A note on mean convergence of Lagrange interpolation in $L_p(0 < p \leq 1)$

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Abstract

Let $w := \exp(-Q)$, where Q is of faster than smooth polynomial growth at ∞, for example $w_{k,\alpha}(x) := \exp\left(-\exp_k(|x|^\alpha)\right)$, $\alpha > 1$. We obtain a necessary and sufficient condition for mean convergence of Lagrange interpolation for such weights in $L_p(0 < p \leq 1)$ completing earlier investigations by the first author and D.S. Lubinsky in $L_p(1 < p < \infty)$.

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1 Introduction and Statement of Results

Let

$$\chi_n := \left\{\xi_{1,n},\, \xi_{2,n}, \ldots, \xi_{n,n}\right\},\ n \geq 1$$

be an arbitrary real interpolation matrix and $f : \mathbb{R} \to \mathbb{R}$ a given continuous function. Then if Π_{n-1} denotes the class of polynomials of degree $\leq n-1$, $n \geq 1$ and $\ell_{j,n}(\chi_n) \in \Pi_{n-1}$, $1 \leq j \leq n$ are the fundamental polynomials of Lagrange interpolation at the ξ_j, $1 \leq j \leq n$ satisfying for $1 \leq k \leq n$

$$\ell_{j,n}(\chi_n)(\xi_{j,n}) = \begin{cases} 1, & j = k \\ 0, & \text{otherwise}, \end{cases}$$

then the Lagrange interpolation polynomial of degree $n-1$ to f with respect to χ_n is denoted by $L_n(f, \chi_n)$ and admits the representation

$$L_n(f, \chi_n)(x) := \sum_{j=1}^{n} f(\xi_{j,n})\ell_{j,n}(\chi_n)(x),\ x \in \mathbb{R}.$$

In this note, we obtain a necessary and sufficient condition which ensures mean convergence of Lagrange interpolation in $L_p(0 < p \leq 1)$ for a class of even Erdős weights such as the weight $w_{k,\alpha}$ on the real line. This completes earlier work of the first author and D.S. Lubinsky in $L_p(1 < p < \infty)$, see [1] and [2]. To formulate our results, we need to define an admissible class of weights and a suitable interpolation matrix. Our class of weights w will be called admissible and we shall write $w \in \mathcal{E}$ if w is of the form $w^2 := e^{-2Q}$ where:

- $Q : \mathbb{R} \to [0, \infty)$ is even and continuous.
- $Q^{(2)}$ exists and $Q^{(j)}$, $j = 0, 1, 2$ is non negative in $(0, \infty)$.
- The function
$$T(x) := 1 + \frac{xQ''(x)}{Q'(x)}$$
is increasing in $(0, \infty)$ with
$$\lim_{x \to \infty} T(x) = \infty,$$
and
$$T(0^+) := \lim_{x \to 0^+} T(x) > 1.$$
- There exists $A > 0$ such that for sufficiently large x
$$\frac{1}{A} \leq \frac{T(x)}{\frac{xQ'(x)}{Q(x)}} \leq A.$$
- For every $\epsilon > 0$, there exists a positive constant A_1 so that uniformly for large enough x,
$$T(x) \leq A_1 (Q(x))^{\epsilon}.$$

Our class of weights is broad enough to easily cover the classical examples below, see [4]:
$$w_{k,\alpha}(x) := \exp\big(- Q_{k,\alpha}(x)\big) \tag{1.1}$$
where
$$Q_{k,\alpha}(x) := \exp_k(|x|^{\alpha}),\ k \geq 1, \alpha > 1.$$
$$w_{D,B}(x) := \exp\big(- Q_{D,B}(x)\big) \tag{1.2}$$
where
$$Q_{D,B}(x) = \exp\big(\log(D + x^2)\big)^{B},\ B > 1.$$
Here, $\exp_k(;) = \exp\big(\exp(\exp(;))\big)$ denotes the kth iterated exponential and D is a large enough but fixed absolute constant.

Given an admissible weight w, see [3], we may define orthonormal polynomials

$$p_n(x) := p_n(w^2, x) = \gamma_n x^n + \cdots, \ \gamma_n = \gamma_n(w^2) > 0, \ x \in \mathbb{R}$$

satisfying

$$\int_{\mathbb{R}} p_n(w^2, x) p_m(w^2, x) w^2(x) dx = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$$

and with zeros denoted by

$$-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty.$$

For each $n \geq 1$ and for the given weight w, we define the interpolatory matrix

$$U_n := \{x_{j,n} : 1 \leq j \leq n\}. \tag{1.3}$$

In [1, Theorem 1.2], one of us and D.S. Lubinsky showed the following result:

Theorem 1.1 Let $w \in \mathcal{E}$, $1 < p < \infty$, $\Delta \in \mathbb{R}$ and $\kappa > 0$. Then for

$$\lim_{n\to\infty} \left\| (f - L_n(f, U_n))) \, w \, (1+Q)^{-\Delta} \right\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying,

$$\lim_{|x|\to\infty} |fw|(x) \, (\log|x|)^{1+\kappa} = 0$$

it is necessary and sufficient that,

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}.$$

In this note we prove:

Theorem 1.2 *Let $w \in \mathcal{E}$, $0 < p \leq 1$, $\Delta \in \mathbb{R}$, and $\kappa > 0$. Then for*

$$\lim_{n\to\infty} ||(f - L_n(f, U_n)) w (1+Q)^{-\Delta}||_{L_p(\mathbb{R})} = 0 \tag{1.4}$$

to hold for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\lim_{|x|\to\infty} |f(x)| w(x) (\log|x|)^{1+\kappa} = 0 \tag{1.5}$$

it is necessary and sufficient that $\Delta > 0$.

