infty) \to \mathbb{R}^n$ be continuously differentiable. If $\dot{y}$ is absolutely integrable on $[0, \infty)$, then $\lim_{t\to\infty} y(t)$ exists.*

The following corollary is an application of Lemma 4.1 to the solutions of (1):

**Corollary 4.2** *Suppose $x \in \mathcal{A}$. Then $\lim_{t\to\infty} \psi(t, x)$ exists in $\mathcal{D}$. In addition, if $\mathcal{O}_x$ is bounded relative to $\mathcal{G}$, then $x \in \mathcal{R}$, that is, $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{R}$.*

*Proof* Denote $y = \psi^x$. Since $x \in \mathcal{A}$, it follows that $\dot{y}$ is absolutely integrable. Lemma 4.1 now implies that $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \psi(t, x)$ exists. Next, suppose $x \in \mathcal{B}$ so that $\mathcal{O}_x$ is bounded relative to $\mathcal{G}$. Then, $\lim_{t\to\infty} \psi(t, x) \in \overline{\mathcal{O}_x} \subseteq \mathcal{G}$, and hence, by definition, $x \in \mathcal{R}$. □

The following result provides a Lyapunov-based sufficient condition for convergence. The sufficient condition involves an inequality that guarantees finite arc length of orbits and hence, by Corollary 4.2, convergence.

**Theorem 4.3** *Suppose there exists a lower semicontinuous function $V : \mathcal{G} \to \mathbb{R}$ such that $\dot{V}(z) \leq 0$ for all $z \in \mathcal{G}$. Let $\mathcal{M}$ be the largest invariant set contained in $\dot{V}^{-1}(0)$ and suppose there exists a relatively open set $\mathcal{U} \subseteq \mathcal{G}$ containing $\mathcal{M}$ and $c \in (0, \infty)$ such that, for every $z \in \mathcal{U}$,*

$$c\dot{V}(z) + \|f(z)\| \leq 0. \tag{2}$$

*Then, for every $x \in \mathcal{G}$ such that $\mathcal{O}_x$ is bounded relative to $\mathcal{G}$, $x \in \mathcal{A}$, and $\lim_{t\to\infty} \psi(t, x)$ exists and is contained in $\mathcal{G}$, that is, $\mathcal{B} \subseteq \mathcal{A} \cap \mathcal{R}$. In particular, if $\mathcal{B} = \mathcal{G}$, then the system (1) is convergent relative to $\mathcal{G}$.*

*Proof* Let $x \in \mathcal{B}$ and denote $y = \psi^x$. Let $m$ be the minimum value of the lower semicontinuous function $V$ on the compact set $\overline{\mathcal{O}_x}$. Since $V \circ y$ is nonincreasing, $m = \lim_{t\to\infty} V(y(t))$. Since $\mathcal{G}$ is locally compact, the hypotheses on $V$ imply that every relatively bounded solution converges to $\mathcal{M}$ [35, Thm. VIII.6.1, c)], [36, Thm. 1]. Thus, there exists $T > 0$ such that $y(t) \in \mathcal{U}$ for all $t \geq T$. Consequently, $c\dot{V}(y(t)) + \|f(y(t))\| \leq 0$ for every $t > T$. The last inequality implies that the function $g : [0, \infty) \to \mathbb{R}$ defined by $g(t) = \int_T^{T+t} \|\dot{y}(\tau)\| \mathrm{d}\tau + c[V(y(T+t)) - V(\psi(T, x))]$ is nonincreasing (see [31, Lem. 5.6], [32, Thm. 2.1]). Since $g(0) = 0$, for any given $t \in [0, \infty)$, we have $\int_T^{T+t} \|f(y(\tau))\| \mathrm{d}\tau \leq c[V(\psi(T, x)) - V(y(T+t))] \leq c[V(\psi(T, x)) - m]$. It follows from the last inequality that $\dot{y}$ is absolutely integrable, that is, $x \in \mathcal{A}$. Since $x \in \mathcal{B}$ was chosen to be arbitrary, it follows that $\mathcal{B} \subseteq \mathcal{A}$. Corollary 4.2 now implies that $\mathcal{B} = \mathcal{B} \cap \mathcal{A} \subseteq \mathcal{R}$. Thus $\mathcal{B} \subseteq \mathcal{A} \cap \mathcal{R}$. In particular, if $\mathcal{B} = \mathcal{G}$, then $\mathcal{G} = \mathcal{R}$, that is, the system (1) is convergent relative to $\mathcal{G}$. □

## 5 Arc length and stability

In this section, we relate properties of the arc length function to stability. More specifically, we show that if the arc length function is continuous at an equilibrium, then the equilibrium is Lyapunov stable, while if the arc length function is continuous in a neighborhood of the equilibrium, then the equilibrium is semistable. This fact leads to an arc-length-based Lyapunov result for Lyapunov stability, semistability and asymptotic stability.

The following result, which relates the stability of an equilibrium to the continuity properties of the arc length function in a neighborhood of the equilibrium, forms the basis for subsequent results in this section.

**Theorem 5.1** *The following statements hold:*

(i) *Every equilibrium in $\mathcal{C}$ is Lyapunov stable relative to $\mathcal{G}$.*
(ii) *Every equilibrium in* int $\mathcal{C}$ *is semistable relative to $\mathcal{G}$.*
(iii) *Every isolated equilibrium in* int $\mathcal{C}$ *is asymptotically stable relative to $\mathcal{G}$.*

*Proof* (i) Let $x \in \mathcal{C}$ be an equilibrium and let $\mathcal{U}_\varepsilon \subseteq \mathcal{G}$ be a relatively open neighborhood of $x$. Choose $\varepsilon > 0$ such that $\{z \in \mathcal{G} : \|z - x\| < \varepsilon\} \subseteq \mathcal{U}_\varepsilon$. Since $x$ is an equilibrium, $S(x) = 0$. Since $S$ is continuous at $x$ relative to $\mathcal{G}$, there exists a relatively open neighborhood $\mathcal{V} \subseteq \mathcal{G}$ of $x$ such that $S(z) < \varepsilon/2$ for every $z \in \mathcal{V}$. Let $\mathcal{U}_\delta = \{z \in \mathcal{V} : \|z - x\| < \varepsilon/2\}$. Then $\mathcal{U}_\delta$ is relatively open in $\mathcal{G}$ and, for every $z \in \mathcal{U}_\delta$ and $t \geq 0$, $\|\psi(t, z) - x\| \leq \|\psi(t, z) - z\| + \|z - x\| = \| \int_0^t f(\psi(\tau, z))\mathrm{d}\tau \| + \|z - x\| \leq \int_0^\infty \|f(\psi(\tau, z))\|\mathrm{d}\tau + \|z - x\| = S(z) + \|z - x\| < \varepsilon$, that is, $\psi(t, z) \in \mathcal{U}_\varepsilon$. Hence, we conclude that $x$ is Lyapunov stable relative to $\mathcal{G}$.

(ii) Let $x$ be an equilibrium in $\operatorname{int} \mathcal{C}$. By (i) above, every equilibrium in $\operatorname{int} \mathcal{C}$ is Lyapunov stable relative to $\mathcal{G}$. In particular, $x$ is Lyapunov stable relative to $\mathcal{G}$. Let $\mathcal{U}_\varepsilon \subseteq \mathcal{G}$ be a relatively open and relatively bounded neighborhood of $x$ such that $\overline{\mathcal{U}_\varepsilon} \subset \mathcal{C}$. Let $\mathcal{U}_\delta \subseteq \mathcal{G}$ be a relatively open neighborhood of $x$ such that $\psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$ for every $t \geq 0$. For every $z \in \mathcal{U}_\delta$, $\mathcal{O}_z \subseteq \mathcal{U}_\varepsilon$ is bounded relative to $\mathcal{G}$. Thus, $\mathcal{U}_\delta \subseteq \mathcal{C} \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$. Hence, Corollary 4.2 implies that $\psi_\infty$ is defined on $\mathcal{U}_\delta$. By our construction of $\mathcal{U}_\delta$, it follows that $\psi_\infty(z) \in \overline{\mathcal{U}_\varepsilon} \subset \mathcal{C}$ for every $z \in \mathcal{U}_\delta$. It now follows from (i) that, for every $z \in \mathcal{U}_\delta$, $\psi_\infty(z)$ is Lyapunov stable relative to $\mathcal{G}$. Semistability of $x$ now follows.

(iii) The result follows from (ii) above by noting that every isolated equilibrium that is semistable relative to $\mathcal{G}$ is also asymptotically stable relative to $\mathcal{G}$. □

We note that the proof of (i) of Theorem 5.1 above is a concise version of the argument used in the proof of Theorem 3 of [22]. Our next result gives arc-length-based sufficient Lyapunov conditions for Lyapunov stability, semistability, and asymptotic stability of an equilibrium.

**Theorem 5.2** *Let $x \in \mathcal{G}$, and suppose there exists a continuous function $V : \mathcal{V} \to \mathbb{R}$ defined on a relatively open neighborhood $\mathcal{V} \subseteq \mathcal{G}$ of $x$, and $c \in (0, \infty)$ such that (2) is satisfied for every $z \in \mathcal{V}$. Then the following statements hold:*

(i) *If $x$ is a local minimizer of $V$ relative to $\mathcal{G}$, then $x \in \mathcal{C}$, and $x$ is a Lyapunov stable equilibrium relative to $\mathcal{G}$.*
(ii) *If $x$ and every equilibrium in $\mathcal{V}$ is a local minimizer of $V$ relative to $\mathcal{G}$, then $x \in \operatorname{int} \mathcal{C}$ and $x$ is a semistable equilibrium relative to $\mathcal{G}$.*
(iii) *If $x$ is a local minimizer of $V$ relative to $\mathcal{G}$ and an isolated equilibrium, then $x \in \operatorname{int} \mathcal{C}$ and $x$ is asymptotically stable relative to $\mathcal{G}$.*

*Proof* (i) Suppose $x$ is a local minimizer of $V$ relative to $\mathcal{G}$. Let $\mathcal{U} \subseteq \mathcal{V}$ be a relatively open and relatively bounded neighborhood of $x$ such that $V(x) \leq V(z)$ for all $z \in \overline{\mathcal{U}}$. Let $r \stackrel{\text{def}}{=} \min_{z \in \operatorname{bd} \mathcal{U}} \|z - x\|$, and note that $r > 0$. Choose a relatively open neighborhood $\mathcal{U}_\delta \subseteq \mathcal{U}$ such that $|V(z) - V(x)| < r/4c$ and $\|z - x\| < r/4$ for every $z \in \mathcal{U}_\delta$.

Consider $z \in \mathcal{U}_\delta$. Inequality (2) implies that the function $g : [0, \infty) \to \mathbb{R}$ defined by $g(t) \stackrel{\text{def}}{=} \int_0^t \|f(\psi(\tau, z))\|\mathrm{d}\tau + c[V(\psi(t, z)) - V(z)]$ has a nonpositive upper right Dini derivative, and hence nonincreasing (see [31, Lem. 5.6], [32, Thm. 2.1]), at every $t \in [0, \infty)$ such that $\psi(t, z) \in \mathcal{U}$. We claim that $\psi(t, z) \in \mathcal{U}$ for every $t \geq 0$. To arrive at a contradiction, suppose there exists $t \geq 0$ such that $\psi(t, z) \notin \mathcal{U}$. Then, by the continuity of $\psi$, there exists $T > 0$ such that $\psi(T, z) \in \operatorname{bd} \mathcal{U}$, and $\psi(t, x) \in \mathcal{U}$ for every $t \in [0, T)$. Since $g(0) = 0$, we have $\int_0^T \|f(\psi(t, z))\|\mathrm{d}t \leq c[V(z) - V(\psi(T, z))] \leq$

$c[V(z) - V(x)] < \frac{r}{4}$. Therefore, $r \leq \|\psi(T, z) - x\| \leq \|\psi(T, z) - z\| + \|z - x\| < \| \int_0^T f(\psi(t, z))\mathrm{d}t\| + \frac{r}{4} \leq \int_0^T \|f(\psi(t, z))\|\mathrm{d}t + \frac{r}{4} < \frac{r}{2}$, which is a contradiction. The contradiction proves that $\psi(t, z) \in \mathcal{U}$ for every $t \geq 0$. It now follows that $g$ is nonincreasing on $[0, \infty)$. Hence, for every $t \geq 0$, we have $\int_0^t \|f(\psi(\tau, z))\|\mathrm{d}\tau \leq c[V(z) - V(\psi(t, z))] \leq c[V(z) - V(x)]$. On letting $t \to \infty$, the last inequality implies that $z \in \mathcal{A}$ and $S(z) \leq c[V(z) - V(x)]$.

Since $z \in \mathcal{U}_\delta$ was chosen arbitrarily, it follows that $\mathcal{U}_\delta \subseteq \mathcal{A}$, and

$$0 \leq S(z) \leq c[V(z) - V(x)] \tag{3}$$

for every $z \in \mathcal{U}_\delta$. In particular, setting $z = x$ in (3) yields $S(x) = 0$, so that $x$ is an equilibrium. Also, the inequality (3) implies that $\lim_{z\to x} S(z) = 0 = S(x)$. Thus, $S$ is continuous at $x$ and $x \in \mathcal{C}$. It now follows from (i) of Theorem 5.1 that $x$ is Lyapunov stable relative to $\mathcal{G}$.

(ii) Suppose $x$ and every equilibrium in $\mathcal{V}$ is a local minimizer of $V$ relative to $\mathcal{G}$, and let $\mathcal{U}_\varepsilon$ be a relatively open neighborhood of $x$ such that $\overline{\mathcal{U}_\varepsilon} \subseteq \mathcal{V}$. Since $\mathcal{G}$ is locally compact, we may assume that $\mathcal{U}_\varepsilon$ is relatively bounded in $\mathcal{G}$. By (i) $x$ is a Lyapunov stable equilibrium and contained in $\mathcal{C} \subseteq \operatorname{int} \mathcal{A}$. Hence, there exists a relatively open neighborhood $\mathcal{U}_\delta \subseteq \mathcal{G}$ of $x$ such that $\mathcal{U}_\delta \subseteq \mathcal{A}$ and $\psi_t(\mathcal{U}_\delta) \subseteq \mathcal{U}_\varepsilon$ for every $t \geq 0$.

We claim that $\mathcal{U}_\delta \in \mathcal{C}$. To prove this, consider $z \in \mathcal{U}_\delta$. Then $\mathcal{O}_z \subset \mathcal{U}_\varepsilon$ is relatively bounded in $\mathcal{G}$, while $z \in \mathcal{A}$. Hence, by Corollary 4.2, $z \in \mathcal{R}$ and $\psi_\infty(z)$ is defined. By construction, $\psi_\infty(z)$ is an equilibrium contained in $\overline{\mathcal{U}_\varepsilon}$, and hence $\psi_\infty(z)$ is a local minimizer of $V$. By (i), $\psi_\infty(z) \in \mathcal{C}$. By (ii) of Proposition 3.2, $z \in \mathcal{C}$. Since $z \in \mathcal{U}_\delta$ was chosen arbitrarily, it follows that $\mathcal{U}_\delta \subseteq \mathcal{C}$.

We have shown that $\mathcal{U}_\delta$ is a relatively open neighborhood of $x$ that is contained in $\mathcal{C}$. It follows that $x \in \operatorname{int} \mathcal{C}$. The result now follows from (ii) of Theorem 5.1.

(iii) Suppose $x$ is a local minimizer of $V$ relative to $\mathcal{G}$ and an isolated equilibrium. Semistability follows from (ii) above. The result then follows by noting that an isolated semistable equilibrium is asymptotically stable. □

The arc-length-based results of this section and the previous section depend on integrability of $\|f(\cdot)\|$ along the trajectories and the continuity properties of the corresponding integral, namely, arc length. We briefly mention previous work in which stability is related to properties of integrals computed along the solution.

Reference [37] explores connections between integrability, asymptotic behavior, and stability in a more general situation in which (1) is equipped with an output function $h : \mathbb{R}^n \to \mathbb{R}$. The main result of [37] is the integral invariance principle (Theorem 1.2 in [37]) which states that if $x \in \mathbb{R}^n$ has a bounded orbit and the function $h \circ \psi^x$ is in $L^p$ for some $p > 0$, then the positive limit set of $x$ is contained in the largest invariant subset of the zero-level set of $h$. Reference [37] further shows that, under the additional assumption of zero-state observability, the continuous dependence of the $p$-norm of $h \circ \psi^x$ on $x$ implies local asymptotic stability of the zero state. Letting $h(x) = \|f(x)\|$ and $p = 1$, the integral-invariance principle implies that if the orbit of $x \in \mathbb{R}^n$ has finite arc length (and is thus bounded), then the positive limit set of $x$ is contained in the set of equilibria $\mathcal{E}$. For this special choice of the output function $h$, zero-state observability implies that the zero state is the unique equilibrium.

Reference [38] gives integral characterizations for uniform asymptotic stability of a set. We believe that an integral version of (2) along with the results of [38] yields asymptotic stability of the set of equilibrium $\mathcal{E}$, in the case where $\mathcal{E}$ is compact.

The results of [37,38] thus yield conclusions on the attractivity and stability of the set of equilibria. On the other hand, our results yield conclusions on convergence of trajectories and semistability of individual equilibria that are not necessarily isolated. As illustrated in [3], stability of the set of equilibria implies neither convergence of individual trajectories to limit points, nor stability of individual equilibria. Thus, in spite of the connections outlined in the paragraphs above, we emphasize that our results are independent of the results of [37,38].

Finally, it is interesting to compare (i) and (iii) of Theorem 5.2 with the theorems of Lyapunov for Lyapunov stability and asymptotic stability, respectively. Lyapunov's theorems require the Lyapunov function to be positive definite at the equilibrium, that is, the Lyapunov function is required to have a local strict minimizer at the equilibrium. On the other hand, (i) and (iii) of Theorem 5.2 require the Lyapunov function to have a local nonstrict minimizer at the equilibrium and satisfy the inequality (2). This feature makes it possible to apply the same Lyapunov function to analyze the stability of more than one equilibrium and makes our results especially suited for stability analysis of systems having a continuum of equilibria, as we illustrate in the next section.

## 6 Application examples

In this section, we present three examples to illustrate possible applications of our results. The first example involves a system considered in [3], the second involves a system of three interacting agents, while the third example considers a chemical reaction network. All three examples involve systems having a continuum of equilibria.

### 6.1 An example from [3]

Consider the system $\dot{y}(t) = f(y(t))$, where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the continuous vector field given by

$$\begin{aligned} f(x) = {} & \operatorname{sign}(x_1^2 + x_2^2 - 1)|x_1^2 + x_2^2 - 1|^\alpha f_r(x) \\ & + \operatorname{sign}(x_1^2 + x_2^2 - 1)|x_1^2 + x_2^2 - 1|^\beta f_\theta(x), \end{aligned} \tag{4}$$

with $\alpha, \beta \geq 1$ and the vector fields $f_r$ and $f_\theta$ given by

$$f_r(x) = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}, \quad f_\theta(x) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}. \tag{5}$$

The vector fields $f_r$ and $f_\theta$ point in the radial and circumferential directions, respectively, and thus the parameters $\alpha$ and $\beta$ determine the rates at which solutions move in these directions, respectively. This can be seen more clearly by rewriting (4) in terms of polar coordinates $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \tan^{-1}(x_2/x_1)$ as

$$\dot{r} = -r\operatorname{sign}(r^2-1)|r^2-1|^{\alpha}, \tag{6}$$
$$\dot{\theta} = -\operatorname{sign}(r^2-1)|r^2-1|^{\beta}. \tag{7}$$

It can be seen from Eqs. (6) and (7) that the set of equilibria $\mathcal{E} = f^{-1}(0)$ consists of the origin $x = 0$ and the unit circle $\mathrm{S}^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$. As phase portraits given in [3] show, all solutions of the system starting from nonzero initial conditions $y(0)$ that are not on the unit circle approach the unit circle. Solutions starting outside the unit circle spiral in clockwise toward the unit circle while solutions starting inside the unit circle spiral out counterclockwise. Consequently, all solutions are bounded and, for every choice of $\alpha$ and $\beta$, all solutions converge to the set of equilibria. However, it has been shown in [3] that in the case where $\alpha \geq \beta + 1$, the trajectories that converge to the unit circle spiral around an infinite number of times, and the system is not convergent.

It was shown in [3] that the system is convergent and all nonzero equilibria are semistable in the case $\alpha \leq \beta$. However, the results of [3] were not applicable in the case $\beta < \alpha < \beta+1$. In this example, we use Theorem 4.3 to show that the conclusions of [3] hold for the larger parameter range $\alpha < \beta + 1$.

Suppose $\alpha < \beta+1$. Let $\gamma = \min\{\alpha, \beta\}$ and $\delta = \max\{\alpha, \beta\}$. Consider the Lyapunov function $V(x) = (\gamma+1-\alpha)^{-1}|\sqrt{x_1^2+x_2^2}-1|^{\gamma+1-\alpha}$. Since $\alpha-\beta < 1$, it follows that $\gamma + 1 - \alpha > 0$, and hence $V$ is continuous. With a slight abuse of notation, we write $V(x) = (\gamma+1-\alpha)^{-1}|r-1|^{\gamma+1-\alpha}$ and compute the derivative of $V$ along the solutions of (6)–(7) as $\dot{V}(x) = -r(r+1)^{\alpha}|r-1|^{\gamma}$, which is seen to take nonpositive values everywhere. Moreover, the set $\dot{V}^{-1}(0)$ consists solely of equilibrium points and is thus invariant. We next compute $\|f(x)\| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = r\sqrt{|r^2-1|^{2\alpha} + |r^2-1|^{2\beta}} = r|r^2-1|^{\gamma}\sqrt{1+|r^2-1|^{2(\delta-\gamma)}} = r(r+1)^{\gamma}|r-1|^{\gamma}\sqrt{1+|r^2-1|^{2(\delta-\gamma)}}$. Consider the open neighborhood $\mathcal{U} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 2\}$ of $\dot{V}^{-1}(0)$. For every $x \in \mathcal{U}$, we have $\dot{V}(x) \leq -r|r-1|^{\gamma}$, while $\|f(x)\| \leq \sqrt{2}(1+\sqrt{2})^{\gamma} r|r-1|^{\gamma}$. Thus every $x \in \mathcal{U}$ satisfies (2) for $c = \sqrt{2}(1+\sqrt{2})^{\gamma}$. Since it was shown in [3] that all orbits of the system are bounded, Theorem 4.3 implies that the system is convergent with respect to $\mathcal{G} = \mathbb{R}^2$ in the case $\alpha < \beta+1$. Every nonzero equilibrium of the system lies on the unit circle and is easily seen to be a local minimizer of the function $V$. Since the open set $\mathcal{U}$ contains all nonzero equilibria, (ii) of Theorem 5.2 implies that every nonzero equilibrium of the system is semistable in the case $\alpha < \beta + 1$.