Corollary 1.3 *Let $0 < p < \infty$, $\Delta \in \mathbb{R}$, and $\kappa > 0$. Then* (1.4) *holds for every continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying* (1.5) *if and only if*

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}.$$

2 Proofs

The proof of Theorem 1.2

To prove the sufficiency of the condition $\Delta > 0$, we reduce the problem to an application of Theorem 1.1. This idea first appeared in [5]. Throughout, C will denote a positive absolute constant which may take on different values at different times.

We choose $q > 1$ with $1 < pq < 4$. Since $\Delta > 0$, we may choose $\Delta_1 \in \mathbb{R}$ satisfying $\Delta > \Delta_1 > 0$. Then

$$\begin{aligned}
&||(f - L_n(f, U_n)w(1+Q)^{-\Delta}||^p_{L_p(\mathbb{R})} \\
&\leq \left(\int \left|(f - L_n(f, U_n)(x)w(x)(1+Q(x))^{-\Delta_1}\right|^{pq} dx\right)^{1/q} \times \\
&\times \left(\int \left|(1+Q(x))^{-(\Delta-\Delta_1)}\right|^{pq'} dx\right)^{1/q'}.
\end{aligned}$$

Since $1 < pq$ and $\Delta_1 > 0 = \max\{0, \frac{2}{3}(\frac{1}{4} - \frac{1}{pq})\}$, we have from Theorem 1.1 that

$$\lim_{n\to\infty} \left(\int \left|(f - L_n(f, U_n)(x)w(x)(1+Q(x))^{-\Delta_1}\right|^{pq} dx\right)^{1/q} = 0.$$

Since $(\Delta - \Delta_1)pq' > 0$, we have

$$\left(\int \left|(1+Q(x))^{-(\Delta-\Delta_1)}\right|^{pq'} dx\right)^{1/q'} < \infty.$$

Therefore

$$\lim_{n\to\infty} ||(f - L_n(f, U_n))w(1+Q)^{-\Delta}||_{L_p(\mathbb{R})} = 0.$$

To estabish the necessity in Theorem 1.2, we proceed much as in the proof of [5, Theorem 1.1] and [1, Theorem 1.2]. Let $\delta > 1 + \kappa$ and X be the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with

$$||f||_X := \sup_{x\in\mathbb{R}} |f(x)|w(x)(\log(2+|x|))^{\delta} < \infty.$$

Moreover, let Y be the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ with

$$||f||_Y := ||fw(1+Q)^{-\Delta}||^p_{L_p(\mathbb{R})} < \infty. \tag{2.1}$$

We then note that Y is not a normed space with respect to (2.1) for $0 < p \leq 1$ but is a metric space with metric

$$d(f, g) := ||(f - g)w(1+Q)^{-\Delta}||^p_{L_p(\mathbb{R})}.$$

Now each $f \in X$ satisfies (1.5) so that

$$\lim_{n\to\infty} ||f - L_n(f, U_n)]||_Y = 0.$$

That is, for each $f \in X$, there exists $\eta > 0$ such that for all $n \geq C$

$$||f - L_n(f, U_n)||_Y \leq \eta.$$

By the generalized uniform boundedness principle, see [6, pp 189-190], the norm of the operator $I - L_n(; U_n)$ is uniformly bounded. That is, for every $f \in X$ with $||f||_X \leq 1$ and $n \geq 1$, there exists a constant M such that

$$||f - L_n(f, U_n)||_Y \leq M||f||_X^p. \tag{2.2}$$

In particular, as $L_1(f, U_n) = f(0)$ (recall $p_1(x) = x$), we derive from (2.2) that for every continous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, and for every $n \geq 1$

$$||L_n(f, U_n)||_Y \leq 2M||f||_X^p \tag{2.3}$$

provided the right hand side of (2.3) is finite. We stress that M does not depend on f or n.

For every $u > 0$, let a_u denote the positive root of the equation

$$u := \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t) dt}{\sqrt{1 - t^2}}$$

which exists and is increasing with u, see [1]. Moreover, choose continuous functions g_n, $n \geq 1$ with

$$g_n = 0 \text{ in } [0, \infty) \cup (-\infty, -a_n/2),$$
$$||g_n||_X = 1, \tag{2.4}$$

and for $x_{jn} \in [-a_n/2, 0)$,

$$(g_n w)(x_{jn})(\log(2 + |x_{jn}|))^\delta \text{sign}(p_n'(x_j n)) = 1.$$

Then for $x \in [1, a_n]$, much as in the proof of the necessary condition of [1, Theorem 1.2], we have for large enough n that

$$|L_n(g_n, U_n)(x)| \geq C a_n^{1/2} |p_n(x)| (\log a_n)^{-\delta}.$$

It follows that using (2.3), (2.4) and an application of [1, Lemma 5.1], that we have uniformly for large enough n,

$$\begin{aligned} 1 = ||g_n||_X^p &\geq C \quad ||L_n(g_n, U_n)||_Y \\ &\geq C \quad a_n^{p/2} (\log a_n)^{-\delta p} ||p_n w (1 + Q)^{-\Delta}||_{L_p([1, a_n])}^p \\ &\geq C \quad a_n (\log a_n)^{-\delta p} Q(a_n)^{\min\{0, -\Delta p\}}. \end{aligned}$$

If $\Delta \leq 0$, the above equation implies that

$$a_n (\log a_n)^{-\delta p} \leq C$$

for every large enough n but this is impossible as a_n increases with n. Thus necessarily $\Delta > 0$. This completes the proof of Theorem 1.2 □.

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