### 6.2 A consensus example

Consider the system on $\mathbb{R}^3$ described by the equations

$$\dot{x}_1 = c_{21}\sigma(x_2 - x_1) + c_{31}\sigma(x_3 - x_1), \tag{8}$$
$$\dot{x}_2 = c_{32}\sigma(x_3 - x_2) + c_{12}\sigma(x_1 - x_2), \tag{9}$$
$$\dot{x}_3 = c_{13}\sigma(x_1 - x_3) + c_{23}\sigma(x_2 - x_3), \tag{10}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Equations (8)–(10) represent the collective dynamics of a group of three agents which interact by exchanging information. The states of the agents are described by the scalar variables $x_1$, $x_2$ and $x_3$. The coefficients $\{c_{ij} : i, j \in \{1, 2, 3\}, i \neq j\}$ appearing in (8)–(10) represent the topology of the information exchange between the agents. More specifically, given distinct $i, j \in \{1, 2, 3\}$, the coefficient $c_{ij}$ is 1 if the agent $j$ receives information from the agent $i$, and zero otherwise. The communication topology between the agents can be represented by a graph $\mathfrak{G}$ having three nodes such that $\mathfrak{G}$ has a directed edge from node $i$ to node $j$ if and only if the agent $j$ can receive information from agent $i$. It is clear that the coefficients $c_{ij}$ are elements of the adjacency matrix of the graph $\mathfrak{G}$. We assume the communication topology to be fixed, so that the coefficients $c_{ij}$ are constant.

In this example, we will use results from the previous sections to analyze the collective behavior represented by (8)–(10). More specifically, we are interested in the consensus behavior of the agents. For this purpose, we make the assumption that the function $\sigma$ is strictly increasing and satisfies $\sigma(-a) = -\sigma(a)$ for all $a \in \mathbb{R}$.

Recall that the directed communication graph $\mathfrak{G}$ is *weakly connected* if the underlying undirected graph is connected. In the case of three agents, the weak connectedness of $\mathfrak{G}$ is equivalent to the assumption that every agent receives information from, or delivers information to, at least one other agent. The graph $\mathfrak{G}$ is said to have a directed spanning tree if there exists a node $i$ such that, for every other node $j \neq i$, there exists a directed path from $i$ to $j$. We will use the results of this paper to show that, under our assumptions on $\sigma$, the following statements hold:

**Claim 1:** If the communication graph $\mathfrak{G}$ is weakly connected, then every trajectory of the system (8)–(10) converges to an equilibrium.

**Claim 2:** If the communication graph $\mathfrak{G}$ has a directed spanning tree, then every trajectory of the system (8)–(10) converges to a semistable consensus state.

First, we claim that every orbit of (8)–(10) is bounded, that is, $\mathcal{B} = \mathbb{R}^3$. To prove our claim, consider the function $U : \mathbb{R}^3 \to \mathbb{R}$ defined by $U(x) \stackrel{\text{def}}{=} \max\{|x_1|, |x_2|, |x_3|\}$. The function $U$ is clearly proper and positive definite. Consider $x \in \mathbb{R}^3$. Without loss of generality, assume $U(x) = |x_1|$. Then, either $x_1 \geq \max\{0, x_2, x_3\}$ or $x_1 \leq \min\{0, x_2, x_3\}$. In the first case, (8) implies that $\dot{x}_1 \leq 0$, while in the second case (8) implies that $\dot{x}_1 \geq 0$. Using similar arguments for the cases $U(x) = |x_2|$ and $U(x) = |x_3|$, we conclude that $U$ decreases along the trajectories of (8)–(10). Since $U$ is proper, it follows that $\mathcal{B} = \mathbb{R}^3$.

To apply our results, define the Lipschitz function $V : \mathbb{R}^3 \to \mathbb{R}$ by

$$V(x) = \max_{i,j} |x_i - x_j| + \max\{|x_i - x_j| : i \neq j, c_{ij} + c_{ji} > 0\}, \tag{11}$$

and let $f : \mathbb{R}^3 \to \mathbb{R}^3$ denote the right-hand side of (8)–(10). The first term on the right-hand side of (11) is the maximum pairwise separation between the agents, while the second term is the maximum pairwise separation between communicating pairs of agents, where the communication may be uni-directional or bidirectional.

To prove Claim 1, assume that the graph $\mathfrak{G}$ is weakly connected. We claim that $\dot{V}(x) + \|f(x)\|_\infty \leq 0$ for every $x \in \mathbb{R}^3$, where $\|\cdot\|_\infty$ denotes the $\infty$-norm on $\mathbb{R}^3$

defined by $\|u\|_\infty = \max\{|u_1|, |u_2|, |u_3|\}$. Since $V^{-1}(0) = \{x : x_1 = x_2 = x_3\} = f^{-1}(0)$, our claim holds for all $x \in \mathbb{R}^3$ satisfying $x_1 = x_2 = x_3$.

Next, consider $x \in \mathbb{R}^3$ having all distinct components. By relabeling if necessary, we may assume that $x_1 < x_2 < x_3$, so that $f_1(x) \geq 0$ while $f_3(x) \leq 0$. In addition, assume that $x_2 - x_1 \neq x_3 - x_2$. There exists a connected open neighborhood $\mathcal{U}$ of $x$ such that $z_1 < z_2 < z_3$ and $z_2 - z_1 \neq z_3 - z_2$ for all $z \in \mathcal{U}$.

First, consider the case $c_{13} + c_{31} > 0$. Then $V(z) = 2(z_3 - z_1)$ for all $z \in \mathcal{U}$. Consequently, $V$ is differentiable on $\mathcal{U}$. This allows us to compute the upper right Dini derivative of $V$ at $x$ along $f$ as $\dot{V}(x) = 2(f_3(x) - f_1(x))$. Hence, $\dot{V}(x) + |f_1(x)| = 2f_3(x) - f_1(x) \leq 0$, and $\dot{V}(x) + |f_3(x)| = f_3(x) - 2f_1(x) \leq 0$. Our assumptions on $\sigma$ and $x$ imply that $|f_2(x)| \leq c_{12}\sigma(x_2 - x_1) + c_{32}\sigma(x_3 - x_2) \leq (c_{12} + c_{32})\sigma(x_3 - x_1)$, so that $\dot{V}(x) + |f_2(x)| \leq 2f_3(x) - 2f_1(x) + (c_{12} + c_{32})\sigma(x_3 - x_1) = [-2(c_{13} + c_{31}) + c_{12} + c_{32}]\sigma(x_3 - x_1) + 2c_{23}\sigma(x_2 - x_3) - 2c_{21}\sigma(x_2 - x_1)$, which is nonpositive since $c_{13} + c_{31} \geq 1$. Thus we conclude that, in the case where $c_{13} + c_{31} > 0$, $\dot{V}(x) + \|f(x)\|_\infty \leq 0$.

Next, consider the case $c_{13} + c_{31} = 0$. The assumption of weak connectedness on $\mathfrak{G}$ implies that $c_{12} + c_{21} \geq 1$ and $c_{23} + c_{32} \geq 1$. First, assume $x_2 - x_1 > x_3 - x_2$. Then, as a consequence of our assumptions on $\sigma$, we have $f_2(x) - f_1(x) = c_{12}\sigma(x_1 - x_2) + c_{32}\sigma(x_3 - x_2) - c_{21}\sigma(x_2 - x_1) \leq (c_{32} - c_{12} - c_{21})\sigma(x_2 - x_1)$, which is nonpositive since $c_{12} + c_{21} \geq 1$. Also, $z_2 - z_1 > z_3 - z_2$ for every $z \in \mathcal{U}$. Hence $V(z) = (z_3 - z_1) + (z_2 - z_1)$ for every $z \in \mathcal{U}$. The function $V$ is thus differentiable on $\mathcal{U}$, and we can easily compute $\dot{V}(x) = f_3(x) + f_2(x) - 2f_1(x)$. We compute $\dot{V}(x) - f_2(x) = f_3(x) - 2f_1(x) \leq 0$. Next, $\dot{V}(x) + f_2(x) = f_3(x) + 2f_2(x) - 2f_1(x) \leq 0$ by our computation above. We have thus shown that $\dot{V}(x) + |f_2(x)| \leq 0$. Next, $\dot{V}(x) + |f_1(x)| = f_3(x) + f_2(x) - f_1(x) \leq 0$. Finally, $\dot{V}(x) + |f_3(x)| = f_2(x) - 2f_1(x) \leq f_2(x) - f_1(x) \leq 0$. We conclude that $\dot{V}(x) + \|f(x)\|_\infty \leq 0$. Analogous arguments can be used to show that $\dot{V}(x) + \|f(x)\|_\infty \leq 0$ also holds if $x_2 - x_1 < x_3 - x_2$.

We have thus far shown that the inequality $\dot{V}(x) + \|f(x)\|_\infty \leq 0$ holds for every $x$ in the set $\mathcal{Q} = \{x : \max_i x_i > \min_i x_i,\ x_j - \min_i x_i \neq \max_i x_i - x_j, j = 1, 2, 3\}$ which is open and dense in $\mathbb{R}^3$. Note that the function $V$ is the pointwise maximum of a finite collection $\{V_1, \ldots, V_r\}$ of distinct functions, where each $V_m$ is a linear functions of the form $x \mapsto (x_i - x_j) + (x_k - x_l)$. For each $x \in \mathbb{R}^3$, denote $\mathcal{I}(x) = \{i : V_i(x) = V(x)\}$, and note that $\mathcal{I}(x)$ is a singleton for each $x \in \mathcal{Q}$. Then, for each $x \in \mathbb{R}^3$, $\dot{V}(x) = \max_{k \in \mathcal{I}(x)} \dot{V}_k(x)$ (see page 38 of [39]). Consider $x \in \mathbb{R}^3 \backslash \mathcal{Q}$, and choose $\varepsilon > 0$. Suppose $k \in \mathcal{I}(x)$ is such that $\dot{V}(x) = \dot{V}_k(x)$. Since $V_k$ is continuously differentiable and $\mathcal{Q}$ is dense, there exists $z \in \mathcal{Q}$ such that $\left|[\dot{V}_k(z) + \|f(z)\|_\infty] - [\dot{V}_k(x) + \|f(x)\|_\infty]\right| < \varepsilon$. We may further choose $z$ such that $\mathcal{I}(z) = \{k\}$. Then $\dot{V}(z) = \dot{V}_k(z)$. We therefore conclude that $\dot{V}(x) + \|f(x)\|_\infty = \dot{V}_k(x) + \|f(x)\|_\infty \leq \dot{V}_k(z) + \|f(z)\|_\infty + \varepsilon = \dot{V}(z) + \|f(z)\|_\infty + \varepsilon \leq \varepsilon$. Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\dot{V}(x) + \|f(x)\|_\infty \leq 0$ also holds for all $x \notin \mathcal{Q}$.

Since the inequality (2) is satisfied everywhere with $c = 1$ and $\|\cdot\| = \|\cdot\|_\infty$, it follows that $\dot{V}^{-1}(0) = f^{-1}(0) = \mathcal{E}$. Since $\mathcal{B} = \mathbb{R}^3$, Theorem 4.3 implies that $\mathcal{R} = \mathcal{A} = \mathbb{R}^3$, that is, the system (8)–(10) is convergent relative to $\mathbb{R}^3$ and all orbits have finite arc length. This proves Claim 1.

To prove Claim 2, assume that the graph $\mathfrak{G}$ has a directed spanning tree. We claim that the set of equilibria $\mathcal{E}$ of the system (8)–(10) is the set of consensus states $\{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\}$. Our assumptions on $\sigma$ readily imply that if $x \in \mathbb{R}^3$ satisfies $x_1 = x_2 = x_3$, then $x \in \mathcal{E}$. To complete the proof of our claim, let $x \in \mathcal{E}$. By relabeling the components if necessary, we may assume that $x_1 = \min_i x_i$ and $x_3 = \max_i x_i$, so that $x_1 \leq x_2 \leq x_3$. To arrive at a contradiction, suppose $x_1 < x_2$. Then, $\sigma(x_3 - x_1) \geq \sigma(x_2 - x_1) > 0$. Since $x \in \mathcal{E}$, we conclude from (8) that $c_{31} = c_{21} = 0$. Since both terms on the right-hand side of (10) are nonpositive and $\sigma(x_1 - x_3) < 0$, $x \in \mathcal{E}$ implies that $c_{13} = 0$. Since the existence of a spanning tree implies weak connectedness of the graph $\mathfrak{G}$, it follows that $c_{12} = 1$. However, the right-hand side of (9) has to be zero. Hence, it follows that $c_{32} = 1$ and $\sigma(x_3 - x_2) = -\sigma(x_1 - x_2) \neq 0$. Now $x \in \mathcal{E}$ immediately implies that, in (10), $c_{23} = 0$. Our conclusions thus far on the coefficients $c_{ij}$ contradict our assumption that the communication graph has a directed spanning tree. The contradiction implies that $x_1 = x_2$. Using similar arguments starting from the assumption $x_2 < x_3$ shows that $x_2 = x_3$. Thus, $\mathcal{E} = \{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\}$.

Since $\mathcal{E} = V^{-1}(0)$ and $V$ takes only nonnegative values, it follows that every equilibrium point of (8)–(10) is a local minimizer of $V$. Hence, (ii) of Theorem 5.2 implies that every equilibrium point of (8)–(10) is semistable. In summary, every trajectory of (8)–(10) converges to a consensus state, and every consensus state is semistable. This proves Claim 2.

Before proceeding to the next example, we remark that our proof of convergence in this example required us to show that the inequality (2) was satisfied. A simpler alternative would be to check the sufficient condition for convergence provided by Theorem 2.3 of [19], which requires us to verify that the Jacobian of the right-hand side of (8)–(10) has two eigenvalues with nonzero real parts at every equilibrium. Note, however, that the results of [19] may not always be applicable to this example under our assumptions on the function $\sigma$. For instance, if $\sigma(a - b) = (a - b)^{\frac{1}{3}}$, then the Jacobian of the right-hand side of (8)–(10) is undefined at all equilibrium points, while if $\sigma(a - b) = (a - b)^3$, then the Jacobian is zero at all equilibrium points. In both cases, the results of [19] do not apply.

### 6.3 An example from chemical kinetics

In the Michaelis–Menten chemical reaction, a substrate S is converted into a product P through an intermediate complex C in the presence of an enzyme E. The reaction is depicted as

$$\mathrm{S} + \mathrm{E} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \mathrm{C} \overset{k_3}{\rightarrow} \mathrm{P} + \mathrm{E},$$

where $k_i > 0$, $i = 1, 2, 3$, are chemical rate constants. In this example, we use Theorems 4.3 and 5.2 to show that the concentrations of species S, P, C, and E in this chemical reaction converge to equilibrium values.

Letting $y_1(t)$, $y_2(t)$, $y_3(t)$, and $y_4(t)$ denote the instantaneous nonnegative concentrations of the species S, C, E, and P, respectively, the law of mass action kinetics yields [4,7]

$$\dot{y}(t) = y_2(t)v_1 + y_1(t)y_3(t)v_2, \tag{12}$$

where $v_1 = \begin{bmatrix} k_2 & -(k_2+k_3) & k_2+k_3 & k_3 \end{bmatrix}^{\mathrm{T}}$ and $v_2 = \begin{bmatrix} -k_1 & k_1 & -k_1 & 0 \end{bmatrix}^{\mathrm{T}}$. Equation (12) is of the form (1), where $f : \mathbb{R}^4 \to \mathbb{R}^4$ is given by $f(x) = x_2 v_1 + x_1 x_3 v_2$.

The nonnegative orthant $\mathcal{G} = \{x \in \mathbb{R}^4 : x_i \geq 0, i = 1, \ldots, 4\}$ is positively invariant under the dynamics (12) [7]. Since the vectors $v_1$ and $v_2$ are linearly independent, it is easy to see that the set of equilibrium concentrations in $\mathcal{G}$ is $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1 = \{x \in \mathcal{G} : x_1 = 0, x_2 = 0, x_3 > 0\}$ and $\mathcal{E}_2 = \{x \in \mathcal{G} : x_1 \geq 0, x_2 = 0, x_3 = 0\}$.

It is easy to verify that the function $U : \mathcal{G} \to \mathbb{R}$ given by $U(x) = x_1 + 2x_2 + x_3 + x_4$ is proper and satisfies $\dot{U} \equiv 0$. It follows that every orbit in $\mathcal{G}$ is relatively bounded in $\mathcal{G}$ (see, for instance, [3, Cor. 3.1]).

Choose $\alpha \in (1, 1 + (k_3/k_2))$, and define $V : \mathcal{G} \to \mathbb{R}$ by $V(x) = \alpha x_1 + x_2$. Then, $V(x) \geq 0$ for every $x \in \mathcal{G}$ and $V^{-1}(0) = \overline{\mathcal{E}_1}$. Thus, every point in $\mathcal{E}_1$ is a local minimizer of $V$ relative to $\mathcal{G}$. Since $\dot{V} : \mathcal{G} \to \mathbb{R}$ is given by $\dot{V}(x) = [\alpha k_2 - (k_2 + k_3)]x_2 + k_1(1-\alpha)x_1 x_3$, it follows that $\dot{V}^{-1}(0) = \mathcal{E}$ and $\dot{V}(x) \leq 0$ for every $x \in \mathcal{G}$.

Let $P \in \mathbb{R}^{4\times 4}$ be given by $P \stackrel{\text{def}}{=} [v_1 \ v_2 \ e_3 \ e_4]$, where $e_3$ and $e_4$ are the third and fourth columns, respectively, of the $4 \times 4$ identity matrix. Note that $P$ is invertible. Next, define $D \in \mathbb{R}^{4\times 4}$ to be the diagonal matrix $D \stackrel{\text{def}}{=} \operatorname{diag}(k_2 + k_3 - \alpha k_2, (\alpha - 1)k_1, 1, 1)$. Finally, define a norm on $\mathbb{R}^4$ by $\|x\| = \|DP^{-1}x\|_1$, $x \in \mathbb{R}^4$, where $\|\cdot\|_1$ denotes the 1-norm on $\mathbb{R}^4$. It is easy to check that $\|f(x)\| = (k_2 + k_3 - \alpha k_2)x_2 + (\alpha - 1)k_1 x_1 x_3$ for all $x \in \mathcal{G}$. Consequently, $\dot{V}(x) + \|f(x)\| = 0$ for all $x \in \mathcal{G}$. Thus, (2) is satisfied with $c = 1$. It now follows from Theorem 4.3 that every orbit of the system (12) has finite arc length, and the system (12) is convergent relative to the nonnegative orthant. Next, every equilibrium in $\overline{\mathcal{E}}_1$ is a local minimizer of $V$ relative to the nonnegative orthant. Moreover, every $x \in \mathcal{E}_1$ has a neighborhood $\mathcal{V}$ such that $\mathcal{V} \cap \mathcal{E} \subseteq \mathcal{E}_1$, so that every equilibrium in $\mathcal{V}$ is a local minimizer of $V$ relative to the nonnegative orthant. Hence, it follows from (i) and (ii) of Theorem 5.2 that every equilibrium in $\overline{\mathcal{E}}_1$ is Lyapunov stable relative to the nonnegative orthant while every equilibrium in $\mathcal{E}_1$ is semistable relative to the nonnegative orthant.

## 7 Converse arc length results

In this section, we explore to what extent the converses of the arc-length-based results presented in the previous two sections hold.

### 7.1 Converse results for convergence

In this subsection, we present a partial converse to Theorem 4.3. Recall that Theorem 4.3 depends on Corollary 4.2, which in turn is based on Lemma 4.1. In general, the converse of Lemma 4.1 does not hold. Consequently, the converse of Corollary 4.2 does not hold in general as the following counterexample shows.

*Example 7.1* Consider the system (1) with $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x) = \sqrt{x_1^2 + x_2^2} f_r(x) + f_\theta(x)$, where the vector fields $f_r$ and $f_\theta$ are given by (5). In this

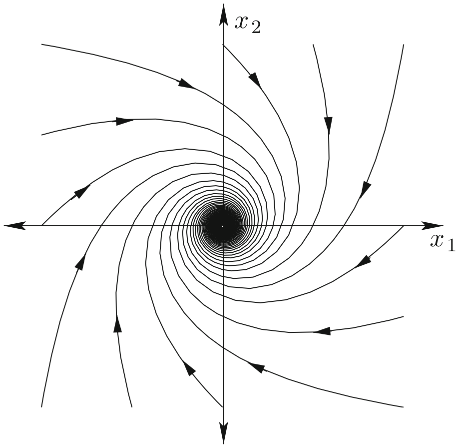


**Fig. 2** Convergent trajectories with infinite arc length

case, the system (1) has a unique equilibrium at $x = 0$. By considering the Lyapunov function $V(x) = x_1^2 + x_2^2$ and computing $\dot{V}(x) = -2(x_1^2 + x_2^2)^{3/2}$, it can be easily verified that the equilibrium $x = 0$ is globally asymptotically stable. Hence, the system (1) is convergent relative to $\mathcal{G} \stackrel{\text{def}}{=} \mathbb{R}^2$.

It is convenient to introduce the function $r : \mathbb{R}^2 \to \mathbb{R}$ given by $r(x) = \sqrt{x_1^2 + x_2^2}$. Then, for every $x \in \mathbb{R}^2$, $\dot{r}(x) = -(r(x))^2$. This differential equation for $r$ can be easily integrated to yield $r(\psi(t, x)) = r(x)/(1 + r(x)t)$ for every $x \in \mathbb{R}^2$ and $t \geq 0$.

It is easy to verify that, for every $x \in \mathbb{R}^2$, $\|f(x)\|_2 = r(x)\sqrt{1 + (r(x))^2} \geq r(x)$, where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^2$. Hence, for every $t \geq 0$ and $x \in \mathbb{R}^2$, we have $\int_0^t \|f(\psi(\tau, x))\|_2 \mathrm{d}\tau \geq \int_0^t r(\psi(\tau, x))\mathrm{d}\tau = \ln(1 + r(x)t)$. Clearly, the integral $\int_0^t \|f(\psi(\tau, x)\|\mathrm{d}\tau$ is unbounded in $t$ for every $x \in \mathbb{R}^2$. Thus, the system considered in this example is convergent, yet the orbit of every nonequilibrium point has infinite arc length. Figure 2 depicts the phase portrait of this system.

Intuitively, we expect that a convergent trajectory that does not curl up upon itself like the trajectories depicted in Fig. 2, will necessarily have finite arc length. Our next result provides a partial converse to Lemma 4.1 by formalizing this idea. The condition that a trajectory should not curl up upon itself is formalized in terms of the set of limit points of the unit tangent vector to the trajectory as time diverges to infinity. The proof requires a result which is stated and proved as Lemma A.1 in the appendix. Before stating the next result, we recall relevant definitions and introduce the necessary notation.

We recall that a set $\mathcal{K} \subseteq \mathcal{G}$ is *connected* if and only if every pair of relatively open sets $\mathcal{U}_i \subseteq \mathcal{G}$, $i = 1, 2$, satisfying $\mathcal{K} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{K} \neq \varnothing$, $i = 1, 2$, have a nonempty intersection. Also, a *connected component* of the set $\mathcal{K} \subseteq \mathcal{G}$ is a connected subset of $\mathcal{K}$ that is not properly contained in any connected subset of $\mathcal{K}$.

Given a set $\mathcal{K} \subseteq \mathbb{R}^n$, let $\mathrm{co}\, \mathcal{K}$ denote the union of the convex hulls of the connected components of $\mathcal{K}$. Given $\mathcal{K} \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we denote $\mathrm{dist}(x, \mathcal{K}) = \inf_{y \in \mathcal{K}} \|x - y\|$. Finally, let $\mathrm{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^n$.

**Lemma 7.2** *Let $y : [0, \infty) \to \mathbb{R}^n$ be a continuously differentiable function such that $\dot{y}(t) \neq 0$ for all $t \geq 0$. Let $\mathcal{L}$ denote the set of all subsequential limits of sequences of the form $\left\{\frac{1}{\|\dot{y}(t_i)\|}\dot{y}(t_i)\right\}$, where $\{t_i\}$ is a divergent sequence in $[0, \infty)$. Suppose $0 \notin \mathrm{co}\,\mathcal{L}$. If $y([0, \infty))$ is bounded, then $\dot{y}$ is absolutely integrable and $\lim_{t\to\infty} y(t)$ exists.*

*Proof* By Lemma A.1 of the appendix, $\mathcal{L}$ is connected and compact. Hence, $\mathrm{co}\,\mathcal{L}$ is the convex hull of $\mathcal{L}$ and, by compactness of $\mathcal{L}$, is closed. Therefore, $r \overset{\mathrm{def}}{=} \mathrm{dist}(0, \mathrm{co}\,\mathcal{L}) > 0$. Let $\mathcal{U} \overset{\mathrm{def}}{=} \{v \in \mathbb{R}^n : \mathrm{dist}(v, \mathrm{co}\,\mathcal{L}) < r/2\}$. It is easy to verify that $\overline{\mathcal{U}}$ is convex and does not contain 0. The Hahn–Banach separation theorem [40, Thm. 4.6.3] now implies that there exists $p \in \mathbb{R}^n$ and $\delta > 0$ such that $\|p\| = 1$ and, for every $u \in \mathcal{U}$, $p^{\mathrm{T}}u \geq \delta$. Let $M = \max\{p^{\mathrm{T}}u : u \in \mathbb{R}^n, \|u\| = 1\}$,[1] so that $p^{\mathrm{T}}u \leq M\|u\|$ for every $u \in \mathbb{R}^n$.

For every $t \in [0, \infty)$, denote $g(t) = \frac{1}{\|\dot{y}(t)\|}\dot{y}(t)$. Since $\mathcal{U}$ is an open neighborhood of $\mathcal{L}$, by Lemma A.1, there exists $T_1 > 0$ such that $g(t) \in \mathcal{U}$ for every $t > T_1$. It follows that $p^{\mathrm{T}}g(t) \geq \delta$ for every $t > T_1$. Now, suppose $y([0, \infty))$ is bounded, and let $N > 0$ be such that $\|y(t) - y(0)\| < N$ for every $t \geq 0$. Then, for every $t \geq T_1$, it follows that $MN > M\|y(t) - y(0)\| \geq p^{\mathrm{T}}[y(t) - y(0)] = \int_0^t p^{\mathrm{T}}\dot{y}(\tau)\mathrm{d}\tau = \int_0^{T_1} p^{\mathrm{T}}\dot{y}(\tau) + \int_{T_1}^t p^{\mathrm{T}}g(\tau)\|\dot{y}(\tau)\|\mathrm{d}\tau \geq \int_0^{T_1} p^{\mathrm{T}}\dot{y}(\tau) + \delta\int_{T_1}^t \|\dot{y}(\tau)\|\mathrm{d}\tau$.

The arguments above show that the increasing function $t \mapsto \int_0^t \|\dot{y}(\tau)\|\mathrm{d}\tau$ is bounded above. It follows that $\dot{y}$ is absolutely integrable. Lemma 4.1 now implies that $\lim_{t\to\infty} y(t)$ exists. □

The next result follows as a corollary of Lemma 7.2, and is a partial converse to Corollary 4.2. To state the result, we will need to consider the limiting direction set of a vector field introduced in [3].

Let $x \in \mathcal{G}\backslash\mathrm{int}\,\mathcal{E}$. Then, a vector $v \in \mathrm{S}^{n-1}$ is a *limiting direction* of $f$ at $x$ relative to $\mathcal{G}$ if there exists a sequence $\{x_i\}$ in $\mathcal{G}\backslash\mathcal{E}$ such that $x_i \to x$ and $\lim_{i\to\infty} \frac{1}{\|f(x_i)\|}f(x_i) = v$. The *limiting direction set* $\mathcal{L}_x$ of $f$ at $x$ relative to $\mathcal{G}$ is the set of all limiting directions of $f$ at $x$ relative to $\mathcal{G}$. Clearly, $\mathcal{L}_x$ is nonempty and compact. Moreover, for every $\varepsilon > 0$, there exists a relatively open neighborhood $\mathcal{U}_\varepsilon \subseteq \mathcal{G}$ of $x$ such that, for every $z \in \mathcal{U}_\varepsilon\backslash\mathcal{E}$, $\mathrm{dist}\left(\frac{1}{\|f(z)\|}f(z), \mathcal{L}_x\right) < \varepsilon$.

**Corollary 7.3** *Suppose $0 \notin \mathrm{co}\,\mathcal{L}_z$ for every $z \in \mathcal{E}\backslash\mathrm{int}\,\mathcal{E}$. Then, $\mathcal{R} = \mathcal{A} \cap \mathcal{B}$, that is, a trajectory of (1) converges to a limit in $\mathcal{G}$ if and only if its orbit is bounded relative to $\mathcal{G}$ and has finite arc length.*

*Proof* Consider $x \in \mathcal{R}$, and let $y = \psi^x$.

First suppose there exists $t \geq 0$ such that $\dot{y}(t) = 0$. Then, $y(t)$ is an equilibrium and hence $\dot{y}(t + h) = 0$ for every $h > 0$. Consequently, $S(x)$ is defined and hence $x \in \mathcal{A}$.

Next suppose that $\dot{y}(t) \neq 0$ for every $t \geq 0$, and let $\mathcal{L}$ denote the set of all subsequential limits of sequences of the form $\left\{\frac{1}{\|\dot{y}(t_i)\|}\dot{y}(t_i)\right\}$, where $\{t_i\}$ is a divergent

[1] Note that $M = \|p\|$ if $\|\cdot\|$ is the Euclidean norm.

sequence in $[0, \infty)$. Let $z = \psi_\infty(x)$, so that $z \in \mathcal{E}$. Since $\dot{y}(t) \neq 0$ for all $t \geq 0$, it follows that $z \notin \operatorname{int} \mathcal{E}$, and hence $0 \notin \operatorname{co} \mathcal{L}_z$. Since $y$ satisfies (1), $\mathcal{L}$ is contained in $\mathcal{L}_z$, so that $\operatorname{co} \mathcal{L} \subseteq \operatorname{co} \mathcal{L}_z$. We conclude that $0 \notin \operatorname{co} \mathcal{L}$. It now follows from Lemma 7.2 that $S(x) = \int_0^\infty \|\dot{y}(\tau)\| \mathrm{d}\tau$ exists and $x \in \mathcal{A}$. Since $x \in \mathcal{R}$ was chosen to be arbitrary, it follows that $\mathcal{R}$, which is contained in $\mathcal{B}$, is also contained in $\mathcal{A}$, that is, $\mathcal{R} \subseteq \mathcal{A} \cap \mathcal{B}$. The reverse inclusion follows from Corollary 4.2. □

It is interesting to revisit Example 7.1 in light of Corollary 7.3.

*Example 7.1 revisited.* All trajectories of the system considered in Example 7.1 converge to the origin. However, every nontrivial orbit of the system has infinite arc length. In this case, Corollary 7.3 implies that at least one connected component of the limiting direction set at the origin contains 0 in its convex hull. We will next verify that this is indeed the case.

Writing $x_1 = r \cos\theta$ and $x_2 = r \sin\theta$ yields

$$\|f(x)\|^{-1} f(x) = -\frac{r}{\sqrt{1+r^2}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \frac{1}{\sqrt{1+r^2}} \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$$

for the system considered in Example 7.1. It follows that every limit point of $\|f(x)\|^{-1} f(x)$ as $x \to 0$ is of the form $[\sin\alpha, -\cos\alpha]^{\mathrm{T}}$, where $\alpha \in \mathbb{R}$. Conversely, for every $\alpha \in \mathbb{R}$, the sequence $\{x_k\}$ given by $x_k = [k^{-1}\cos\alpha, k^{-1}\sin\alpha]^{\mathrm{T}}$ is such that $x_k \to 0$ and $\|f(x_k)\|^{-1} f(x_k) \to [\sin\alpha, -\cos\alpha]^{\mathrm{T}}$ as $k \to \infty$. It follows that the limiting direction set $\mathcal{L}_0$ at the origin is the unit circle $\mathrm{S}^1$. Since $\mathrm{S}^1$ is connected, $\operatorname{co} \mathcal{L}_0$ is simply the convex hull of $\mathrm{S}^1$, which contains 0.

The following theorem provides a partial converse to Theorem 4.3:

**Theorem 7.4** *Suppose* $0 \notin \operatorname{co} \mathcal{L}_z$ *for every* $z \in \mathcal{E}$. *If* (1) *is convergent relative to* $\mathcal{G}$, *then there exists a lower semicontinuous function* $V : \mathcal{G} \to [0, \infty)$ *such that the inequality* (2) *holds on* $\mathcal{G}$ *with* $c = 1$.

*Proof* Suppose (1) is convergent relative to $\mathcal{G}$, that is, $\mathcal{G} = \mathcal{R}$. By Corollary 7.3, $\mathcal{G} = \mathcal{R} \subseteq \mathcal{A} \subseteq \mathcal{G}$, that is, $\mathcal{G} = \mathcal{A}$. Letting $V : \mathcal{G} \to [0, \infty)$ be given by $V(x) = S(x)$, it follows from Proposition 3.1 that $V$ is lower semicontinuous on $\mathcal{G}$ and $\dot{V}(x) = -\|f(x)\|$ for every $x \in \mathcal{G}$. Thus, the inequality (2) is satisfied on $\mathcal{G}$ with $c = 1$. □

## 7.2 Converse results for stability

In general, the converses of the statements (i), (ii) and (iii) in Theorem 5.1 do not hold. For instance, the system considered in Example 7.1 had an asymptotically stable equilibrium such that the arc length function was undefined at every point other than the equilibrium. The following example demonstrates a semistable equilibrium such that the arc length function is defined everywhere on a neighborhood of the equilibrium and yet unbounded on every neighborhood of the equilibrium:

*Example 7.5* Let $g : [0, \infty) \times \mathbb{R} \to [0, \infty)$ be the continuous function given by

$$\begin{aligned} g(a, b) &= b^2, && a \le b, \\ &= 1 + (b^2 - 1)(b - a + 1), && b \le a < b + 1, \\ &= 1, && a \ge b + 1. \end{aligned}$$

For each $b \in \mathbb{R}$, the function $a \mapsto g(a, b)$ is globally bounded and piecewise linear. Hence, for each $b \in \mathbb{R}$, the function $a \mapsto [g(a, b)]^{-1}$ is integrable on every compact interval of $[0, \infty)$ and the function $a \mapsto \int_0^a [g(\tau, b)]^{-1} \mathrm{d}\tau$ is continuous. Also, $g(a, b) \ge 0$ for every $(a, b) \in [0, \infty) \times \mathbb{R}$, with equality if and only if $(a, b) = (0, 0)$. Let $r : \mathbb{R}^3 \to [0, \infty)$ be the function defined by $r(x) = \sqrt{x_1^2 + x_2^2}$. By the Urysohn lemma [41, Thm. 4.3.1], there exists a continuous function $h : \mathbb{R}^3 \to [0, 1]$ such that $h(x) = 0$ for every $x \in \mathbb{R}^3$ satisfying $r(x) \ge x_3 + 1$, and $h(x) = 1$ for every $x \in \mathbb{R}^3$ satisfying $r(x) \le x_3$. Consider the system (1), where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f(x) = g(r(x), x_3) \begin{bmatrix} -x_1 \\ -x_2 \\ 0 \end{bmatrix} - h(x) \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}. \tag{13}$$

Letting $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^3$, it follows that

$$\|f(x)\| = r(x)\sqrt{(g(r(x), x_3))^2 + (h(x))^2},$$

which implies that the set of equilibria is $\mathcal{E} = \{x \in \mathbb{R}^3 : r(x) = 0\}$.

For every $x \in \mathbb{R}^3$, let $\mathcal{G}_x$ denote the set $\{z \in \mathbb{R}^3 : z_3 = x_3\}$. It is easy to show that, for every $x \in \mathbb{R}^3$, $\mathcal{G}_x$ is positively invariant. Since the function $r$ is proper relative to $\mathcal{G}_x$ and $\dot{r}(x) = -r(x)g(r(x), x_3) \le 0$ for every $x \in \mathbb{R}^3$, it follows that, for every $x \in \mathbb{R}^3$, every orbit in $\mathcal{G}_x$ is bounded relative to $\mathcal{G}_x$.

We claim that $\mathcal{G}_x \subseteq \mathcal{A}$ for every $x \in \mathbb{R}^3$. To prove this, consider the continuous function $V : \mathbb{R}^3 \to \mathbb{R}$ given by $V(x) = \int_0^{r(x)} [g(\tau, x_3)]^{-1} \mathrm{d}\tau$. The derivative of $V$ along the solutions of (1) is given by $\dot{V}(x) = -r(x)$, so that $\dot{V}^{-1}(0) = \mathcal{E}$. Consider $x \in \mathbb{R}^3$, and let $M = \max\{1, x_3^2\}$. Then, $g(r(z), z_3) \le M$ for every $z \in \mathcal{G}_x$. Hence, for every $z \in \mathcal{G}_x$, $\|f(z)\| = r(z)\sqrt{(g(r(z), x_3))^2 + (h(z))^2} \le -\dot{V}(z)\sqrt{M^2 + 1}$. Theorem 4.3 now applies with $c = \sqrt{M^2 + 1}$ and $\mathcal{G} = \mathcal{U} = \mathcal{G}_x$. Hence, Theorem 4.3 implies that $\mathcal{G}_x \subseteq \mathcal{A}$.

Since $\mathcal{G}_x \subseteq \mathcal{A} \cap \mathcal{B}$ for every $x \in \mathbb{R}^3$ and since $\mathbb{R}^3 = \cup_{x \in \mathbb{R}^3} \mathcal{G}_x$, it follows from Corollary 4.2 that $\mathbb{R}^3 = \mathcal{R}$, that is, the system (1) is convergent relative to $\mathbb{R}^3$.

Next we claim that every equilibrium of the system (1) is Lyapunov stable relative to $\mathbb{R}^3$. To see this, consider $x \in \mathcal{E}$, and let $V_x : \mathbb{R}^3 \to \mathbb{R}$ be the function $V_x(z) = (r(z))^2 + (z_3 - x_3)^2$. Then, $V_x$ is positive definite at $x$ while $\dot{V}_x(z) = -2(r(z))^2 g(r(z), z_3)$ is negative semidefinite at $x$. Hence it follows that $x$ is Lyapunov stable. Thus, every solution of (1) converges to a Lyapunov stable equilibrium and the system (1) is semistable. In particular, 0 is semistable relative to $\mathbb{R}^3$.

Next consider $x \in \mathbb{R}^3$ satisfying $r(x) \leq x_3$ and $x_3 > 0$. Since every $z \in \mathbb{R}^3$ satisfying $r(z) \leq z_3$ satisfies $h(z) = 1$, $g(r(z), z_3) = z_3^2$ and $\dot{r}(z) = -z_3^2 r(z) \leq 0$, it follows that $\psi(t, x) \in \{z \in \mathbb{R}^3 : r(z) \leq z_3\}$ for every $t \geq 0$. In addition, (13) implies that $\psi(t, x) \in \{z \in \mathbb{R}^3 : z_3 = x_3\}$ for all $t \geq 0$. Hence, for every $t \geq 0$, $\dot{r}(\psi(t, x)) = -r(\psi(t, x))x_3^2$, so that $r(\psi(t, x)) = \mathrm{e}^{-x_3^2 t} r(x)$ for every $t \geq 0$. Therefore, for every $t \geq 0$, $\|f(\psi(t, x))\| = \mathrm{e}^{-x_3^2 t} r(x)\sqrt{1 + x_3^4}$. It is easy to evaluate $S(x) = r(x)\sqrt{1 + x_3^4}/x_3^2$.

Now consider the sequence $\{x_k\}$, where, for each $k = 1, 2, \ldots$, $x_k = [k^{-1}, 0, k^{-1}]^{\mathrm{T}}$. For each $k$, $x_k \in \{z \in \mathbb{R}^3 : r(z) \leq z_3\}$. Hence, $S(x_k) = k\sqrt{1 + k^{-4}}$. It is easy to see that $x_k \to 0$ while $S(x_k) \to \infty$ as $k \to \infty$. Thus, the origin is a semistable equilibrium contained in the interior of the set $\mathcal{A}$, and a point of discontinuity of the arc length function. This demonstrates that the converse of Theorem 5.1 does not hold in general.

While Example 7.5 demonstrates that the converse of Theorem 5.1 does not hold in general, the following result shows that the converse does hold in the case where no connected component of the limiting direction set at the equilibrium contains 0 in its convex hull.

**Theorem 7.6** *Let $x \in \mathcal{E}\backslash \operatorname{int} \mathcal{E}$ and suppose $0 \notin \operatorname{co} \mathcal{L}_x$. Then the following hold:*

(i) *If $x$ is Lyapunov stable relative to $\mathcal{G}$, then $x \in \mathcal{C}$.*
(ii) *If $x$ is semistable relative to $\mathcal{G}$, then $x \in \operatorname{int} \mathcal{C}$.*

*Proof* (i) Suppose $x$ is Lyapunov stable relative to $\mathcal{G}$. By Lemma A.2 from the appendix, there exists a bounded open neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$ of $\mathcal{L}_x$ such that $0 \notin \overline{\operatorname{co} \mathcal{U}}$. Denote $r_1 = \inf\{\|w\| : w \in \overline{\operatorname{co} \mathcal{U}}\}$ so that $r_1 > 0$.

Choose $\varepsilon > 0$ and let $\mathcal{V}_\varepsilon \subseteq \mathcal{G}$ be a relatively open neighborhood of $x$ such that $\|f(z)\|^{-1} f(z) \in \mathcal{U}$ for every $z \in \mathcal{V}_\varepsilon \backslash \mathcal{E}$ and $\|z - x\| < r_1 \varepsilon/2$ for every $z \in \mathcal{V}_\varepsilon$. By Lyapunov stability, there exists a relatively open neighborhood $\mathcal{V}_\delta \subseteq \mathcal{G}$ of $x$ such that $\psi_t(\mathcal{V}_\delta) \subseteq \mathcal{V}_\varepsilon$ for all $t \geq 0$.

First, consider $z \in \mathcal{V}_\delta \cap \mathcal{E}$. Then, $z$ satisfies $|S(z) - S(x)| = S(z) = 0 < \varepsilon$.

Next, consider $z \in \mathcal{V}_\delta \backslash \mathcal{E}$ and let $T = \sup\{t > 0 : f(\psi(t, z)) \neq 0\}$. Define $g : [0, T) \to \mathbb{R}^n$ by $g(t) = \|f(\psi(t, z))\|^{-1} f(\psi(t, z))$. The function $g$ is continuous and hence $g([0, T))$ is a connected subset of $\mathcal{U}$. Let $\mathcal{W}$ denote the connected component of $\mathcal{U}$ containing $g([0, T))$. Since $\mathcal{W}$ is connected and bounded, $\overline{\operatorname{co} \mathcal{W}}$ is convex and compact. Since $\mathcal{W} \subseteq \mathcal{U}$, it follows that $0 \notin \overline{\operatorname{co} \mathcal{W}}$. Let $r_2 = \inf_{z \in \overline{\operatorname{co} \mathcal{W}}} \|z\|$, so that $r_2 \geq r_1$. On applying Theorem 4.6.3 of [40], it follows that there exists $p \in \mathbb{R}^n$ such that $\|p\|_{\mathrm{i}} = 1$ and $p^{\mathrm{T}} w \geq r_2$ for all $w \in \overline{\operatorname{co} \mathcal{W}}$, where $\|p\|_{\mathrm{i}} \overset{\text{def}}{=} \max\{|p^{\mathrm{T}} w| : w \in \mathbb{R}^n, \|w\| = 1\}$ denotes the induced norm of the linear function[2] $w \mapsto p^{\mathrm{T}} w$. Therefore, for every $t \in [0, T)$,

[2] If $\|\cdot\|$ is the Euclidean norm, then $\|p\|_{\mathrm{i}} = \|p\|$.

$$\begin{aligned} r_1\varepsilon &> \|\psi(t,z)-x\| + \|x-z\| \\ &\geq \|p\|_{\mathrm{i}}\|\psi(t,z)-z\| \\ &\geq p^{\mathrm{T}}(\psi(t,z)-z) \\ &= \int_0^t p^{\mathrm{T}} g(\tau)\|f(\psi(\tau,z))\|\mathrm{d}\tau \\ &\geq r_2\int_0^t \|f(\psi(\tau,z))\|\mathrm{d}\tau. \end{aligned}$$

Letting $t \to T$, it follows that $z \in \mathcal{A}$ and $S(z) < r_1\varepsilon/r_2 \leq \varepsilon$.

Thus we have shown that, for every $\varepsilon > 0$, there exists a relatively open neighborhood $\mathcal{V}_\delta \subseteq \mathcal{G}$ such that $\mathcal{V}_\delta \subseteq \mathcal{A}$ and $|S(z) - S(x)| = S(z) < \varepsilon$ for all $z \in \mathcal{V}_\delta$. It follows that $x \in \mathcal{C}$.

(ii) Suppose $x$ is semistable relative to $\mathcal{G}$. By Lemma A.2, there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$ of $\mathcal{L}_x$ such that $0 \notin \overline{\mathrm{co}\,\mathcal{U}}$. Let $\mathcal{V}$ be an open neighborhood of $\mathcal{L}_x$ such that $\overline{\mathcal{V}} \subseteq \mathcal{U}$. There exists a relatively open neighborhood $\mathcal{V}_1 \subseteq \mathcal{G}$ of $x$ such that every solution in $\mathcal{V}_1$ converges to a Lyapunov stable equilibrium and, for every $z \in \mathcal{V}_1\backslash\mathcal{E}$, $\|f(z)\|^{-1}f(z) \in \mathcal{V}$. It follows that $\mathcal{L}_z \subseteq \overline{\mathcal{V}} \subseteq \mathcal{U}$ for every $z \in \mathcal{V}_1\backslash\mathrm{int}\,\mathcal{E}$. Since $0 \notin \mathrm{co}\,\mathcal{U}$, it follows that $0 \notin \mathrm{co}\,\mathcal{L}_z$ for every $z \in \mathcal{V}_1\backslash\mathrm{int}\,\mathcal{E}$. By (i) above, it follows that every equilibrium in $\mathcal{V}_1\backslash\mathrm{int}\,\mathcal{E}$ is contained in $\mathcal{C} \subseteq \mathrm{int}\,\mathcal{A}$. Since every equilibrium in $\mathrm{int}\,\mathcal{E}$ is clearly in $\mathcal{C}$, it follows that every equilibrium in $\mathcal{V}_1$ is contained in $\mathcal{C} \subseteq \mathrm{int}\,\mathcal{A}$. Let $\mathcal{V}_2 \subset \mathcal{G}$ be a relatively open neighborhood of $x$ such that $\overline{\mathcal{V}_2} \subseteq \mathcal{V}_1$. By Lyapunov stability of $x$, there exists a relatively open neighborhood $\mathcal{V}_3 \subseteq \mathcal{G}$ of $x$ such that $\psi_t(\mathcal{V}_3) \subseteq \mathcal{V}_2$ for every $t \geq 0$. It follows that $\psi_\infty(\mathcal{V}_3) \subseteq \overline{\mathcal{V}_2} \subseteq \mathcal{V}_1$. Now consider $z \in \mathcal{V}_3$. Then, $\psi_\infty(z)$ is an equilibrium in $\mathcal{V}_1$ and hence contained in $\mathcal{C}$. It now follows from (ii) of Proposition 3.2 that $z \in \mathcal{C}$. Since $z \in \mathcal{V}_3$ was chosen arbitrarily, we conclude that $\mathcal{V}_3 \subseteq \mathcal{C}$. Thus, $\mathcal{V}_3$ is a relatively open neighborhood of $x$ that is contained in $\mathcal{C}$, and hence $x \in \mathrm{int}\,\mathcal{C}$. □

The following result yields a partial converse to Theorem 5.2:

**Theorem 7.7** *Let $x \in \mathcal{E}\backslash\mathrm{int}\,\mathcal{E}$ and suppose $0 \notin \mathrm{co}\,\mathcal{L}_x$. Then the following hold:*

(i) *If $x$ is Lyapunov stable relative to $\mathcal{G}$, then there exists a relatively open neighborhood $\mathcal{V} \subseteq \mathcal{G}$ of $x$ and a lower semicontinuous function $V : \mathcal{V} \to \mathbb{R}^n$ such that $x$ is a local minimizer of $V$ relative to $\mathcal{G}$, $V$ is continuous at $x$, and $\dot{V}$ is continuous on $\mathcal{V}$ and satisfies the inequality (2) on $\mathcal{V}$ with $c = 1$.*

(ii) *If $x$ is semistable relative to $\mathcal{G}$, then there exists a relatively open neighborhood $\mathcal{V} \subseteq \mathcal{G}$ of $x$ and a continuous function $V : \mathcal{V} \to \mathcal{G}$ such that every equilibrium in $\mathcal{V}$ is a local minimizer of $V$, and $\dot{V}$ is continuous on $\mathcal{V}$ and satisfies the inequality (2) on $\mathcal{V}$ with $c = 1$.*

*Proof* (i) Suppose $x$ is Lyapunov stable relative to $\mathcal{G}$. It follows from (i) of Theorem 7.6 that $x \in \mathcal{C}$, that is, the arc length function $S$ is defined on a relatively open neighborhood $\mathcal{V} \subseteq \mathcal{G}$ of $x$ and continuous at $x$. Define $V : \mathcal{V} \to \mathbb{R}$ by $V(z) = S(z)$

for every $z \in \mathcal{V}$. Then, $V(z) \geq 0$ for every $z \in \mathcal{V}$ while $V(x) = 0$. Thus, $x$ is a local minimizer of $V$ relative to $\mathcal{G}$. By Proposition 3.1, $V$ is lower semicontinuous on $\mathcal{V}$, while $\dot{V}$ is given by $\dot{V}(z) = -\|f(z)\|$, $z \in \mathcal{V}$, and is clearly continuous. Inequality (2) is thus satisfied on $\mathcal{V}$ with $c = 1$.

(ii) Suppose $x$ is semistable relative to $\mathcal{G}$. By (ii) of Theorem 7.6, $x \in \operatorname{int} \mathcal{C}$, that is, the arc length function $S$ is defined and continuous on a relatively open neighborhood $\mathcal{V} \subseteq \mathcal{G}$ of $x$. Define $V : \mathcal{V} \to \mathbb{R}$ by $V(z) = S(z)$ for every $z \in \mathcal{V}$. Then, $V(z) \geq 0$ for every $z \in \mathcal{V}$ while $V(z) = 0$ for every $z \in \mathcal{V} \cap \mathcal{E}$. Thus, every equilibrium in $\mathcal{V}$ is a local minimizer of $V$ relative to $\mathcal{G}$. By Proposition 3.1, $\dot{V}$ is given by $\dot{V}(z) = -\|f(z)\|$, $z \in \mathcal{V}$, which is clearly continuous. Inequality (2) is thus satisfied on $\mathcal{V}$ with $c = 1$. □

*Remark 7.8* It is easy to see that any equilibrium in the relative interior of $\mathcal{E}$ is semistable relative to $\mathcal{G}$. Also, the arc length function is identically zero, and hence continuous, in a neighborhood of an equilibrium in int $\mathcal{E}$. Hence, Theorems 7.6 and 7.7 are stated only for equilibria that are not in the relative interior of $\mathcal{E}$.

*Remark 7.9* Theorems 7.6 and 7.7 state that the converses of statements (i) and (ii) of Theorems 5.1 and 5.2, respectively, hold under the assumption that no connected component of the limiting direction set contains 0 in its convex hull. Statements (i) and (ii) of Theorems 5.1 and 5.2 deal with Lyapunov stability and semistability, respectively, while statement (iii) in each of these theorems deals with asymptotic stability. It is natural to ask if the converses of statements (iii) in Theorems 5.1 and 5.2 hold under the assumption mentioned above. However, a well-known result from [42] states that in the case where $\mathcal{G} = \mathbb{R}^n$, the image of every open neighborhood of an asymptotically equilibrium under the vector field $f$ contains 0 in its interior. This implies that if $x$ is an asymptotically stable equilibrium of (1), then $\mathcal{L}_x = \mathrm{S}^{n-1}$, so that $0 \in \operatorname{co} \mathcal{L}_x$. Thus, the assumption made in Theorems 7.6 and 7.7 on the limiting direction set does not apply in the case of asymptotically stable equilibria.

## 8 Conclusion

We have shown that properties of the arc length function can be used to deduce convergence of trajectories and stability of equilibria. It is possible to infer properties of the arc length function by using Lyapunov functions. This leads to arc-length-based Lyapunov tests for convergence and stability. These tests do not require the Lyapunov function to be positive definite, and can yield stability conclusions for a continuum of equilibria using a single Lyapunov function. This makes our results especially well suited to applications that naturally involve a continuum of equilibria, as our examples illustrate. The converses of our results hold under the condition, captured in terms of the limiting direction set, that trajectories do not curl up upon themselves.

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## A Appendix

*Proof of Lemma 4.1* First we note that $\lim_{t\to\infty} y(t)$ exists if and only if, for every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $\|y(t+h) - y(t)\| < \varepsilon$ for every $h \geq 0$ and $t > T_\varepsilon$.

For every $t, h \geq 0$, we have

$$\|y(t+h) - y(t)\| \leq \int_t^{t+h} \|\dot{y}(\tau)\| \mathrm{d}\tau \leq \int_t^{\infty} \|\dot{y}(\tau)\| \mathrm{d}\tau. \tag{14}$$

Suppose $\dot{y}$ is absolutely integrable. Then, for every $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that the second integral in Eq. (14) is less than $\varepsilon$ for all $t > T_\varepsilon$. Hence, we conclude that $\lim_{t\to\infty} y(t)$ exists. □

The assertions of the following lemma are stated without proof on page 129 of [43]:

**Lemma A.1** *Let $y : [0, \infty) \to \mathbb{R}^n$ be continuously differentiable and suppose that $\dot{y}(t) \neq 0$ for all $t \geq 0$. Let $\mathcal{L}$ denote the set of all subsequential limits of sequences of the form $\left\{\frac{1}{\|\dot{y}(t_i)\|}\dot{y}(t_i)\right\}$, where $\{t_i\}$ is a increasing divergent sequence in $[0, \infty)$. Then $\mathcal{L}$ is nonempty, compact and connected, and, for every open neighborhood $\mathcal{U} \subseteq \mathbb{R}^n$ of $\mathcal{L}$, there exists $T > 0$ such that $\frac{1}{\|\dot{y}(t)\|}\dot{y}(t) \in \mathcal{U}$ for all $t > T$.*

*Proof* Define $g : [0, \infty) \to \mathrm{S}^{n-1}$ by $g(t) = \|\dot{y}(t)\|^{-1}\dot{y}(t)$. Our hypotheses on $y$ imply that $g$ is continuous. It is easy to show that $\mathcal{L} = \cap_{t\geq 0}\mathcal{B}_t$, where, for every $t \in [0, \infty)$, $\mathcal{B}_t \stackrel{\text{def}}{=} \overline{g([t, \infty))}$. For every $t \in [0, \infty)$, $\mathcal{B}_t$ is a closed subset of the compact set $\mathrm{S}^{n-1}$. Hence, for every $t \in [0, \infty)$, $\mathcal{B}_t$ is compact. Also, the family of sets $\{\mathcal{B}_t\}_{t\geq 0}$ is nested in the sense $\mathcal{B}_t \subseteq \mathcal{B}_h$ for every $t \geq h \geq 0$. Hence, Theorem 3.5.9 in [41] implies that $\mathcal{L}$ is nonempty and compact.

For each $t \in [0, \infty)$, $\mathcal{B}_t$ is the closure of the continuous image of a connected set, and hence connected. To show that $\mathcal{L}$ is connected, consider two disjoint open sets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^n$ such that $\mathcal{L} \cap \mathcal{U} \neq \varnothing$ and $\mathcal{L} \cap \mathcal{V} \neq \varnothing$. Since $\mathcal{L} = \cap_{t\geq 0}\mathcal{B}_t$, for each $t$, $\mathcal{B}_t \cap \mathcal{U} \neq \varnothing$ and $\mathcal{B}_t \cap \mathcal{V} \neq \varnothing$. For each $t$, let $\mathcal{M}_t \stackrel{\text{def}}{=} \mathcal{B}_t \backslash (\mathcal{U} \cup \mathcal{V})$. Then, for each $t \in [0, \infty)$, $\mathcal{M}_t$ is a closed subset of the compact set $\mathcal{B}_t$ and hence compact. Since $\mathcal{B}_t$ is connected for each $t$, it follows that $\mathcal{M}_t \neq \varnothing$. Also, the family of sets $\{\mathcal{M}_t\}_{t\geq 0}$ is nested in the sense $\mathcal{M}_t \subseteq \mathcal{M}_h$ for every $t \geq h \geq 0$. Hence, Theorem 3.5.9 in [41] implies that $\cap_{t\geq 0}\mathcal{M}_t \neq \varnothing$. Also, $\cap_{t\geq 0}\mathcal{M}_t \subseteq (\cap_{t\geq 0}\mathcal{B}_t)\backslash(\mathcal{U} \cup \mathcal{V}) = \mathcal{L}\backslash(\mathcal{U} \cup \mathcal{V})$. It follows that $\mathcal{L} \nsubseteq \mathcal{U} \cup \mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ were chosen arbitrarily, it follows that $\mathcal{L}$ is connected.

Next, let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open neighborhood of $\mathcal{L}$, and, for every $t \in [0, \infty)$, define $\mathcal{N}_t \stackrel{\text{def}}{=} \mathcal{B}_t \backslash \mathcal{U}$. Then, for every $t \in [0, \infty)$, $\mathcal{N}_t$ is a closed subset of the compact set $\mathcal{B}_t$, and hence compact. Moreover, $\{\mathcal{N}_t\}_{t\geq 0}$ is a nested family of sets. Suppose $\mathcal{N}_t$ is nonempty for every $t \geq 0$. Then Theorem 3.5.9 in [41] implies that $\cap_{t\geq 0}\mathcal{N}_t$ is nonempty. However, by construction, $\cap_{t\geq 0}\mathcal{N}_t = \cap_{t\geq 0}(\mathcal{B}_t\backslash\mathcal{U}) = (\cap_{t\geq 0}\mathcal{B}_t)\backslash\mathcal{U} = \mathcal{L}\backslash\mathcal{U} = \varnothing$. This contradiction implies that there exists $T \geq 0$ such that $\mathcal{N}_T = \varnothing$, that is, $\mathcal{B}_T \subseteq \mathcal{U}$. Since $g(t) \in \mathcal{B}_T$ for every $t \geq T$, it follows that $\|\dot{y}(t)\|^{-1}\dot{y}(t) \in \mathcal{U}$ for every $t \geq T$. □

For the next result, it will be useful to recall some basic facts on set convergence. The *limit superior* of a sequence $\{\mathcal{W}_k\}$ of subsets of $\mathbb{R}^n$ denoted $\limsup_{k\to\infty}\mathcal{W}_k$, is the set of all subsequential limits of sequences $\{w_k\}$ in $\mathbb{R}^n$ such that $w_k\in\mathcal{W}_k$ for every $k$, while the *limit inferior* of the sequence, denoted by $\liminf_{k\to\infty}\mathcal{W}_k$, is the set of limits of convergent sequences $\{w_k\}$ in $\mathbb{R}^n$ such that $w_k\in\mathcal{W}_k$ for every $k$. The sequence $\{\mathcal{W}_k\}$ *converges* to the set $\mathcal{W}\subseteq\mathbb{R}^n$ if $\mathcal{W}=\liminf_{k\to\infty}\mathcal{W}_k=\limsup_{k\to\infty}\mathcal{W}_k$. It follows from [44, Thm. 1.1.7], [45, Thm. 4.18] that every sequence of nonempty subsets of a bounded subset of $\mathbb{R}^n$ has a subsequence that converges to a nonempty set. The limit of a convergent sequence of connected subsets of a bounded set is also connected. See, for instance, Lemma A.1 in [3].

**Lemma A.2** *Suppose* $x\in\mathcal{G}\backslash\mathrm{int}\,\mathcal{E}$ *and* $0\notin\mathrm{co}\,\mathcal{L}_x$. *Then, for every* $\epsilon>1$, *there exists an open neighborhood* $\mathcal{U}\subseteq\mathbb{R}^n$ *of* $\mathcal{L}_x$ *such that* $0\notin\overline{\mathrm{co}\,\mathcal{U}}$ *and every* $w\in\overline{\mathrm{co}\,\mathcal{U}}$ *satisfies* $\|w\|\leq\epsilon$.

*Proof* For each $k=1,2,\ldots,$ let $\mathcal{U}_k=\{w\in\mathbb{R}^n:\mathrm{dist}(z,\mathcal{L}_x)<1/k\}$. For every $k$, $\mathcal{U}_k$ is a bounded open set containing $\mathcal{L}_x$, $\overline{\mathcal{U}_k}$ is compact, $\overline{\mathcal{U}_{k+1}}\subset\mathcal{U}_k$ and $\cap_k\overline{\mathcal{U}_k}=\mathcal{L}_x$. We claim that there exists $k$ such that $0\notin\overline{\mathrm{co}\,\mathcal{U}_k}$.

To prove our claim by contradiction, suppose that $0\in\overline{\mathrm{co}\,\mathcal{U}_k}$ for every $k$. Then, for each $k$, there exist a connected component of $\mathcal{W}_k$ of $\mathcal{U}_k$ and a vector $v_k$ contained in the convex hull of $\mathcal{W}_k$ such that the sequence $\{v_k\}$ converges to 0. Each $\mathcal{W}_k$ is a subset of the bounded set $\mathcal{U}_1$. Hence, there exists an increasing sequence $\{k_j\}_{j=1}^{\infty}$ of integers such that the subsequence $\{\mathcal{W}_{k_j}\}_{j=1}^{\infty}$ converges. Let $\mathcal{W}=\lim_{j\to\infty}\mathcal{W}_{k_j}$. Then, $\mathcal{W}$ is the limit of a sequence of connected subsets of the bounded set $\mathcal{U}_1$, and is hence connected [3, Lem. A.1].

Next, consider $w\in\mathcal{W}$. There exists a sequence $\{w_j\}$ such that $w_j\in\mathcal{W}_{k_j}\subseteq\mathcal{U}_{k_j}$ for every $j$, and $\lim_{j\to\infty}w_j=w$. Since $\{\mathcal{U}_k\}$ is a decreasing sequence of sets, for every $k$, the sequence $\{w_j\}$ is eventually contained in $\mathcal{U}_k$. Hence, $w\in\overline{\mathcal{U}_k}$ for every $k$, that is $w\in\cap_k\overline{\mathcal{U}_k}=\mathcal{L}_x$. Since $w\in\mathcal{W}$ was arbitrary, it follows that $\mathcal{W}\subseteq\mathcal{L}_x$. Hence, the connected set $\mathcal{W}$ is contained in a connected component of $\mathcal{L}_x$.

Carathéodory's theorem [46, Thm. 17.1] implies that, for every $j$, there exist vectors $w_j^i\in\mathcal{W}_{k_j}$, $i=1,\ldots,n$, and scalars $\lambda_j^i\in[0,1]$, $i=1,\ldots,n$, such that $\lambda_j^1+\cdots+\lambda_j^n=1$ and $\lambda_j^1w_j^1+\cdots+\lambda_j^nw_j^n=v_{k_j}$. For each $i=1,\ldots,n$, let $\lambda^i\in[0,1]$ and $w^i$ be subsequential limits of the bounded sequences $\{\lambda_j^i\}_{j=1}^{\infty}$ and $\{w_j^i\}_{j=1}^{\infty}$, respectively. Then, for every $i$, $w^i\in\lim_{j\to\infty}\mathcal{W}_{k_j}=\mathcal{W}$, while $\lambda^1+\cdots+\lambda^n=1$ and $\lambda^1w^1+\cdots+\lambda^nw^n=\lim_{j\to\infty}v_{k_j}=0$. Thus, $0\in\mathrm{co}\,\mathcal{W}\subseteq\mathrm{co}\,\mathcal{L}_x$, which is a contradiction. Hence, we conclude that there exists $k$ such that $0\notin\overline{\mathrm{co}\,\mathcal{U}_k}$.

Next let $\epsilon>1$, and choose an integer $i$ such that $i>\max\{k,(\epsilon-1)^{-1}\}$. Finally, let $\mathcal{U}=\mathcal{U}_i$. Then, every $w\in\mathcal{U}$ satisfies $\mathrm{dist}(w,\mathcal{L}_x)<i^{-1}<(\epsilon-1)$, while $\overline{\mathrm{co}\,\mathcal{U}}\subseteq\overline{\mathrm{co}\,\mathcal{U}_k}$, so that $0\notin\overline{\mathrm{co}\,\mathcal{U}}$. Consider $w\in\mathcal{U}$. Since $\mathcal{L}_x$ is compact, there exists $z\in\mathcal{L}_x$ such that $\|w-z\|=\mathrm{dist}(w,\mathcal{L}_x)<\epsilon-1$. Then, $\|w\|\leq\|w-z\|+\|z\|<(\epsilon-1)+1=\epsilon$. Since $w\in\mathcal{U}$ was chosen to be arbitrary, it follows that every $w\in\mathcal{U}$ satisfies $\|w\|<\epsilon$. Since the open ball $\{w\in\mathbb{R}^n:\|w\|<\epsilon\}$ is convex and contains $\mathcal{U}$, it follows that $\overline{\mathrm{co}\,\mathcal{U}}\subseteq\{w\in\mathbb{R}^n:\|w\|\leq\epsilon\}$. Thus, $\mathcal{U}$ is an open neighborhood of $\mathcal{L}_x$ such that $0\notin\overline{\mathrm{co}\,\mathcal{U}}$ and every $w\in\overline{\mathrm{co}\,\mathcal{U}}$ satisfies $\|w\|\leq\epsilon$. □

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